Maximal Admission of Joint Range of Motion Based on Redundancy Resolution for Kinematically Redundant Manipulators

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Abstract—This paper addresses a inverse kinematics problem for kinematically redundant manipulators. In the problem, using the redundant degrees-of-freedom we can execute a secondary task which has no effect on a primary task. We usually design the secondary task as a performance criterion function to avoid obstacles, kinematic singularities, or joint limits, etc. For avoiding joint limits, however, many performance criterion functions are expected to converge a particular angle within joint range of motion. There also exists a performance criterion function which cannot avoid joint limits essentially. In this paper we propose a motion planning method with maximal admission of joint range of motion for kinematically redundant manipulators. The effectiveness of our proposed method is demonstrated by a numerical example that each joint avoids the neighborhood of joint limits strictly.

Keywords—kinematically redundant manipulators; joint limits; inverse kinematics; redundancy resolution; motion planning

I. INTRODUCTION

Almost all robotic systems are practically subject to some constraints such as actuator saturation and joint limits. If such constraints are not considered in designing a controller, it can deteriorate performance of the control system drastically and, at worst, lead to instability and breakdown of the system. Accordingly, a control design in consideration of these constraints is one of inherent problems on practical robotic systems.

A manipulator with more degrees-of-freedom (DoF) than are required to perform a given task (e.g., at the end-effector) is called a kinematically redundant manipulator. The redundant DoF provides the task execution with flexibility and adaptability. We can obtain a joint motion from a task motion by solving the so-called inverse kinematic problem. However, the problem for a kinematically redundant manipulator admits an infinite number of solutions. Hence a performance criterion is generally introduced to resolve the redundant DoF in the solutions.

We can exploit the redundant DoF to execute a secondary task which has no effect on a primary task. We can design a performance criterion function to perform a desired secondary task. The secondary task which has performed so far is to avoid obstacles [1], [2], kinematic singularities [3], and joint limits [3]–[11], to improve the manipulability [12], to minimize each joint torque [13]–[15], etc.

This paper focuses on avoiding joint limits as the secondary task. Summarizing the conventional performance criterion functions for this task, many are designed so that a joint angle converges to a particular angle within joint range of motion. This concept is negative because we cannot exploit the joint range of motion effectively. Although the functions based on the alternative concept also exist, these have possibility to exceed joint limits essentially.

In this paper we present a motion planning method to avoid joint limits for kinematically redundant manipulators. This method is based on a new performance criterion function. The main feature of the proposed function is to admit joint range of motion maximally. This function adopts the Gradient Projection Method (GPM) [4] to obtain the redundancy resolution. The effectiveness of our proposed method is demonstrated in some simulations for a three DoF planar manipulator. The difference between the proposed and conventional functions is also discussed from the viewpoint of the definition and the simulation results.

The rest of the paper is organized as follows. Section 2 recalls the inverse kinematics of redundant manipulators and one of the solutions, the GPM [4]. In Section 3, a new performance criterion function for the redundancy resolution is presented and is also discussed in comparison with the conventional functions. In Section 4, we give some simulation results to validate the proposed function. Finally, in Section 5, the main contributions of the paper are summarized.

II. INVERSE KINEMATICS OF REDUNDANT MANIPULATORS AND THESE SOLUTIONS

This section recalls the inverse kinematics of redundant manipulators and one of the solutions, the GPM [4]. We consider the case where a n-DoF manipulator with n actuated (active) joints executes a m (< n)-DoF primary task.
A. Inverse Kinematics of Redundant Manipulators

Let \( q \in \mathbb{R}^n \) and \( r \in \mathbb{R}^m \) be the joint vector and the task vector, respectively. The both are related as follows.

\[
r = f(q)
\]

Differentiating (1) with respect to time, we obtain the forward kinematics

\[
\dot{r} = \frac{\partial f}{\partial q} \dot{q} = J(q) \dot{q}.
\]

The matrix \( J \in \mathbb{R}^{m \times n} \) is the so-called task Jacobian matrix (also called analytic Jacobian matrix).

The inverse kinematics is to solve (2) with respect to \( q \). Here we cannot exploit the inverse of \( J \) because \( J \) is not square. Usually, regarding the inverse kinematics as to solve

\[
\begin{align*}
\min_{q} ||\dot{r} - J(q)\dot{q}||, \\
\end{align*}
\]

we obtain the general solution

\[
\dot{q} = J^+(q) \dot{r} + \{I_n - J^+(q) J(q)\} \phi,
\]

where \( J^+ = J^\top (J J^\top)^{-1} \in \mathbb{R}^{n \times m} \) denotes the pseudo-inverse of \( J \). \( (I_n - J^+ J) \in \mathbb{R}^{n \times n} \) is the orthogonal projection operator into the null-space of \( J \), and \( \phi \in \mathbb{R}^3 \) is an arbitrary vector. The solution \( \dot{q} \) which satisfies

\[
\begin{align*}
\min_{q} ||\dot{q}||
\end{align*}
\]

in addition to (3) is equal to the one that \( \phi = 0 \) in (4).

B. Gradient Projection Method

Liégeois [4] proposes a redundancy resolution scheme so as to maximize a given performance criterion function using the GPM [16]. This scheme gives the solution

\[
\dot{q} = J^+(q) \dot{r} + k_r \{I_n - J^+(q) J(q)\} \left( \frac{\partial V}{\partial q} \right) \top,
\]

where \( V \) is a performance criterion function and \( k_r (> 0) \) is a scalar parameter. This equation is equal to (4) with the \( \phi = k_r (\partial V/\partial q) \top \).

Now we suppose that the primary task is to track a desired trajectory \( r^d \) and the secondary one is to avoid joint limits. Then, based on (5), a desired joint velocity \( \dot{q}^d \) is described as

\[
\dot{q}^d = J^+(q) \dot{r}^d + \{I_n - J^+(q) J(q)\} \left( \frac{\partial V}{\partial q} \right) \top,
\]

which includes \( k_r \) in \( V \) for the sake of the following discussion. We need to select \( V \) appropriately so as to achieve the joint limits avoidance as the secondary task.

III. MOTION PLANNING METHOD TO ADMIT JOINT RANGE OF MOTION MAXIMALLY

In this section a new performance criterion function for the redundancy resolution is presented and is also discussed in comparison with the conventional functions.

A. Related Studies

We review related studies which treat avoidance of joint limits as the secondary task. For avoiding joint limits, performance criterion functions which have proposed so far [3]–[11] are classified roughly into four typical types.

To the authors’ knowledge, a performance criterion function for avoiding joint limits had firstly been proposed by Liégeois [4]. The Liégeois’s function is described as

\[
V(q) = -k_r \sum_{i=1}^{n} \left( \frac{q_i - q_{i \text{ mid}}}{\Delta q_i} \right)^2,
\]

where \( q_i \text{ max} \), \( q_i \text{ min} \) are the upper and lower limits of the \( i \)-th joint, \( q_{i \text{ mid}} := (q_i \text{ max} + q_i \text{ min})/2 \), and \( \Delta q_i := q_i \text{ max} - q_i \text{ min} \), respectively.

Tsai [6] proposes the following performance criterion function based on the exponential function.

\[
V(q) = 1 - \exp \left[ k_r \prod_{i=1}^{n} \frac{(q_i - q_{i \text{ max}})(q_i - q_{i \text{ min}})}{\Delta q_i^2} \right]
\]


The following performance criterion function is proposed by Zghal, et al. [7].

\[
V(q) = -\frac{1}{4} k_r \sum_{i=1}^{n} \frac{\Delta q_i^2}{(q_i - q_{i \text{ max}})(q_i - q_{i \text{ min}})}
\]

Chan et al. [8] introduce the Zghal, et al. ’s function not to the GPM but to the Weighted Least-Norm (WLN) solution which is proposed by Whitney [17]. And they propose an algorithm that the weighted matrix is not fixed but is switched according to the variation of the gradient of the performance criterion function.

Marchand, et al. [9] propose the following switched performance criterion function.

\[
V(q) = \begin{cases} 
- \frac{1}{2} k_r \sum_{i=1}^{n} \frac{(q_i - q_i^{\text{max}})^2}{\Delta q_i}, & \text{if } q_i > q_i^{\text{max}} \\
- \frac{1}{2} k_r \sum_{i=1}^{n} \frac{(q_i - q_i^{\text{min}})^2}{\Delta q_i}, & \text{if } q_i < q_i^{\text{min}}, \\
0, & \text{otherwise} 
\end{cases}
\]

where \( q_i^{\text{max}} := q_i \text{ max} - \rho \Delta q_i, \ q_i^{\text{min}} := q_i \text{ min} + \rho \Delta q_i, \ \rho \in (0, 1/2) \).

Chaumette et al. [10] point out some problems on \( k_r \) which is usually determined by trial and error, and propose an iterative solution instead of the GPM. Mansard et al. [11] also propose a solution that increases DoF of the secondary task to improve control performance for achieving the task. Both solutions exploit the performance criterion function (10) to implement avoidance of the joint limits in an application to visual servoing.

The profiles of the four performance criterion functions on \( q_i \) are shown in Fig. 1. The value of each \( k_r \) is appropriately

\footnote{When \( k_r \) is defined to be a negative number, Liégeois’s scheme is also regarded as the redundancy resolution to minimize a given performance criterion function.}
Redundancy Resolution

B. Maximal Admission of Joint Range of Motion Based on

Every performance criterion function except the Marc-

Every performance criterion function except the Zghal, et al.‘s function is convex upward, which has its

The former feature means that a joint variable converges to the midpoint in joint range of motion. A performance criterion function which has such feature is expected to avoid joint limits, but it is negative concept because we cannot exploit the joint range of motion effectively. The latter feature indicates that a joint variable potentially exceeds its limit depending on the selection of \( k_r \), the desired joint trajectories which achieve the primary task and so on.

B. Maximal Admission of Joint Range of Motion Based on Redundancy Resolution

As stated the previous subsection, using the conventional performance criterion function, we can negatively achieve the secondary task, the joint limits avoidance, or not. We therefore consider that a desired concept for avoiding joint limits is to avoid joint limits strictly and to exploit the joint range of motion maximally. Let us call this concept maximal admission of joint range of motion. And we propose the following conditions which should be satisfied as a performance criterion function to perform maximal admission of joint range of motion:

1) Let \( N_i^{\text{max}} \) := \{ \( q_i \mid q_i^{\text{max}} < q_i < q_i^{\text{max}} \) \} and \( N_i^{\text{min}} \) := \{ \( q_i \mid q_i^{\text{min}} < q_i < q_i^{\text{min}} \) \}, where \( q_i^{\text{max}} \) and \( q_i^{\text{min}} \) are the upper and lower limits on the \( i \)-th joint, \( q_i^{\text{max}} := q_i^{\text{max}} - \rho \Delta q_i \), \( q_i^{\text{min}} := q_i^{\text{min}} + \rho \Delta q_i \), \( \rho \in (0, 1/2) \), and \( \Delta q_i := q_i^{\text{max}} - q_i^{\text{min}} \), respectively. Function \( V_i(q_i) \) satisfies

\[
\frac{\partial V_i}{\partial q_i} \rightarrow \begin{cases} 
0 & \text{as } q_i \rightarrow q_i^{\text{max}}, \text{ for } q_i \in N_i^{\text{max}} \\
-\infty & \text{as } q_i \rightarrow q_i^{\text{min}}, \text{ for } q_i \in N_i^{\text{min}} 
\end{cases}
\]

and

\[
\frac{\partial V_i}{\partial q_i} \rightarrow \begin{cases} 
\infty & \text{as } q_i \rightarrow q_i^{\text{min}}, \text{ for } q_i \in N_i^{\text{min}} \\
0 & \text{as } q_i \rightarrow q_i^{\text{max}}, \text{ for } q_i \in N_i^{\text{max}} 
\end{cases}
\]

2) Let \( \mathcal{M}_i := \{ q_i \mid q_i^{\text{min}} < q_i < q_i^{\text{max}} \} \). For \( q_i \in \mathcal{M}_i \), \( \mathcal{N}_i^{\text{max}} + \mathcal{N}_i^{\text{min}} \), \( \partial V_i/\partial q_i = 0 \).

Condition 1) means to avoid each joint limit only in its assigned neighborhood. Condition 2) also means to perform only the primary task everywhere except in the assigned neighborhoods. All the conventional performance criterion functions (7)–(10) don’t satisfy the both conditions.

We proposed the following function as one candidate of performance criteria which satisfies the above conditions.

\[
V(q) = k_r \sum_{i=1}^{n} V_i(q_i),
\]

\[
V_i(q_i) = \begin{cases} 
-\tan^4(\alpha_i(q_i - q_i^{\text{max}})), & \text{if } q_i \in \mathcal{N}_i^{\text{max}} \\
-\tan^4(\alpha_i(q_i - q_i^{\text{min}})), & \text{if } q_i \in \mathcal{N}_i^{\text{min}} \\
0, & \text{otherwise}
\end{cases}
\]

where \( \alpha_i := \pi/(2\rho\Delta q_i) \), \( k_r \) and \( \rho \) are two design parameters. It is trivial that \( V_i \) is smooth on \( \mathcal{M}_i \), from the definition. And the gradient vector of this function is derived as follows:

\[
\frac{\partial V}{\partial q} = \begin{bmatrix} \frac{\partial V}{\partial q_1} & \frac{\partial V}{\partial q_2} & \cdots & \frac{\partial V}{\partial q_n} \end{bmatrix},
\]

\[
\frac{\partial V}{\partial q_i} = \begin{cases} 
-4k_r\alpha_i \tan^3(\alpha_i(q_i - q_i^{\text{max}})) \cos^2(\alpha_i(q_i - q_i^{\text{max}})), & \text{if } q_i \in \mathcal{N}_i^{\text{max}} \\
-4k_r\alpha_i \tan^3(\alpha_i(q_i - q_i^{\text{min}})) \cos^2(\alpha_i(q_i - q_i^{\text{min}})), & \text{if } q_i \in \mathcal{N}_i^{\text{min}} \\
0, & \text{otherwise}
\end{cases}
\]

The profile of the proposed function is depicted in Fig. 2.

In this paper the proposed function (11) adopts the GPM [4]. This function can also adopt the other solution, e.g., the WLN solution [17], the Chaumette et al.’s iterative solution [10], the Mansard et al.’s solution [11], etc.

IV. A NUMERICAL EXAMPLE

In this section we show some simulation results to validate the proposed performance criterion function.

A three DoF planar manipulator (\( n = 3 \)) depicted in Fig. 3 is considered. The primary task is to control position of the end-tip (\( m = 2 \)). Let \( r = [r_1, r_2]^T := [x, y]^T \) and \( q := \)
\[
[q_1, q_2, q_3]^T \text{ be the position vector of the end-tip and the joint angle vector. Then } f \text{ in (1) and } J \text{ in (2) are}
\]
\[
f(q) = \begin{bmatrix}
\ell_1 C_1 + \ell_2 C_{12} + \ell_3 C_{123} \\
\ell_1 S_1 + \ell_2 S_{12} + \ell_3 S_{123}
\end{bmatrix},
\]
\[
J(q) = \begin{bmatrix}
-\ell_1 S_1 - \ell_2 S_{12} - \ell_3 S_{123} & -\ell_2 S_{12} - \ell_3 S_{123} & -\ell_3 S_{123} \\
\ell_1 C_1 + \ell_2 C_{12} + \ell_3 C_{123} & \ell_2 C_{12} + \ell_3 C_{123} & \ell_3 C_{123}
\end{bmatrix},
\]
where \( \ell_i, \ i = 1, 2, 3 \) denotes the length of each link, \( S_{ijk} = \sin(\theta_i + \theta_j + \theta_k) \) and \( C_{ijk} = \cos(\theta_i + \theta_j + \theta_k) \).

The primary task is to track the following three-order time polynomial trajectory at the end-effector.
\[
r^{d}(t) = \begin{bmatrix}
a_{10} + a_{11}t + a_{12}t^2 + a_{13}t^3 \\
a_{20} + a_{21}t + a_{22}t^2 + a_{23}t^3
\end{bmatrix},
\]
where,
\[
a_{10} = r_{i}(0), \quad a_{11} = \dot{r}_{i}(0),
\]
\[
a_{12} = \frac{1}{t_f} \left[ 3(r_{i}(t_f) - r_{i}(0)) - 2\dot{r}_{i}(0) + \ddot{r}_{i}(t_f) \right] t_f,
\]
\[
a_{13} = \frac{1}{t_f^2} \left[ -2(r_{i}(t_f) - r_{i}(0)) + \ddot{r}_{i}(t_f) + \dddot{r}_{i}(0) \right] t_f
\]
and \( t_f \) is the terminal time.

The primary task is to move from \( r(0) \approx (1.13, 6.50) \) (m) \( \dot{r}(0) = 0 \) to \( r(t_f) = (-r_{i}(0), r_2(0), \dot{r}(0), \ddot{r}(0), \dddot{r}(0)) \) using a third-order time polynomial trajectory. The physical and design parameters are as follows: \( \ell_1 = \ell_2 = 1 \) (m), \( \ell_3 = 0.3 \) (m), \( \rho = 0.1 \). The joint limits are also shown in TABLE I.

The simulation results are shown in Fig. 4. For comparison, the results in the case of the Zghal, et al.’s and Marchand, et al.’s functions are included. Only in the case of the proposed function, each joint angle can avoid its limits not depending on \( k_r \). These results exhibit the effectiveness of the proposed function.

![Three DoF manipulator](image)

**Fig. 3.** Three DoF manipulator.

**TABLE I**

<table>
<thead>
<tr>
<th>( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q_{1max} ) (deg)</td>
<td>180</td>
<td>150</td>
<td>180</td>
</tr>
<tr>
<td>( q_{2max} ) (deg)</td>
<td>-180</td>
<td>-150</td>
<td>-180</td>
</tr>
</tbody>
</table>

**V. CONCLUSION**

We have presented a motion planning method to avoid joint limits for kinematically redundant manipulators. This method can admit joint range of motion maximally as a secondary task. In other words, we can exploit joint range of motion effectively. Some simulation results showed the effectiveness of our proposed method. We can apply the proposed method, for instance, in a visual servoing application.

**REFERENCES**

Fig. 4. Simulation results.