Abstract
The central issue of the hidden subgroup problem (HSP) is to bound the number of identical copies of coset states necessary to identify the hidden subgroup. We show general upper and lower bounds for this identification and its variant by information-theoretic arguments. These general bounds are tight for HSPs with all the candidate subgroups having the same prime order. In particular, our results provide a unified approach that gives the same lower bounds as previously known results for several important instances including the symmetric group by Hallgren et al. and certain semidirect product groups such as the dihedral and Heisenberg groups by Bacon et al. It also gives a new lower bound for a general class of direct product groups. Furthermore, we also show matching upper bounds for those instances by evaluating our general upper bound. Our proofs are more general and simpler than the existing proofs.

1 Introduction
1.1 Background
The hidden subgroup problem is one of the central issues in quantum computation, which was introduced for revealing the structure behind exponential speedups in quantum computation [34].

Definition 1.1 (Hidden Subgroup Problem (HSP)) Let $G$ be a finite group. For a hidden subgroup $H \leq G$, 

we define a map \( f_H \) from \( G \) to a finite set \( S \) with the property that \( f_H(g) = f_H(gh) \) if and only if \( h \in H \). Given \( f_H : G \to S \) and a generator set of \( G \), Hidden Subgroup Problem (HSP) is the problem of outputting a set of generators for the hidden subgroup \( H \). We say that HSP over \( G \) is efficiently solvable if we can construct an algorithm in time polynomial in \( \log |G| \).

The nature of many existing quantum algorithms relies on efficient solutions to Abelian HSPs (i.e., HSPs over Abelian groups) \[43, 27, 7, 8\]. In particular, Shor’s celebrated quantum algorithms for factoring and discrete logarithm essentially consist of reductions to certain Abelian HSPs and efficient solutions to the Abelian HSPs \[42\]. Besides his results, many efficient quantum algorithms for important number-theoretic problems were based on solutions to Abelian HSPs such as Pell’s equation \[15\] and unit group of a number field \[16, 40\].

Recently, non-Abelian HSPs have also received much attention. It is well known that the graph isomorphism problem can be reduced to the HSP over the symmetric group \[7, 5\] (more strictly, the HSP over \( S_n \setminus S_2 \) \[9\]). Moreover, Regev showed that we can construct an efficient quantum algorithm for the unique shortest vector problem if we find an efficient solution to HSP over the dihedral group under certain conditions \[37\].

While the efficient quantum algorithm for general Abelian HSPs has been already given \[27, 34\], the non-Abelian HSPs are extremely harder than the Abelian ones. There actually exist efficient quantum algorithms for HSPs over several special classes of non-Abelian groups \[39, 12, 18, 13, 14, 22, 23, 30, 3\]. Nonetheless, most of important cases of non-Abelian HSPs are not known to have efficient solutions such as the dihedral and symmetric HSPs. Thus, finding efficient algorithms for non-Abelian HSPs is one of the most challenging issues in quantum computation.

The main approach to the non-Abelian HSPs is based on a generic framework called the standard method. To our best knowledge, all the existing quantum algorithms for HSPs except for \[11\] obey this framework. The heart of the standard method is a reduction to a quantum state identification for the so-called coset states, which contain information of the hidden subgroup.

**Definition 1.2 (Coset State and Standard Method)** Let \( G \) be any finite group and let \( H \) be the hidden subgroup of \( G \). We then define the coset state \( \rho_H \) for \( H \) as \( \rho_H = \frac{1}{|G|} \sum_{g \in G} |gH\rangle \langle gH| = \frac{|H|}{|G|} \sum_{g \in G/H} |gH\rangle \langle gH|, \) where \( |gH\rangle = \frac{1}{\sqrt{|H|}} \sum_{h \in H} |gh\rangle \).

**Standard Method with \( k \) Coset States**

1. Prepare two registers with a uniform superposition over \( G \) in the first register and all zeros in the second register: \( \frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle |0\rangle \).
2. Compute \( f_H(g) \) and store the result to the second register: \( \frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle |f_H(g)\rangle \).
3. Discard the second register: \( \rho_H = \frac{|H|}{|G|} \sum_{g \in G/H} |gH\rangle \langle gH| \).
4. Repeat (1)–(3) \( k \) times and then apply a quantum measurement to \( k \) copies of \( \rho_H \).

Thus the main task for solving HSP based on the standard method is to find an efficiently implementable quantum measurement extracting the information of the hidden subgroup from the coset states.

Many researchers have broadly studied hard instances of non-Abelian HSPs from positive and negative aspects based on the standard method. In particular, they have focused on how many coset states are sufficient and necessary to identify the hidden subgroup with high probability. In several classes of the non-Abelian HSPs for which efficient algorithms are unknown, it is shown that we can identify any hidden subgroup by (a possibly inefficient) classical post-process using the classical information obtained by the quantum Fourier transforms.
to polynomially many coset states [10, 18, 14, 30]. For the most general case, Ettinger, Høyer and Knill gave a quantum algorithm solving any HSP with polynomially many queries to the function $f_H$ by using more general quantum operation than the quantum Fourier transform [11].

Some positive results recently revealed the importance of joint measurements over multiple coset states. Note that the joint measurement over $k$ coset states is much more powerful than $k$ repetitions of a single-shot measurement using only one coset state. The joint measurement plays an essential role in a subexponential-time quantum algorithm for HSP over the dihedral group by Kuperberg [29] and its space-efficient variant by Regev [38], efficient quantum algorithms for HSPs over a class of semidirect product groups, including the Heisenberg group, by Bacon, Childs and van Dam [3], and subexponential-time quantum algorithms on direct product groups by Alagic, Moore and Russell [2]. These results suggest that it is indispensable to consider the necessary number of coset states in order to identify hidden subgroups.

The necessary number of coset states in the standard method has been intensively studied as well as the sufficient number to solve HSP. In the case of the dihedral group $\mathbb{Z}_N \rtimes \mathbb{Z}_2$, Bacon, Childs and van Dam showed that $\Theta(N)$ coset states are necessary and sufficient to identify a hidden subgroup with high probability by proving the optimality of their quantum measurement [4]. They also extended their approach to a certain class of semidirect product groups $A \rtimes \mathbb{Z}_p$, where $A$ is any Abelian group and $p$ is a prime [3]: $\Theta(\log |A|/ \log p)$ coset states are necessary and sufficient to solve HSPs over the semidirect product groups.

Particularly, the difficulty of the HSP over the symmetric group $S_n$ has been shown by a number of results. Hallgren, Russell and Ta-Shma first proved that it is impossible to efficiently distinguish between two kinds of coset states with the so-called weak Fourier sampling method on a single coset state [18]. Grigni et al. next proved that it is also impossible to efficiently distinguish between them even by the strong Fourier sampling method on a single state with random choices of bases for the representations of $S_n$ [14]. Kempe and Shalev also generalized the negative results of [18] and [14] on the hardness of SHSP in terms of the quantum Fourier sampling methods by using the permutation group theory [26]. For the case of a single coset state, Moore, Russell and Schulman finally proved that any quantum algorithm on a single coset state cannot solve the symmetric HSP by combining the representation theory with information-theoretic arguments [33]. Moore and Russell also extended their result to the case of two coset states [32]. They proved that any quantum algorithm operating over two coset states requires at least $\exp(\Omega(\sqrt{n}/ \log n))$ coset states. Hallgren et al. most recently gave the tight lower bound of $\Omega(n \log n)$ [17], which matches the upper bound given by Moore and Russell [31]. Their argument is also applicable to lower bounds for over a general class of finite groups satisfying some representation-theoretic conditions.

### 1.2 Our Contributions

Many previous results deeply relied on highly mathematical arguments like the group representation theory and the advanced finite group theory. This paper proposes simple proof techniques for bounds of the number of coset states necessary and sufficient to identify hidden subgroups over general finite groups.

We consider the Quantum State Identification introduced by Sen [41], which is a general problem of identifying an unknown state from a candidate set. (A restricted version of this problem was also discussed in [36].) The problem of identifying a hidden subgroup based on the standard method, named the Coset State Identification, can be regarded as a special case of the general problem.
Definition 1.3 (Quantum State Identification (QSI) and Coset State Identification (CSI))  Let $S = \{\rho_1, \ldots, \rho_N\}$ be a candidate set of $d$-dimensional quantum states. Given a black box that generates an unknown state $\sigma$ in $S$, the Quantum State Identification for $S$ is the problem of deciding $i \in \{1, \ldots, N\}$ such that $\sigma = \rho_i$. In particular, if the candidate set $\mathcal{S}_H$ is a set of coset states $\{\rho_H, \ldots, \rho_{H_N}\}$ for subgroups $\mathcal{H} = \{H_1, \ldots, H_N\}$, we call this problem the Coset State Identification (CSI) for $\mathcal{S}_H$.

We then give a general lower bound for QSI with a simple information-theoretic argument.

Theorem 1.4 (Lower Bound for QSI)  Let $S$ be any set of $d$-dimensional quantum states. Then, to solve QSI for $S$ with constant success probability, $\Omega\left(\frac{\log |S|}{\log d + \log \max_{\rho_S} \|\rho\|}\right)$ copies of the instance quantum state are necessary.

Evaluating the $l_2$-norm of the coset states, we can also obtain the following theorem on the lower bounds for CSI by the above theorem.

Theorem 1.5 (Lower Bound for CSI)  Let $\mathcal{H}$ be any set of subgroups of a finite group $G$. Then, to solve CSI for $\mathcal{S}_H$ with constant success probability, $\Omega\left(\frac{\log |\mathcal{H}|}{\log \max_{\mathcal{H} \in \mathcal{H}} |\mathcal{H}|}\right)$ coset states are necessary.

Note that the lower bound for CSI with any candidate set of subgroups of a finite group $G$ implies that for the number of the coset states to solve the HSP over $G$ based on the standard method. Using Theorem 1.5, it is quite easy to obtain the lower bounds for HSP over a specific finite group. We just find a large candidate set of the hidden subgroups of low order.

We also present general upper bound for CSI by using the pretty good measurement (also known as the squire root measurement or least squares measurement) [19].

Theorem 1.6 (Upper Bound for CSI)  Let $\mathcal{H}$ be any set of subgroups of a finite group. Then, to solve CSI for $\mathcal{S}_H$ with constant success probability, $O\left(\frac{\log |\mathcal{H}|}{\log \min_{\mathcal{H} \in \mathcal{H}} |\mathcal{H}|}\right)$ coset states are sufficient. In particular, $O\left(\frac{\log |\mathcal{H}|}{\log \min_{\mathcal{H} \in \mathcal{H}} |\mathcal{H}|}\right)$ coset states are sufficient if $|H|$ is a prime for every $H \in \mathcal{H}$.

Here, recall that we have $|H \cap H'| = 1$ for two subgroups $H$ and $H'$ if $|H|$ and $|H'|$ are primes and $H \neq H'$. This theorem is useful to estimate upper bounds for the sufficient number of coset states for a specific candidate set.

We also consider another version of distinction problems for coset states. Special cases of the following problem named the Triviality of Coset State have been discussed for the limitations of the standard method in many previous results, especially on the symmetric group [18, 14, 26, 32, 33, 17].

Definition 1.7 (Triviality of Coset State (TCS))  Let $\mathcal{H} = \{H_1, \ldots, H_N\}$ be a set of non-trivial subgroups of a finite group $G$, i.e., $H_i \neq \{id\}$ for every $i$. We then define a set of coset states $\mathcal{S}_H = \{\rho_{H_1}, \ldots, \rho_{H_N}\}$ corresponding to the set $\mathcal{H}$. Given a black box that generates an unknown state $\sigma$ that is either in $\mathcal{S}_H$ (i.e., a coset state for the non-trivial subgroup) or equal to $I/|G|$ (i.e., a coset state for the trivial subgroup), the Triviality of Coset State for $\mathcal{S}_H$ is the problem of deciding whether $\sigma$ is in $\mathcal{S}_H$ or equal to $I/|G|$. We say that a quantum algorithm solves TCS with constant advantage if it correctly decides whether a given state is in $\mathcal{S}_H$ or equal to $I/|G|$ with success probability at least $1/2 + \delta$ for some constant $\delta \in (0, 1/2]$.

Note that this problem might be efficiently solvable even if we cannot identify the hidden subgroup. Actually, if we can give a solution to TCS for $\mathcal{H}_{Sym} = \{H < S_n : H = \langle h \rangle, h^2 = id, h(i) \neq i (i = 1, \ldots, n)\}$, i.e., a
set of all the subgroups generated by the involution composed of \(n/2\) disjoint transpositions, we can also solve
the rigid graph isomorphism problem, i.e., the problem of finding an isomorphism between two graphs having
no non-trivial automorphisms, and the decisional graph automorphism problem, i.e., the problem of deciding
whether a given graph has non-trivial automorphisms or not \([28]\). We then obtain the following lower bound
for TCS over a general finite group using another simple argument.

**Theorem 1.8 (Lower Bound for TCS)** Let \(\mathcal{H}\) be any set of subgroups of a finite group. Then \(\Omega \left( \frac{\log |\mathcal{H}|}{\log \max_{H \in \mathcal{H}} |H|} \right)\) coset states are necessary to solve TCS for \(S_{\mathcal{H}}\) with constant advantage.

We also give general upper bound for the sufficient number of coset states to solve TCS. Our theorem
generalizes the result by Moore and Russell \([31]\), which gives an upper bound for TCS when the candidate set
is a conjugates of any subgroup. Note that this upper bound are tight for TCS up to a constant factor if every
\(H \in \mathcal{H}\) has the same order.

**Theorem 1.9 (Upper Bound for TCS)** Let \(\mathcal{H}\) be any set of subgroups of a finite group. Then \(O \left( \frac{\log |\mathcal{H}|}{\log \min_{H \in \mathcal{H}} |H|} \right)\) coset states are sufficient to solve TCS for \(S_{\mathcal{H}}\) with constant advantage.

Summarizing our bounds for CSI and TCS, we obtain the following corollary, which shows that TCS is as
hard as CSI from a viewpoint of the number of the coset states under some conditions.

**Corollary 1.10 (Tight Bounds for CSI and TCS)** Let \(\mathcal{H}\) be any set of subgroups of a finite group satisfying
that \(|H| = p\) for every \(H \in \mathcal{H}\), where \(p\) is a prime. Then, \(\Theta \left( \frac{\log |\mathcal{H}|}{\log p} \right)\) coset states are necessary and sufficient to
solve CSI for \(S_{\mathcal{H}}\) with constant success probability and to solve TCS for \(S_{\mathcal{H}}\) with constant advantage.

These results generalize or contain many previous results on the upper and lower bounds by simple and
unified arguments. Theorem 1.5 immediately leads to the lower bound \(\Omega(n\log n)\) for the number of the coset
states necessary to solve the HSP over the symmetric group \(S_n\), which was originally obtained by Hallgren et al. \([17]\) and \(\Omega(\log |A|/\log p)\) over a general class of semidirect product groups \(A \rtimes \mathbb{Z}_p\), including the dihedral
and Heisenberg groups, where \(A\) is any Abelian group and \(p\) is any prime, which was originally due to Bacon,
Childs, and van Dam \([3]\). While these results are based on different arguments for each group, our proofs are
rooted on a unified general argument. Our technique furthermore gives a new result on a general class of direct
product groups. We show that \(\Omega(n)\) coset states are necessary to solve HSP over a direct product group \(G^n\),
where \(G\) is an arbitrary finite group of size \(\geq 2\), based on the standard method. This generalizes the results on
certain classes of direct product groups by \([1, 17]\). In addition, we can easily obtain the matching upper bounds
to the above three lower bounds by Theorem 1.6.

As mentioned above, \(S_{\mathcal{H}_{\text{sym}}}\) is an important candidate set in TCS related to the rigid graph isomorphism
problem and the decisional graph automorphism problem. Applying Theorem 1.8 to this instance, we show
that \(\Omega(n\log n)\) coset states are necessary to solve TCS for \(S_{\mathcal{H}_{\text{sym}}}\). We also give a matching upper bound by
Theorem 1.9.

We moreover apply our arguments used in the above theorems to evaluation of the information-theoretic
security of the quantum encryption schemes proposed by Kawachi et al. \([24, 25]\). They proposed two quantum
encryption schemes: One is a single-bit encryption scheme, which has a computational security proof based on
the worst-case hardness of the decisional graph automorphism problem, and the other is a multi-bit encryption
scheme, which has no security proof. Since their schemes make use of quantum states quite similar to coset states over the symmetric group as the encryption keys and ciphertexts, our proof techniques are applicable to the security evaluation of their schemes. We prove that the success probability of any computationally unbounded adversary distinguishing between any two ciphertexts is at most $\frac{1}{2} + 2^{-\Omega(n)}$ in their log $m$-bit encryption scheme with the security parameter $n$ if the adversary has only $o\left(\frac{n \log n}{m \log m}\right)$ encryption keys.

2 Information-Theoretic Bounds for CSI and TCS

In this section, we present our main results on the general bounds for CSI and TCS, and applications to several important instances of HSP. We first introduce basic notions and useful lemmas for our proofs in Section 2.1. We then prove the general lower bounds for the number of coset states in order to solve QSI, CSI and TCS with high probability in Section 2.2. We also give the general upper bounds for CSI and TCS in Section 2.3. We finally show upper and lower bounds for several important instances by applying our main results to them in Section 2.4.

2.1 Basic Notions and Useful Lemmas

Any quantum operations for extracting classical information from quantum states can be generally described by the positive operator-valued measure (POVM) $\{M_i\}_{i \in S}$. A POVM $M = \{M_i\}_{i \in S}$ associated with a set of outcomes $S$ is a set of Hermitian matrices satisfying that $M_i \geq 0$ ($i \in S$) and $\sum_{i \in S} M_i = I$. Then the probability of obtaining outcome $k \in S$ by the POVM $M$ from a quantum state $\rho$ is given by $\text{tr}(M_k \rho)$.

The trace norm of a matrix $X \in \mathbb{C}^{d \times d}$ is useful to estimate success probability of quantum state distinction for two states, and is defined as $\|X\|_\text{tr} = \max\{Y, X\} = \text{tr} \sqrt{X^* X}$, where $\|Y\|_1$ is the $l_2$-norm of a matrix $Y$ and $\langle Y, X \rangle = \text{tr} Y^* X$ is the matrix inner product. It is well known that for any two quantum states $\rho_0$ and $\rho_1$ the average success probability of the optimal POVM distinguishing between two quantum states is equal to $\frac{1}{2} + \frac{1}{4} \|\rho_0 - \rho_1\|_\text{tr}$, i.e., $\frac{1}{4} \max_{M = \{M_0, M_1\}} (\text{tr} M_0 \rho_0 + \text{tr} M_1 \rho_1) = \frac{1}{2} + \frac{1}{4} \|\rho_0 - \rho_1\|_\text{tr}$. See [6] for more details on the matrix analysis and [35] [20] for basics of the quantum information theory.

We make use of the pretty good measurement in order to prove the general upper bound for CSI. The following lemma shown by Hayashi and Nagaoka [21] is useful to estimate the error probability of the pretty good measurement. (See also Lemma 4.5 in [20].)

Lemma 2.1 (Hayashi and Nagaoka [21]) For any Hermitian matrices $S$ and $T$ satisfying that $I \geq S \geq 0$ and $T \geq 0$, it holds that $I - \sqrt{S + T}^{-1} S \sqrt{S + T}^{-1} \leq 2(I - S) + 4T$, where $\sqrt{S + T}^{-1}$ is the generalized inverse matrix of $\sqrt{S + T}$.

In our several proofs, we need to calculate the rank of a coset state. The following lemma gives the estimation of the rank.

Lemma 2.2 For any coset state for a subgroup $H$ of a finite group $G$, it holds that $\text{rank}(\rho_H) = \frac{|G|}{|H|}$.

Proof. Let $|\psi\rangle$ be a purification of $\rho_H$ described as $|\psi\rangle = \sqrt{|G|} \sum_{g \in G} |g\rangle_A |f_H(g)\rangle_B$, where $f_H$ is the given function in the definition of HSP. Tracing out the register $A$, we have $\text{rank}(\text{tr}_A |\psi\rangle \langle \psi|) = |G/H|$. Since $\text{rank}(\text{tr}_A |\psi\rangle \langle \psi|) = |G/H|$.
rank \((\text{tr}_C|\psi\rangle\langle\psi|)\), we obtain \(\text{rank}(\rho_H) = \frac{G}{|\mathcal{H}|}\). \(\square\)

### 2.2 Lower Bounds

We next prove the key theorem on lower bounds for QSI. This theorem generally gives the necessary number of identical copies of an unknown quantum state for the identification.

**Theorem 2.3** Let \(S\) be any set of \(d\)-dimensional quantum states. Then \(\Omega\left(\frac{\log |S|}{\log \max_{\rho\in S} \|\rho\| + \log d}\right)\) copies of the instance quantum state are necessary to solve QSI for \(S\) with constant success probability.

**Proof.** Let \(M = \{M_1, \ldots, M_N\}\) be any POVM associated with \(S = \{\rho_1, \ldots, \rho_N\}\) using \(k\) copies of the instance quantum state. By using the fact that \(|\langle X, Y\rangle| \leq ||X||||Y||_{tr}\) for any matrices \(X, Y \in \mathbb{C}^{d \times d}\), the probability of \(M\) obtaining correct outcome is upper bounded by

\[
\frac{1}{N} \sum_{i=1}^{N} \text{tr} M_i \rho_i^\otimes k \leq \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \rho_i^\otimes k \|M_i\|_{tr} = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \rho_i^\otimes k \frac{\text{tr} (\sqrt{M_j^\dagger M_j})}{N} = \frac{1}{N} \sum_{i=1}^{N} \rho_i^\otimes k \left(\frac{\max_j \|\rho_j\|d^k}{N}\right).
\]

One can easily see that \(k\) must be \(\Omega\left(\frac{\log |S|}{\log \max_{\rho\in S} \|\rho\| + \log d}\right)\) to attain constant success probability. \(\square\)

We also present the general lower bounds for CSI by Theorem 2.4. Although it is not so easy to estimate the \(l_2\)-norm of a general quantum state, we can estimate that of the coset state by Lemma 2.2.

**Theorem 2.4** Let \(\mathcal{H}\) be any set of subgroups of a finite group. Then \(\Omega\left(\frac{\log |\mathcal{H}|}{\log \max_{H \in \mathcal{H}} |H|}\right)\) coset states are necessary to solve CSI for \(S_{\mathcal{H}}\) with constant success probability.

**Proof.** Let \(G\) be the underlying finite group. By the argument in Theorem 2.3, the success probability of any quantum algorithm that solves CSI with \(k\) coset states is upper bounded by \(\frac{(\max_{H \in \mathcal{H}} |H|)^k}{|\mathcal{H}|}\). Since the coset state \(\rho_H = \frac{1}{|G/H|} \sum_{g \in G/H} |gH\rangle\langle gH|\) for any subgroup \(H\) is a uniform summation of the matrices \(|gH\rangle\langle gH|\) orthogonal to each other, we obtain \(\|\rho_H\| = 1/\text{rank}(\rho_H)\). It follows that \(\|\rho_H\| = |H|/|G|\) by Lemma 2.2. The success probability is thus at most \(\frac{(\max_{H \in \mathcal{H}} |H|)^k}{|\mathcal{H}|}\), which implies that any quantum algorithm that solves CSI for \(S_{\mathcal{H}}\) requires \(\Omega\left(\frac{\log |\mathcal{H}|}{\log \max_{H \in \mathcal{H}} |H|}\right)\) coset states in order to attain constant success probability. \(\square\)

As mentioned in Section 1, we do not necessarily have to identify any hidden subgroup over the symmetric group in order to solve the rigid graph isomorphism problem and the decisional graph automorphism problem since we can reduce these problems to TCS for \(S_{\mathcal{H}_{\text{sym}}}\) over the symmetric group \(S_n\). Thus, we also show another proof technique to obtain the lower bound for TCS. The following theorem gives the general lower bound for any TCS. We can easily estimate the lower bound for the important instance of TCS by using this theorem.

**Theorem 2.5** Let \(\mathcal{H}\) be any set of subgroups of a finite group. Then \(\Omega\left(\frac{\log |\mathcal{H}|}{\log \max_{H \in \mathcal{H}} |H|}\right)\) coset states are necessary to solve TCS for \(S_{\mathcal{H}}\) with constant advantage.
Proof. We first show that the success probability of solving TCS for \( S_\mathcal{H} \) is upper bounded by that of identification for certain two quantum states. Let \( M = \{ M_0, M_1 \} \) be any POVM associated with \( \{|i\rangle, \mathcal{H}\} \). The success probability of \( M \) is given by \( \min_{\rho \in S_\mathcal{H}} \sum_{\rho \in S_\mathcal{H}} \{ \text{tr} M_1 (I/|G|)^{\otimes k}, \text{tr} M_1 (I/|G|)^{\otimes k} \} \). Also, it holds by the linearity of the trace and the POVM that \( \text{tr} M_1 (I/|G|)^{\otimes k}, \text{tr} M_1 (I/|G|)^{\otimes k} \geq \min_{\rho \in S_\mathcal{H}} \text{tr} M_1 (I/|G|)^{\otimes k} \). Thus, the success probability is at most \( \min_{\rho \in S_\mathcal{H}} \sum_{\rho \in S_\mathcal{H}} \{ \text{tr} M_1 (I/|G|)^{\otimes k}, \text{tr} M_1 (I/|G|)^{\otimes k} \} \). This is equal to the success probability of the identification for \( (I/|G|)^{\otimes k} \) and \( 1/|\mathcal{H}| \sum_{\rho \in S_\mathcal{H}} \text{tr} M_1 (I/|G|)^{\otimes k} \).

Note that we cannot apply Theorem 2.3 to this identification. Instead, we directly evaluate an upper bound of the trace norm of the matrix \( X = 1/|\mathcal{H}| \sum_{\rho \in S_\mathcal{H}} \text{tr} M_1 (I/|G|)^{\otimes k} \). Then the success probability of the identification is at most \( \frac{1}{2} \sum \|X\|_{tr} \) by the property of the trace norm. Naively expanding \( X \), we obtain by the triangle inequality

\[
\|X\|_{tr} = \left\| 1/|\mathcal{H}| \sum_{H \in \mathcal{H}} \frac{1}{|G|^k} \sum_{g_1, ..., g_k \in G} \left( \sum_{h_1, ..., h_k \in H} |g_1, ..., g_k, h_1, ..., h_k\rangle\langle g_1, ..., g_k, h_1, ..., h_k| - |g_1, ..., g_k, h_1, ..., h_k\rangle\langle g_1, ..., g_k, h_1, ..., h_k| \right) \right\|_{tr}
\]

\[
\leq \left\| 1/|\mathcal{H}| \sum_{g_1, ..., g_k \in G} \left( \sum_{H \in \mathcal{H}} \sum_{h_1, ..., h_k \in H} |g_1, ..., g_k, h_1, ..., h_k\rangle\langle g_1, ..., g_k, h_1, ..., h_k| \right) \right\|_{tr}
\]

\[
= \frac{1}{|\mathcal{H}|} \sqrt{\sum_{H \in \mathcal{H}} (|H| - 1)} \leq \sqrt{\left( \max_{H \in \mathcal{H}} |H| \right)^k - 1}/|\mathcal{H}|.
\]

The point of the last equation is that the norm of vector is equal to the square root of the number of the summands. In order to have this trace norm larger than some positive constant, \( k \) must be \( \Omega \left( \frac{\log |\mathcal{H}|}{\log \min_{H \in \mathcal{H}} |H|} \right) \) copies are necessary for constant advantage. \( \square \)

### 2.3 Upper Bounds

We present general upper bounds for CSI and TCS in this section. First, we prove the upper bound for CSI by using the pretty good measurement for \( S_\mathcal{H} \). In this proof, we make use of Lemma 2.1 to estimate the error probability of the pretty good measurement.

**Theorem 2.6** Let \( \mathcal{H} \) be any set of subgroups of a finite group. Then, to solve CSI for \( S_\mathcal{H} \) with constant success probability, \( O\left( \frac{\log |\mathcal{H}|}{\log \min_{H \in \mathcal{H}} |H|} \right) \) coset states are sufficient. In particular, \( O\left( \frac{\log |\mathcal{H}|}{\log \min_{H \in \mathcal{H}} |H|} \right) \) coset states are sufficient if \( |H| \) is a prime for every \( H \in \mathcal{H} \).

**Proof.** Let \( P_H \) be the projection onto the space spanned by \( \text{supp}(\rho_H) \) for \( H \in \mathcal{H} \). We consider the pretty good measurement \( M = \{ \Sigma^{-1/2} P_H \Sigma^{-1/2} \}_{H \in \mathcal{H}} \) for \( S_\mathcal{H} \), where \( \Sigma = \sum_{H \in \mathcal{H}} P_H \). Let \( \gamma_{H,H'} = ||(h, h') \in H \times H' : hh' = id|| \) for \( H, H' \in \mathcal{H} \). We now prove that the error probability of \( M \) is at most \( 4 \sum_{H' \in \mathcal{H}} \frac{(\gamma_{H,H'})^k}{|H'|} \) if the given state is \( \rho_H \).
Since we have
\[ \text{tr} \rho_{PH^H} = \frac{1}{|G|^2} \sum_{g,g' \in G} \sum_{h \in H} \text{tr}(ghg') = \frac{1}{|G|^2} \sum_{g \in G} \sum_{h \in H} \text{tr}(gh) \]

it follows that \( \text{tr} \rho_{PH^H}^k = \frac{\gamma_{H,H^H}}{|G|^k} \). Setting \( S = P_{H^k}^k \) and \( T = \bigcup_{h \in H} P_{H^k}^k \), if the given state is \( \rho_H \), the error probability of \( M \) is

\[ \text{tr}(I - \Sigma_{H \neq H'}^2 P_{H^k}^k) \rho_H^k \leq 2 \text{tr}(I - P_{H^k}^k) \rho_H^k + 4 \text{tr} \left( \sum_{H' \neq H} P_{H^k}^k \right) \rho_H^k = 4 \sum_{H' \neq H} (\text{tr} \rho_{PH^H}^k)^k = 4 \sum_{H' \neq H} \gamma_{H,H^H}^k. \]

We can easily obtain the upper bound of the error probability from the above estimation. Since we have

\[ 4 \max_{H \in \mathcal{H}} \sum_{H' \neq H} \frac{(\gamma_{H,H^H})^k}{|H'|^k} \leq 4 |\mathcal{H}| \max_{H \in \mathcal{H}} \frac{\log \min_{H' \in \mathcal{H}} |H'|^k}{\log |H|^k}, \]

the error probability of \( M \) is at most \( \frac{4 |\mathcal{H}| \max_{H \in \mathcal{H}} \gamma_{H,H^H}}{\log \min_{H' \in \mathcal{H}} |H'|^k} \), which implies that \( O \left( \frac{\log |H|}{\log \min_{H' \in \mathcal{H}} |H'|} \right) \)

\( \mod \) \( \frac{\log |H|}{\log \min_{H' \in \mathcal{H}} |H'|} \)

Theorem 2.7 Let \( \mathcal{H} \) be any set of subgroups of a finite group. Then \( O \left( \frac{\log |H|}{\log \min_{H' \in \mathcal{H}} |H'|} \right) \) coset states are sufficient to solve TCS for \( S_{\mathcal{H}} \) with constant advantage.

Proof. Let \( G \) be the underlying finite group. We consider a projection \( T \) onto the space spanned by \( \bigcup_{H \in \mathcal{H}} \text{supp}(P_{H^k}^k) \). It obviously holds that \( \text{tr} T \rho_H^k = 1 \) for every \( H \in \mathcal{H} \). On the other hand, the error probability is given by \( \text{tr} T(I/|G|^k) \rho_H^k \). Then we have \( \text{tr} T(I/|G|^k) \rho_H^k = \frac{\text{rank}(T)}{|G|^k} \leq \frac{\text{rank}(\rho_H)}{|G|^k} \frac{\text{rank}(P_{H^k}^k)}{|G|^k} \). Since \( \text{rank}(\rho_H) = |G|/|H| \)

by Lemma 2.2 we obtain \( \frac{\sum_{H \in \mathcal{H}} \text{rank}(\rho_H)^k}{|G|^k} = \frac{\sum_{H \in \mathcal{H}} |G|/|H|^k}{|G|^k} \leq \frac{|H|}{\text{min}_{H' \in \mathcal{H}} |H'|^k} \). This implies that at most \( O \left( \frac{\log |H|}{\log \min_{H' \in \mathcal{H}} |H'|} \right) \) coset states are sufficient for constant advantage.

2.4 Applications to Important Instances

Using Theorems 2.4, 2.5, 2.6 and 2.7, we now present upper and lower bounds for CSI and TCS on the following three groups: the semidirect product group \( A \rtimes_{\phi} \mathbb{Z}_p \) for any Abelian group \( A \) and any prime \( p \), the symmetric group \( S_n \), and a direct product group \( G^n \) for any finite group \( G \) of constant size \( \geq 2 \).

Corollary 2.8 Let \( \mathcal{H}_{\text{SDP}} = \{ ((a, 1)) < A \rtimes_{\phi} \mathbb{Z}_p : a \in A \} = \{ (0, 0), (a, 1), (a + \phi(a), 2), \ldots, (\sum_{i=0}^{p-1} \phi^i(a), p - 1) : a \in \mathbb{Z}_p \} \), where \( \phi^{i+1}(a, b) = \phi(\phi^i(a, b)) \) and \( \phi^0(a, b) = (a, b) \) for any \( (a, b) \in A \rtimes_{\phi} \mathbb{Z}_p \). Then \( \Theta(\log |A|/\log p) \) coset states are necessary and sufficient to solve CSI for \( S_{\mathcal{H}_{\text{SDP}}} \) with constant success probability and to solve TCS for \( S_{\mathcal{H}_{\text{SDP}}} \) with constant advantage.
Corollary 2.9  Let $\mathcal{H}_{\text{Sym}} = \{ H < S_n : H = \langle h \rangle, h^2 = id, h(i) \neq i (i = 1, ..., n) \}$. Then $\Theta(n \log n)$ coset states are necessary and sufficient to solve CSI for $S_\mathcal{H}_{\text{Sym}}$ with constant success probability and to solve TCS for $S_\mathcal{H}_{\text{Sym}}$ with constant advantage.

Corollary 2.10  Let $\mathcal{H}_{\text{DP}} = \{ H < G^n : H \text{ is generated by } (h_1, ..., h_n), \text{where } [n/2] \text{ elements of } \{h_1, ..., h_n\} \text{ are } h \text{ for any fixed } h \neq id \text{ and the remaining ones are } id \}$. Then $\Theta(n)$ coset states are necessary and sufficient to solve CSI for $S_\mathcal{H}_{\text{DP}}$ and to solve TCS for $S_\mathcal{H}_{\text{DP}}$ with constant advantage.

Note that $|\mathcal{H}_{\text{SDP}}| = |A|, |\mathcal{H}_{\text{Sym}}| = \Theta(n \log n)$ and $|\mathcal{H}_{\text{DP}}| = \Theta(n)$, and the orders of any subgroups in $\mathcal{H}_{\text{SDP}}, \mathcal{H}_{\text{Sym}}$ and $\mathcal{H}_{\text{DP}}$ are $p, 2$ and at most $|G|$, respectively. Moreover, we have $|H \cap H'| = 1$ for any distinct subgroup $H$ and $H'$ of $\mathcal{H}_{\text{SDP}}, \mathcal{H}_{\text{DP}}$ and $\mathcal{H}_{\text{DP}}$. By our theorems and the above facts, we can easily prove the above corollaries. These results imply that any quantum algorithm based on the standard method which solves HSP over $A \rtimes \mathbb{Z}_p, S_n$ and $G^n$ respectively requires $\Omega(\log |A|/\log p), \Omega(n \log n)$ and $\Omega(n)$ coset states. For the cases of $A \rtimes \mathbb{Z}_p$ and $S_n$, these lower bounds have already been proven based on the different techniques in $[1, 17]$, respectively. On the other hand, the same lower bound for $G^n$ is known only under some representation-theoretic conditions for $G$ $[1, 17]$. We stress that our lower bound holds for a more general class of direct product groups.

In addition, a quantum algorithm for HSP over $S_n \wr S_2 < S_{2n}$ is sufficient to solve the graph isomorphism problem $[21]$. We can also prove that $\Omega(n \log n)$ coset states are necessary to solve HSP over $S_n \wr S_2$ based on the standard method by the almost same argument as the case of $S_n$. (In $[17]$, they proved that showing the lower bounds for $S_n \wr S_2$ can be reduced to that for the symmetric group by a representation-theoretic argument.)

3 Security Evaluation of Quantum Encryption Schemes

Our arguments are applicable not only to bounds for HSP but also to security evaluation of quantum cryptographic schemes. In this section, we apply our arguments to evaluation of the information-theoretic security of the quantum encryption schemes proposed in $[24, 25]$. As mentioned in Section 1, they proposed single-bit and multi-bit quantum encryption schemes. While they gave the complexity-theoretic security to the single-bit scheme under the assumption of the worst-case hardness of the decisonal graph automorphism problem, the multi-bit one has no security proof. Also, they have already proven in $[25]$ that any computationally unbounded quantum algorithm cannot solve a certain quantum state distinction problem that underlies the single-bit scheme with few samples by reducing the solvability of their distinction problem to the result of $[17]$. On the other hand, the security of their encryption schemes, as well as the underlying problem for their multi-bit scheme, are not evaluated yet from a viewpoint of the quantum information theory.

Their schemes make use of certain quantum states for their encryption keys and ciphertexts. We now call these quantum states encryption-key states and cipherstates, respectively. Since their multi-bit encryption scheme contains the single-bit one as a special case if we ignore its efficiency and complexity-theoretic security, we only discuss their multi-bit scheme in this paper.

We now describe their multi-bit encryption scheme in detail. Assume that the message length parameter $m$ divides the security parameter $n$, where $m \in \{2, ..., n\}$. Let $\mathcal{K}_m^n = \{ h : h = (a_1 \cdots a_m) \cdots (a_{n-m+1} \cdots a_n), a_i \in \{1, ..., n\}, a_i \neq a_j (i \neq j) \subset S_n \}$. i.e., a set of the permutations composed of $n/m$ disjoint cyclic permutations,
which is used for the decryption key. In this scheme, we exploit the following quantum state for a message $s$:

$$
\rho_h^{(s)} = \frac{1}{m!} \sum_{\sigma \in S_n} \left( \sum_{k=0}^{m-1} \omega_m^{k} (gh^k) \left( \sum_{l=0}^{m-1} \omega_m^{-l} (gh^l) \right) \right), \text{where } \omega_m = e^{2\pi i/m} \text{ and } h \in \mathcal{K}_n^m.
$$

Note that $\rho_h^{(0)}$ is the coset state for the hidden subgroup $|K_m \rangle$. We now refer to as $(n, m)$-QES their multi-bit encryption scheme with the security parameter $n$ and the message length parameter $m$. The protocol of $(n, m)$-QES is summarized as follows.

**Protocol: $(n, m)$-QES**

1. The receiver Bob chooses his decryption key $h$ uniformly at random from $\mathcal{K}_n^m$ and generates the encryption-key states $\sigma_h = (\rho_h^{(0)}, ..., \rho_h^{(m-1)})$.

2. The sender Alice requests the encryption-key state $\sigma_h$ to Bob. She picks $\rho_h^{(s)}$ up from $\sigma_h$ as the cipherstate corresponding to her classical message $s \in \{0, ..., m - 1\}$ and then sends it to him.

3. Bob decrypts her cipherstate $\rho_h^{(s)}$ with his decryption key $h$. We assume the same adversary model except for Eve’s computational power as the original ones in [24, 25]. Note that the eavesdropper Eve can also request the same encryption-key states to Bob as one of senders. Eve in advance requests the encryption-key states to Bob. When Alice sends to Bob her cipherstate that Eve wants to eavesdrop, Eve picks up Alice’s cipherstate and then tries to extract Alice’s message from the cipherstate with the encryption-key states by computationally unbounded quantum computer, i.e., Eve can apply an arbitrary POVM over the cipherstates and encryption-key states to extract Alice’s message. We actually consider the stronger security notion such that Eve cannot distinguish between even two candidates, i.e., she cannot find a non-negligible gap between $\text{tr} M_1(\rho_h^{(s)} \otimes \sigma_h^{0k})$ and $\text{tr} M_1(\rho_h^{(s)} \otimes \sigma_h^{0k})$ even by the optimal POVM $M = \{M_0, M_1\}$ when Bob chooses $h$ uniformly at random. Since the gap is at most $\frac{1}{2} \| \sum_{h \in \mathcal{K}_n^m} \rho_h^{(s)} \otimes \sigma_h^{0k} - \rho_h^{(s)} \otimes \sigma_h^{0k} \|_2$, this notion can be formalized by the trace norm between them. Then, we say that the cipherstates are information-theoretically indistinguishable within $k$ encryption-key states if $\| \sum_{h \in \mathcal{K}_n^m} \rho_h^{(s)} \otimes \sigma_h^{0k} - \rho_h^{(s)} \otimes \sigma_h^{0k} \|_2 = 2^{-\Omega(k)}$.

For this security notion, we can obtain the following theorem by our information-theoretic arguments. The proof is almost straightforward by Theorem 2.5.

**Theorem 3.1** The cipherstates of $(n, m)$-QES are information-theoretically indistinguishable within $o\left(\frac{n \log n}{m \log m}\right)$ encryption-key states.

**Proof.** Let $l_s = \left\| \sum_{h \in \mathcal{K}_n^m} \rho_h^{(s)} \otimes \sigma_h^{0k} - (1/n!)^\otimes m^{k+1} \right\|_0$. Then the trace norm between two state sequences given in the definition of the information-theoretic indistinguishability is at most $l_s + l_{s'}$ by the triangle inequality. Since the trace norm is invariant under unitary transformations, we can show that $l_s + l_{s'} = 2l_0$ by taking appropriate unitary operators. Then we can prove that $l_0 \leq \sqrt{m^{mk+1} / |\mathcal{K}_n^m|}$ by the argument of Theorem 2.5. Since we have $|\mathcal{K}_n^m| \approx \frac{m^{mk+1} }{e^{-m/m}} \sqrt{\frac{m}{\log m}}$ by the standard counting method and the Stirling approximation, the trace norm is at most $2^{-\Omega(n)}$ if $k = o\left(\frac{n \log n}{m \log m}\right)$.

For example, when we set $m = n^\epsilon$ for any constant $0 < \epsilon < 1$, we obtain the $\epsilon \log n$-bit encryption scheme whose cipherstates are information-theoretically indistinguishable within $o(n^{1-\epsilon})$ encryption-key states.

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