Frequency Weighted $H_{\infty}$ Model Reduction Using LMIs

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Abstract: This paper treats the problem of approximating an $n$th order system by a system of order $r(r < n)$. An algorithm based on the matrix inequality framework is derived to solve the frequency weighted $H_{\infty}$-norm model reduction problem. The aim of the optimization algorithm is to minimize the weighted $H_{\infty}$-norm of the difference between the main system and the reduced one. The feature of the method is that it can be applied to all systems and there is no restriction that the system should be strictly proper or stable. One example is used to compare the proposed method with other existing methods.

Keywords: Model reduction, $H_{\infty}$-norm, Linear matrix inequality, Frequency weighted problem.

1. Introduction

Modern controller design methods such as $H_{\infty}$, produce controllers that have the order equal to the plants order. These control laws are too complex to be applied for practical purposes. In order to achieve simpler controllers, one can reduce the order of the controller model.

Model reduction is a very important topic in control community. It is important when obtaining a low order model, not to sacrifice vital characteristics of the physical system, such as stability, transient response, steady state error and so on. Different methods of obtaining reduced order models with these characteristics have been presented over the last thirty years, each focusing primarily on some properties of the system which has been deemed important, such as balanced realization, balanced truncation and optimal Hankel norm approximation [1].

There were a lot of researchers working on this topic. For instance, in [2] Helmersson used an approach using linear matrix inequality to reduce the order of systems. He reached a very interesting conclusion that, when reducing a model of order $n$ to $k$, the Hankel norm approximation method gives the optimum solution if $k = n - 1$. In other cases, the $H_{\infty}$ model error is bounded by the sum of the $n - 1$ smallest Hankel singular values. If, these values are close to each other, one can expect that the Hankel norm reduction method can be improved in some cases. The method was developed for $H_{\infty}$ model reduction and no weight is applied on the frequency ranges. In [3], it is shown that the lower bounds of the $H_{\infty}$ norm of the associated error system can be analyzed by using LMI techniques. These lower bounds are given in terms of the Hankel singular values of the main system. In [4], the authors improved the method of Helmersson in order not to stick in local minimum. The main disadvantage of their approach was dealing with complicated LMIs and the order of the inequality which should be solved by the method. The other disadvantage of the method was dealing just with strictly proper and stable systems, because they used the $H_2$-norm.

In [5] and [6], model reduction techniques are investigated for state-space symmetric systems and in [7] a randomized algorithm for reduced order $H_{\infty}$ controller design is studied. On that paper, full-order optimal $H_{\infty}$ controllers were randomly generated and by standard methods they were reduced to a lower order.

In [8], the objective is to minimize the Hankel norm of the main system and the reduced order one. In this method, genetic algorithm is used to minimize the Hankel norm difference. In [9], genetic algorithm is used and the objective is to minimize the $H_2$ norm of the main system and the reduced order one. The method combines the least square method and the genetic algorithm to solve the $H_2$ norm problem. One of the disadvantages of this method is that it can just be applied to strictly proper and stable systems. In [10], genetic algorithm was used to minimize the root mean squared error of time responses between the exact discrete-time model and the reduced order model. It can just be applied to discrete-time systems and its purpose is to reduce model in order to achieve good time response.

In [11], authors studied the model reduction of linear continuous-time systems over finite frequency interval using Generalized Kalman-Yakubovich-Popov (GKYP) lemma, via LMIs. It’s clear that their theory is based on GKYP lemma which is based on some complicated LMIs.

In this paper, linear matrix inequalities are used for model order reduction and an algorithmic method based on weighted $H_{\infty}$-norm minimization is described. Briefly, the method of [2] and [3] are extended to the frequency weighted problems. The feature of the method is that it
can be applied to all systems and there is no restriction that the system should be strictly proper or stable.

The paper is outlined as follows. In section 2, some results of model order reduction are summarized. In section 3, linear matrix inequalities are reviewed very briefly. The algorithm is defined step by step in section 4 and illustrated on an example in section 5. Finally, conclusions are given in section 6.

2. Model Order Reduction

The central problem is: given a high-order linear time-invariant model $G(s)$,

$$G(s) = \frac{G_{num}}{G_{den}} = \frac{a_k s^n + a_{k-1} s^{n-1} + \cdots + a_1 s + a_0}{b_k s^n + b_{k-1} s^{n-1} + \cdots + b_1 s + b_0}$$

find a lower order approximation $\hat{G}(s)$ of degree $r (r < n)$ such that the frequency response of these two transfer functions are closed to each other as possible and the input-output behavior is changed as little as possible. Try to match frequency responses by minimizing their difference using:

$$\|G(.) - \hat{G}(.)\|_\infty = \sup_{\omega} |G(j\omega) - \hat{G}(j\omega)|$$

Let the high order system G(s) have an input $u(t)$ and an output $y(t)$. The low order system $\hat{G}(s)$ has the same input, but the output is $\hat{y}(t)$. Calculate the Fourier transforms of these signals as:

$$u_f(\omega) = F(u(t))$$
$$y_f(\omega) = F(y(t))$$
$$\hat{y}_f(\omega) = F(\hat{y}(t))$$

So:

$$y_f(\omega) = G(j\omega)u_f(\omega)$$
$$\hat{y}_f(\omega) = \hat{G}(j\omega)u_f(\omega)$$

If we assume the error as $e(t) = y(t) - \hat{y}(t)$, its fourier transform will be:

$$F(e(t)) = e_f(\omega) = [G(j\omega) - \hat{G}(j\omega)]u_f(\omega)$$

Assume that $\|u_f(\omega)\|^2 = 1$, we can minimize worst case error $\|e_f(\omega)\|^2$ by minimizing [12]:

$$\|G(j\omega) - \hat{G}(j\omega)\|_\infty = \sup_{\omega} |G(j\omega) - \hat{G}(j\omega)|$$

Therefore, it is logical to minimize the $H_\infty$-norm of the difference between the main model and the reduced order one.

3. Linear Matrix Inequalities (LMIs)

A linear matrix inequality (LMI) is any constraint of the form [4]:

$$F(x) = F_0 + x_1 F_1 + \cdots + x_m F_m > 0$$

Where $x = (x_1, x_2, \ldots, x_m)^T \in R^m$ is a vector of $m$ unknown scalars which are optimization variables. $F_1, F_2, \ldots, F_m$ are symmetric matrices satisfying:

$$F_i \in R^{n \times n}, i = 1, 2, \ldots, m$$

Consider the problem of finding the $H_\infty$ norm of a stable system $G$ defined by $G(s) = C(sI - A)^{-1}B + D$. Then $\|G\|_\infty < \gamma$ is equivalent to the existence of a symmetric, positive definite matrix $P > 0$ satisfying:

$$PA + A^TP + PB\begin{bmatrix} C & D \end{bmatrix} < 0$$

This inequality is called the bounded real lemma [2].

4. Model Reduction Algorithm

4.1 The unweighted problem

Assume that the model which should be reduced and the reduced order model are defined in the following state space representations:

$$G(s) = C(sI - A)^{-1}B + D$$

Thus the model reduction problem can be considered as an optimization problem subject to a matrix inequality constraint. Find the smallest possible $\gamma$ with respect to $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ and $P > 0$ such that:

$$\begin{bmatrix} P \hat{A} + A^TP + PB\hat{C} & \hat{D}^T \hat{B}^T \hat{C}^T \\ \hat{C}^T \hat{B}^T & -\gamma I \end{bmatrix} < 0$$

$$\begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}$$

$$\begin{bmatrix} \hat{B}^- & \hat{C}^- \\ \hat{D}^- & \hat{D} \end{bmatrix} \begin{bmatrix} \hat{A}^- & \hat{B}^- \\ \hat{C}^- & \hat{D}^- \end{bmatrix}$$

By partitioning the matrix P into:

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$$

we can rewrite the inequality (14) as [2]:

$$\begin{bmatrix} P_{11}A + A^TP_{11} + P_{12}\hat{A} + \hat{A}^TP_{21} + P_{21}\hat{B} - \hat{B}^TP_{22} & \hat{C}^- \\ \hat{B}^-P_{11} + \hat{B}^-P_{21} + \hat{B}^-P_{12} - \hat{B}^-P_{22} & \hat{C}^- \end{bmatrix} < 0$$

4.2 The Frequency Weighted Case

The ordinary problem is discussed in the previous section, here the frequency weighted $H_\infty$-norm model reduction problem is considered. The same basic ideas and techniques as those described for the unweighted case can be applied to solve the problem. Taking weights into account increases the size of the LMIs involved in the algorithms.

The objective is to find the global solution to the following optimization program:

$$\min_{\hat{G}(s)} \|W(s)(G(s) - \hat{G}(s))\|_\infty$$

Where $W(s)$ is the frequency weight. Suppose that $A_w, B_w, C_w, D_w$ is the minimal realization of $W(s)$. Then, the state space realization of $\hat{G}(s) = W(s)(G(s) - \hat{G}(s))$ is:
\[
\begin{bmatrix}
\hat{A}_w & \hat{B}_w \\
\hat{C}_w & \hat{D}_w
\end{bmatrix} = 
\begin{bmatrix}
A & 0 & 0 & B \\
0 & \bar{A} & 0 & \bar{B} \\
B_wC & \bar{B}_wC & A_w & 0 \\
D_wC & \bar{D}_wC & C_w & D - \bar{B}
\end{bmatrix}
\] (19)

By applying (20) and (21) constraints to the whole algorithm, which are described as below, we can get a reduced order system which is strictly proper and its stability has been guaranteed.
\[
\bar{B} = 0
\] (20)
\[
\bar{A} < 0
\] (21)

Partitioning \( P \) in accordance with \( \hat{A}_w \):
\[
P = \begin{bmatrix}
P_{11} & P_{12} & P_{13} \\
P_{21} & P_{22} & P_{23} \\
P_{31} & P_{32} & P_{33}
\end{bmatrix} > 0
\] (22)

Where:
\[
\begin{align*}
P_{11} & \in R^{n \times n} \\
P_{12} & \in R^{n \times r} \\
P_{13} & \in R^{n \times w}
\end{align*}
\]

Inequality (17) changes to inequality (23) where the sign * placed instead of element (i,j) denotes the transpose of the term (j,i).

This problem is a peculiar nonlinear problem and the algorithms to solve LMIs cannot be applied. As it is nonlinear, for dealing with it, we should change it to linear problems. The first strategy we used, was to keep two blocks \( P_{13} \) and \( P_{23} \) zero in the whole algorithm.

In order to deal with other nonlinearities we suggest solving the LMI in an iterative algorithm and divide it into two simpler optimization procedures, each of them being linear in decision variables.

The proposed algorithm is as follows:
1) Find a \( r \) order approximation of \( G(s) \) using one of the classical techniques.
2) Choose an initial, arbitrary and proper upper bound \( \gamma(\gamma) \) for \( \gamma \).
3) Keep \( \hat{A}, \hat{B}, \hat{C}, \hat{D} \) constant, minimize \( \gamma \) with respect to \( P_{11}, P_{12}, P_{22}, P_{33} \) and subject to inequalities (22), (23) and (24) which is defined as below:
\[
0 < \gamma < \gamma_{ini}
\] (24)
(Define \( \gamma_{opt} \) as the minimum value of \( \gamma \) which is gained in this step).
4) Keeping \( P_{12}, P_{22}, P_{33} \) constant, minimize \( \gamma \) with respect to \( P_{11}, \hat{A}, \hat{B}, \hat{C}, \hat{D} \) subject to inequalities (22), (23) and (24). (Define \( \gamma_{2opt} \) as the minimum value of \( \gamma \) which is gained in this step).
5) Repeat 3 and 4 until the solution converges. In fact, when \( \gamma_{opt} \) and \( \gamma_{2opt} \) have the same value, the solution converges.

In repeating step 3, it’s important to note that in (24), \( \gamma_{ini} \) is replaced by \( \gamma_{opt} \) which is gained from the previous iteration and also in repeating step 4, it’s important to note that in (24), \( \gamma_{ini} \) is replaced by \( \gamma_{2opt} \) which is gained from the previous iteration. If the main system is strictly proper and stable, we must consider (20) and (21) in steps 3 and 4, too.

5. Simulation Results

In this section, we will study an example and compare the obtained results using the proposed method and the other existing methods.

Example: To test the method, consider the fourth order system given below. This is the same example that is chosen by Song, Lee and their coauthors to compare their algorithm with the other methods [13].

\[
A = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 \\
0 & 0 & -3 & 0 \\
0 & 0 & 0 & -4
\end{bmatrix}
\]
\[
B = \begin{bmatrix}
0 \\
1/2 \\
-1/2 \\
-1/2
\end{bmatrix}
\]
\[
C = \begin{bmatrix}
1 & 0 & 1 & 0 \\
4/15 & 1 & 0 & 1
\end{bmatrix}
\]

The input weight is second order and is given by:
\[
A_w = \begin{bmatrix}
-4.5 & 0 \\
0 & -4.5
\end{bmatrix}
\]
\[
B_w = \begin{bmatrix}
3 & 0 \\
0 & 3
\end{bmatrix}
\]
\[
C_w = \begin{bmatrix}
1.5 & 0 \\
0 & 1.5
\end{bmatrix}
\]
\[
D_w = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

If we plot Bode diagrams of the above weighting transfer functions between the inputs and the outputs, the results will show that \( G_w(12) \) and \( G_w(21) \) are zero and \( G_w(11) \) and \( G_w(22) \) are low pass filters (Fig. 1).

![Bode diagrams for the transfer functions of the weight system (W(s)).](image)

(Note that \( G_w(ij) \) indicates the transfer function between the \( i^{th} \) input and the \( j^{th} \) output).
Therefore, the target is that the frequency responses for the 1st input-1st output and the 2nd input-2nd output transfer functions of the reduced system should be similar to the unreduced one from 0 rad/sec to almost 10 rad/sec. The results which are achieved for the first, the second and the third order reduced models are given in the following. The Bode diagrams of the reduced order models are compared with the Bode diagram of the main system in Fig 2-4.

First order model:
\[
\begin{bmatrix}
\hat{A} = [-0.5643] \\
\hat{B} = [0.2571 \quad 1.7063] \\
\hat{C} = [1.2624] \\
\hat{D} = [0 \quad 0]
\end{bmatrix}
\]

Second order model:
\[
\begin{bmatrix}
\hat{A} = [-1.0357 \quad 0.0630] \\
\hat{B} = [0.2268 \quad -2.6720] \\
\hat{C} = [0.0296 \quad -2.6230] \\
\hat{D} = [-0.2751 \quad 1.1721] \\
\hat{C} = [-1.6522 \quad -3.5566] \\
\hat{D} = [0.0310 \quad -0.8372] \\
\end{bmatrix}
\]

Third order model:
\[
\begin{bmatrix}
\hat{A} = [-1.1511 \quad 0.7453 \quad 0.0064] \\
\hat{B} = [0.4057 \quad -2.6123 \quad 0.7132] \\
\hat{C} = [-0.1195 \quad -0.0655 \quad -2.2075] \\
\hat{C} = [0.0741 \quad -2.8194] \\
\hat{C} = [-0.2607 \quad 0.7371] \\
\hat{C} = [0.1215 \quad -0.5330] \\
\hat{C} = [-1.1544 \quad -3.6366 \quad 1.0213] \\
\hat{C} = [-0.3957 \quad 0.5785 \quad 2.7836] \\
\end{bmatrix}
\]

Note that \(G(\text{ij})\) indicates the transfer function between the \(i^{th}\) input and the \(j^{th}\) output.
Due to the results, the Bode diagrams for the 1st input-1st output and the 2nd input-2nd output transfer functions of the reduced systems are similar to the main system from 0 rad/sec to almost 10 rad/sec. Note that the 2nd input-2nd output transfer function, when the reduced order is two, is nonminimum phase. So the proposed method in this paper doesn’t guarantee that the reduced system will be definitely minimum phase. The reduced models of order 1, 2 and 3 are computed by this method of the paper and they are compared with the other methods. The method of this paper was initiated by optimal Hankel norm approximation method. The results are shown in TABLE I. Note that the weighted errors in TABLE I refer to $\|W(s)(\hat{G}(s) - G(s))\|_{\infty}$. From TABLE I it is seen that the proposed method can find better solutions with little errors.

<table>
<thead>
<tr>
<th>Order of the model</th>
<th>Balanced Model Reduction</th>
<th>Frequency Weighted Model Reduction</th>
<th>State Weighted Model Reduction</th>
<th>Proposed method</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.4114</td>
<td>2.1267</td>
<td>2.1147</td>
<td>1.0870</td>
</tr>
<tr>
<td>2</td>
<td>0.3123</td>
<td>0.2660</td>
<td>0.2598</td>
<td>0.1158</td>
</tr>
<tr>
<td>3</td>
<td>0.1363</td>
<td>0.1131</td>
<td>0.1142</td>
<td>0.0584</td>
</tr>
</tbody>
</table>

6. Conclusion

A model reduction algorithm using linear matrix inequalities has been studied and implemented on an example. The algorithm is based on the matrix inequality framework and solves the frequency $H_\infty$-norm model reduction problem. The aim of the optimization algorithm is to minimize the weighted $H_\infty$-norm between the main system and the reduced one. The problem corresponds to an optimization problem which is solved using a two steps iterative scheme in order to change the nonlinear matrix inequality into a linear matrix inequality. The feature of the method is that it can be applied to all systems and there is no restriction that the system should be strictly proper or stable. The numerical example shows the efficiency of the method.

References