Vertex-magic Total Labeling of Generalized Petersen Graphs and Convex Polytopes

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Abstract

A vertex-magic total labeling on a graph with \( v \) vertices and \( e \) edges is a one-to-one map taking the vertices and edges onto the integers \( 1, 2, \ldots, v + e \) with the property that the sum of the label on a vertex and the labels of its incident edges is constant, independent of the choice of vertex. We give vertex-magic total labelings for several classes of regular graphs. The paper concludes with several conjectures and open problems in the area.

1 Introduction

All graphs considered in this paper are finite, simple and undirected. The graph \( G \) has vertex set \( V = V(G) \) and edge set \( E = E(G) \) and we let
\( e = |E| \) and \( v = |V| \). A general reference for graph theoretic notions is [8]. In this paper we will deal only with connected graphs, although the concepts apply equally to graphs with more than one connected component.

A **labeling** for a graph is a map that takes graph elements to numbers (usually positive or non-negative integers). In this paper the domain is the set of all vertices and edges, giving rise to **total labelings**. Other labelings use the vertex set alone ('vertex labelings') or the edge set alone ('edge labelings'). The most complete recent survey of graph labelings is [3].

Various authors have introduced labelings that generalize the idea of a magic square. These are called **magic labelings** and readers are referred to [7] for a discussion of magic labelings and a standardisation of the terminology.

In [6] we introduced the notion of a **vertex-magic total labeling**. This is an assignment of the integers from 1 to \( e + v \) to the vertices and edges of \( G \) so that at each vertex the vertex label and the labels on the edges incident at that vertex add to a fixed constant. More formally, a one-to-one map \( \lambda \) from \( E \cup V \) onto the integers \( \{1, 2, \ldots, e+v\} \) is a **vertex-magic total labeling** if there is a constant \( k \) so that for every vertex \( x \),

\[
\lambda(x) + \sum \lambda(xy) = k \tag{1}
\]

where the sum is over all vertices \( y \) adjacent to \( x \).

Set \( M = e + v \) and let \( S_v \) be the sum of the vertex labels and \( S_e \) the sum of the edge labels. Clearly, since the labels are the numbers \( 1, 2, \ldots, M \), we have as the sum of all labels

\[
S_v + S_e = \sum_{i=1}^{M} i = \binom{M+1}{2}.
\]

At each vertex \( x_i \) we have \( \lambda(x_i) + \sum \lambda(x_iy) = k \). We sum this over all \( v \) vertices \( x_i \). This adds each vertex label once and each edge label twice, so that

\[
S_v + 2S_e = vk. \tag{2}
\]

Combining these two equations gives us

\[
S_e + \binom{M+1}{2} = vk. \tag{3}
\]

The edge labels are all distinct (as are all the vertex labels). The edges
could conceivably receive the $e$ smallest labels or, at the other extreme, the $e$ largest labels, or anything between. Consequently we have

\[
\sum_{1}^{e} i \leq S_e \leq \sum_{v+1}^{M} i.
\]  

(4)

A similar result holds for $S_v$. Combining (3) and (4), we get

\[
\binom{M+1}{2} + \binom{e+1}{2} \leq vk \leq 2\binom{M+1}{2} - \binom{v+1}{2}
\]

which will give the range of feasible values for $k$.

It is clear from the definition of vertex-magic total labeling that when $k$ is given and the edge labels are known, then the vertex labels are determined. So the labeling is completely described by the edge labels. In this paper we give new vertex-magic total labelings of several classes of graphs. The paper concludes with a list of conjectures and open problems.

2 Labelings of generalized Petersen graphs

Suppose $g$ is an edge-labeling of a graph, that is, a one-to-one map from $E$ onto the integers $\{1, 2, \ldots, e\}$. Then the weight $w(x)$ of a vertex $x \in V$ is defined as the sum of labels assigned to all edges incident to $x$.

An edge-labeling $g$ is said to be consecutive if the weights of all vertices constitute a set of consecutive integers. Two edge-labelings $g$ and $g'$ are said to be complementary if there is a constant $c$ such that $w_g(x) + w_{g'}(x) = c$ for all $x \in V$.

Let $I = \{1, 2, \ldots, n\}$ be an index set. For simplicity, we use the convention that $x_{j,n+1} = x_{j,1}$ for $j = 1, 2, \ldots, 6$.

A generalized Petersen graph $P(n,m)$, $1 \leq m < \frac{n}{2}$, consists of an outer $n$-cycle $y_1, y_2, \ldots, y_n$, a set of $n$ spokes $y_i x_i$, $1 \leq i \leq n$, and $n$ inner edges $x_i x_{i+m}$, $1 \leq i \leq n$, with indices taken modulo $n$. The standard Petersen graph is the instance $P(5,2)$. 
$P(n, m)$ is regular of degree 3 and has $v = 2n$ vertices and $e = 3n$ edges; thus $M = 5n$. Using (5) we can readily determine the feasible values of $k$ for the generalized Petersen graphs $P(n, m)$:

$$\frac{17n}{2} + 2 \leq k \leq \frac{23n}{2} + 2.$$ 

It was shown in [1] that for $n \geq 4$, $n$ even and $1 \leq m \leq \frac{n}{2} - 1$, the generalized Petersen graph $P(n, m)$ has a consecutive edge-magic labeling defined by the bijective mapping $g_1 : E(P(n, m)) \rightarrow \{1, 2, \ldots, 3n\}$:

- $g_1(y_iy_{i+1}) = \frac{5n+i+1}{2} \delta(i) + \frac{n-i+2}{2} \delta(i + 1)$
- $g_1(y_ix_i) = \frac{3n-i+1}{2} \delta(i) + [(n + \frac{i}{2} - 2) \rho(i, 4) + (\frac{n+i}{2} - 2) \rho(6, i)] \delta(i + 1)$
- $g_1(x_ix_{i+m}) = \frac{3n+i+1}{2} \delta(i) + [(2n-\frac{i}{2}+3) \rho(i, 4) + (\frac{5n-i}{2}+3) \rho(6, i)] \delta(i+1)$

- for $i \in I$, where

$$\delta(x) = \begin{cases} 
0 & \text{if } x \equiv 0 \bmod 2 \\
1 & \text{if } x \equiv 1 \bmod 2
\end{cases} \quad (6)$$

$$\rho(x, y) = \begin{cases} 
0 & \text{if } x > y \\
1 & \text{if } x \leq y
\end{cases} \quad (7)$$

The weights of vertices under the mapping $g_1$ constitute the sets

$W_1 = \{w_{g_1}(y_i) : i \in I\} = \{\frac{7n}{2} + 2, \frac{9n}{2} + 3, \ldots, \frac{9n}{2} + 1\}$ and

$W_2 = \{w_{g_1}(x_i) : i \in I\} = \{\frac{9n}{2} + 2, \frac{9n}{2} + 3, \ldots, \frac{11n}{2} + 1\}$.

Label the edges of the generalized Petersen graph $P(n, m)$ by the consecutive labeling $g_1$. If $g_2$ is the complementary vertex labeling with values in the set

$$\{|E| + 1, |E| + 2, \ldots, |E| + |V|\} = \{3n + 1, 3n + 2, \ldots, 5n\}$$

then the labelings $g_1$ and $g_2$ combine to give a vertex-magic total labeling of $P(n, m)$. Since the largest labels are assigned to the vertices, it is easily seen that the resulting magic constant $k = \frac{17n}{2} + 2$ is the largest possible.

Thus we have the following
Theorem 1 For \( n \geq 4 \), \( n \) even and \( 1 \leq m \leq \frac{n}{2} - 1 \), the generalized Petersen graph \( P(n, m) \) has a vertex-magic total labeling with \( k = \frac{17n}{2} + 2 \).

In [6] it is proved that if a regular graph \( G \) possesses a vertex-magic total labeling \( \lambda \) with magic constant \( k \), then \( G \) also has a dual labeling \( \lambda' \) having magic constant \( k' = (r+1)(M+1) - k \). Since \( P(n, m) \) is regular, it has a dual labeling. Hence

Corollary 1 For \( n \geq 4 \), \( n \) even and \( 1 \leq m \leq \frac{n}{2} - 1 \), the generalized Petersen graph \( P(n, m) \) has a vertex-magic total labeling with \( k = \frac{23n}{2} + 2 \).

Proof. In this case the dual labeling \( g' \) is defined by

- \( g'_1(u) = |E| + |V| + 1 - g_1(u) \) for any edge \( u \in E(P(n, m)) \),
- \( g'_2(x) = |E| + |V| + 1 - g_2(x) \) for any vertex \( x \in V(P(n, m)) \).

Since the magic sum for \( g \) is \( k = \frac{17n}{2} + 2 \), then \( g' \) is a vertex-magic total labeling with magic sum \( k' = 4M + 4 - k = \frac{23n}{2} + 2 \). \( \square \)

The prism \( D_n \), \( n \geq 3 \), is a trivalent graph which can be defined as the cartesian product \( P_2 \times C_n \) of a path on two vertices with a cycle on \( n \) vertices. We note that the prism \( D_n \) is the generalized Petersen graph \( P(n, 1) \).

Corollary 2 If \( n \) is even, \( n \geq 4 \), then the prism \( D_n \) has a vertex-magic total labeling with \( k = \frac{17n}{2} + 2 \) and another one with \( k = \frac{23n}{2} + 2 \).

3 Labelings of some families of convex polytopes

In this section we shall investigate the graphs of two families of convex polytopes. First we consider the graphs \( R_n \) consisting of \( 2n \) 5-sided faces, \( n \) 6-sided faces and a pair of \( n \)-sided faces, embedded in the plane and labeled as in Figure 2. Using equation (5) where \( v = 6n \) and \( e = 9n \), we can determine the feasible values of the magic constant \( k \) for the graph \( R_n \):
\[ \frac{51}{2} n + 2 \leq k \leq \frac{69}{2} n + 2. \]

The dual graph of \( R_n \) is a planar graph \( B_n \) which has been investigated in [2]. There it was shown that \( B_n \) has a face anti-magic labeling \( g \), i.e., an edge labeling in which the sum around each face is a constant. To obtain a vertex-magic total labeling of \( R_n \) we make use of the labeling \( h_1 : E(R_n) \rightarrow \{1, 2, \ldots, 9n\} \) which is a modification of the labeling \( g \) from [2]. It is defined as follows (with \( \delta \) and \( \rho \) as defined in (6) and (7)):

- \( h_1(x_1, x_{1,i+1}) = [(8n + i)\delta(i) + (n - i)\delta(i + 1)]\rho(i, n - 1) + n\rho(n, i) \)
- \( h_1(x_1, x_{2,i}) = (\frac{5n}{2} + i)\delta(i) + (\frac{5n}{2} - i + 1)\delta(i + 1) \)
- \( h_1(x_{2,i}, x_{3,i}) = (\frac{15n}{2} - i + 1)\delta(i) + (\frac{9n}{2} - i + 2)\delta(i + 1) \)
- \( h_1(x_{3,i}, x_{2,i+1}) = (\frac{13n}{2} + i)\delta(i) + (\frac{9n}{2} - i + 1)\delta(i + 1) \)
- \( h_1(x_{3,i}, x_{4,i}) = (n + 1)\rho(i, 1) + [\frac{3n - i + 3}{2}\delta(i) + (\frac{9n}{2} + i)\delta(i + 1)]\rho(2, i) \)
- \( h_1(x_{4,i}, x_{5,i}) = \frac{11n + i + 1}{2}\delta(i) + (\frac{11n}{2} - i + 1)\delta(i + 1) \)
- \( h_1(x_{5,i}, x_{4,i+1}) = \frac{13n - i + 1}{2}\delta(i) + \frac{15n + i}{2}\delta(i + 1) \)
The edge labeling $h_1$ is consecutive: the weights of vertices in turn assume the values $\frac{21n}{2} + 2, \frac{21n}{2} + 3, \ldots, \frac{33n}{2} + 1$.

If $h_2$ is the complementary vertex labeling with values in the set $\{|E(R_n)| + 1, |E(R_n)| + 2, \ldots, |E(R_n)| + |V(R_n)|\} = \{9n + 1, 9n + 2, \ldots, 15n\}$ then the labelings $h_1$ and $h_2$ combine to give a vertex-magic total labeling of $R_n$ with the magic constant $k = \frac{51n}{2} + 2$ which is the minimum possible.

We have

**Theorem 2** For $n \geq 4$, $n$ even, the plane graph $R_n$ has a vertex-magic total labeling with $k = \frac{51n}{2} + 2$.

Since $R_n$ is regular, we have a dual labeling as before:
Corollary 3 For \( n \geq 4 \), \( n \) even, the plane graph \( R_n \) has a vertex-magic total labeling with \( k = \frac{39n}{2} + 2 \).

The final graphs we investigate are the antiprisms \( A_n \), \( n \geq 3 \), a family of planar graphs that are regular of degree 4. These are Archimedean convex polytopes and, in particular, \( A_3 \) is the octahedron.

We will denote the vertex set of \( A_n \) by \( V = \{x_i : i \in I\} \cup \{y_i : i \in I\} \) and the edge set by \( E = \{(x_i x_{i+1}) : i \in I\} \cup \{(y_i y_{i+1}) : i \in I\} \cup \{(x_i y_i) : i \in I\} \cup \{(y_i x_{i+1}) : i \in I\} \) as in Figure 3.

![Figure 3: The antiprism \( A_n \).](image)

From (5) we get the range of feasible values for \( k \):

\[
\frac{26n+5}{2} \leq k \leq \frac{34n+5}{2}.
\]

Theorem 3 For \( n \geq 4 \), \( n \) even, the antiprism \( A_n \) has a vertex-magic total labeling with \( k = 15n + 2 \).

Proof. We construct an edge labeling \( f_1 \) of \( A_n \), \( n = 2m \), \( m \geq 2 \), in the following way:

- \( f_1(x_i x_{i+1}) = 6n \rho(i, 1) + [(5n + i - 1) \delta(i) + i \delta(i + 1)] \rho(2, i) \)
- \( f_1(y_i y_{i+1}) = [(5n + i) \delta(i) + (2n + i + 1) \delta(i + 1)] \rho(i, n - 1) + (2n + 1) \rho(n, i) \)
- \( f_1(x_i y_i) = (3n + 1) \alpha(1, i, 1) + (5n - 2i + 3) \alpha(2, i, m + 1) + (3n + 3) \alpha(m + 2, i, m + 2) + (5n - 2i + 3) \alpha(m + 3, i, n - 1) + (4n - 1) \alpha(n, i, n) \)
- \( f_1(y_i x_{i+1}) = 2n - 2i + 1 \)
• for $i \in I$, where

$$\alpha(x, y, z) = \begin{cases} 
1 & \text{if } x \leq y \leq z \\
0 & \text{otherwise}
\end{cases}$$ \hspace{1cm} (8)$$

The edge labeling $f_1$ is a one-to-one map from $E(A_n)$ onto the set $\{i : i \in I\} \cup \{n + 2j - 1 : j = 1, 2, \ldots, 2n\} \cup \{5n + i : i \in I\}$. The weights of the vertices under the edge labeling $f_1$ constitute the set

$$W = \{w_{f_1}(x) : x \in V(A_n)\} = \{10n + 2j : j = 1, 2, \ldots, 2n\}.$$ 

If $f_2$ is the complementary vertex labeling with values in the set $\{n + 2j : j = 1, 2, \ldots, 2n\}$ then the labelings $f_1$ and $f_2$ combine to give a vertex-magic total labeling of $A_n$ with the magic constant $k = 15n + 2$. \hfill \Box

Again by duality we have

**Theorem 4** For $n \geq 4$, $n$ even, the antiprism $A_n$ has a vertex-magic total labeling with $k = 15n + 3$.

**Figure 4:** Labeling of the antiprism $A_4$. 

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4 Open problems

We have shown that there exist vertex-magic total labelings for the generalized Petersen graph $P(n, m)$ for $n \geq 4, n$ even. We conjecture that

**Conjecture 1** There is a vertex-magic total labeling for the prism $D_n = P(n, 1)$ for all $n \geq 3$.

More strongly,

**Conjecture 2** There is a vertex-magic total labeling for the generalized Petersen graph $P(n, m)$ for all $n \geq 3$ and $2 \leq m < \frac{n}{2}$.

We have not yet found a construction that will produce a vertex-magic total labeling for the plane graph $R_n$ for $n$ odd. However, we suggest the following

**Conjecture 3** There is a vertex-magic total labeling for the plane graph $R_n$ for all $n \geq 3$.

**Open Problem 1** Find a vertex-magic total labeling for the antiprism $A_n$ for all odd $n \geq 3$.

The ladder $L_n, n \geq 3$, can be viewed as the cartesian product $P_2 \times P_n$ of a path on two vertices and a path on $n$ vertices, or as a prism $D_n$ with two edges deleted.

**Open Problem 2** Find a vertex-magic total labeling for the ladder $L_n$.

*added in proof*: Conjectures 1 and 2 have been proved by Dan McQuillan.

**References**


