ON SUPER $d$-ANTIMAGIC LABELINGS OF DISJOINT UNION OF PRISMS

MARTIN BAČA*
Department of Appl. Mathematics, Technical University
Letná 9, 042 00 Košice, Slovak Republic
e-mail: martin.baca@tuke.sk
and
FOZIA BASHIR
*Abdus Salam School of Mathematical Sciences, GC University
68-B, New Muslim Town, Lahore, Pakistan
e-mail: fozia.gc@hotmail.com

Abstract

We label the vertices, edges and faces by the consecutive integers from 1 to $|V| + |E| + |F|$ in such a way that the label of a face and the labels of the vertices and edges surrounding that face all together add up to a weight of that face. These face weights then form an arithmetic progression with common difference $d$. In this paper we examine the existence of such labeling for a disjoint union of prisms.

Keywords: union of prisms, $d$-antimagic labeling, super $d$-antimagic labeling

2000 Mathematics Subject Classification: 05C78

1. Introduction

We consider finite undirected graphs without loops and multiple edges. Denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph $G$, respectively. Let $|V(G)| = v$ and $|E(G)| = e$.

If we consider a plane graph which is drawn on the Euclidean plane in such a way that edges do not cross each other except at the vertices of the graph then for a plane graph $G = (V, E, F)$, it makes sense to consider its faces, including the unique face of the infinite area. Let $F(G)$ be the face set and let $|F(G)| = f$.

A labeling of a graph is any mapping that sends some set of graph elements to a set of numbers (usually positive integers). Specially, if we have bijections $g : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, v + e\}$ or $h : V(G) \cup E(G) \cup F(G) \rightarrow \{1, 2, \ldots, v + e + f\}$ then the labelings are called respectively a labeling of type $(1, 1, 0)$ (total labeling) or a labeling of type $(1, 1, 1)$. 
The weight of a face under a labeling is the sum of labels (if present) carried by that face and the edges and vertices on its boundary.

A labeling of plane graph \( G \) is called \textit{\( d \)-antimagic} if for every number \( s \) the set of \( s \)-sided face weights is \( W_s = \{ a_s, a_s + d, a_s + 2d, \ldots, a_s + (f_s - 1)d \} \) for some integers \( a_s \) and \( d \geq 0 \), where \( f_s \) is the number of the \( s \)-sided faces. We allow different sets \( W_s \) for different \( s \).

The \( d \)-antimagic labeling of the plane graphs was defined in [2]. In particular for \( d = 0 \), Ko–Wei Lih in [12] called such labeling \textit{magic}. Ko–Wei Lih [12] described magic labelings of type \((1, 1, 0)\) for the wheels, the friendship graphs and the prisms.

The \( d \)-antimagic labelings of type \((1, 1, 1)\) for the generalized Petersen graphs \( P(n, 2) \), the hexagonal planar maps and the grids can be found in [4], [5] and [6].

A \( d \)-antimagic labeling is called \textit{super} if the smallest possible labels appear on the vertices. The super \( d \)-antimagic labelings of disjoint union of prisms for antiprisms are described in [8].

The prism \( D_n, n \geq 3 \), is a cubic graph which can be represented as a Cartesian product \( P_2 \times C_n \) of a path on two vertices with a cycle on \( n \) vertices. Prism \( D_n, n \geq 3 \), consists of an outer \( n \)-cycle \( y_1, y_2, \ldots, y_n \), an inner \( n \)-cycle \( x_1, x_2, \ldots, x_n \), and a set of \( n \) spokes \( x_iy_i, i = 1, 2, \ldots, n \). Note that \( x_{n+1} = x_1 \) and \( y_{n+1} = y_1 \). We denote by \( z_{4,i}, 1 \leq i \leq n, \) the \( 4 \)-sided face bounded by the edges \( x_iy_i, x_i+1y_{i+1}, y_iy_{i+1} \) and \( y_iy_{i+1} \), and we denote by \( z_n \) and \( z'_n \) the inner \( n \)-sided face and outer \( n \)-sided face, respectively.

In [2], it was proved that the prism \( D_n \) has the \( d \)-antimagic labelings of type \((1, 1, 1)\) for \( d \in \{2, 3, 4, 6\} \) and \( n \equiv 3 \mod 4 \). Lin et al. in [13] showed that \( D_n, n \geq 3 \), admits \( d \)-antimagic labelings of type \((1, 1, 1)\) for \( d \in \{2, 4, 5, 6\} \). The \( d \)-antimagic labelings of type \((1, 1, 1)\) for prism \( D_n \) and for several \( d \geq 7 \) are described in [16].

In this paper we deal with the super \( d \)-antimagic labeling of type \((1, 1, 1)\) for the disjoint union of \( m \) copies of prism \( D_n \), denoted by \( mD_n \).

The disconnected graph \( mD_n \), for \( n \geq 3 \), \( m \geq 2 \), consists of the vertex set \( V(mD_n) = \{ x_{i,j}^l, y_{i,j}^l : 1 \leq i \leq n, 1 \leq j \leq m \} \) and the edge set \( E(mD_n) = \{ x_{i,j}^l y_{i,j+1}^l, x_{i,j+1}^l y_{i,j+1}^l, x_{i,j}^l y_{i,j+1}^l : 1 \leq i \leq n, 1 \leq j \leq m \} \), with indices taken modulo \( n \). The face set \( F(mD_n) \) contains \( mn \) \( 4 \)-sided faces \( \{ x_{4,i,j}^l : 1 \leq i \leq n, 1 \leq j \leq m \} \), \( m \) inner \( n \)-sided faces \( \{ z_{4,i,j}^l : 1 \leq j \leq m \} \) and one external face. So, \( v = 2mn \), \( e = 3mn \) and \( f = m(n+1)+1 \).

Suppose that \( h : V(mD_n) \cup E(mD_n) \cup F(mD_n) \rightarrow \{1, 2, \ldots, 6mn + m + 1\} \) is a super \( d \)-antimagic labeling of type \((1, 1, 1)\) of the \( mD_n \). The minimum weight of \( n \)-sided face is

\[
a_n \geq \sum_{i=1}^{n} i + \sum_{i=1}^{n+1} (2mn + i) = (n+1)(2mn + n + 1).
\]

The maximum weight of \( n \)-sided face is

\[
a_n + (m-1)d \leq \sum_{i=1}^{n} (2mn - n + i) + \sum_{i=1}^{n+1} (6mn + m - n + i).
\]
Thus
\[
d \leq \frac{mn(6n+5) - n(2n+1) + m+1}{m-1}.
\]
The upper bound for the possible value of \(d\) is very large in this case.

It is easy to see that the minimum possible weight \(a_4\) of the 4-sided face, under the labeling \(h\), is at least \(10mn + 25\). On the other hand, the maximum possible weight of a 4-sided face is no more than \(38mn + 5m - 11\). Hence,

\[
a_4 + (mn - 1)d \leq 38mn + 5m - 11
\]
and for \(n \geq 3\) we get \(d < 30\).

It was shown in [1] that graph \(mD_n\) admits a super \(d\)-antimagic labeling of type \((1, 1, 1)\) for \(d \in \{0, 1, 2, 3, 4, 5\}\) and for every \(m \geq 2\), \(n \geq 3\), \(n \neq 4\). In this paper we investigate the existence of the super \(d\)-antimagic labeling of type \((1, 1, 1)\) for \(mD_n\) and for \(d \in \{6, 7\}\).

2. Preliminary Results

We will treat the graph \(mD_n\) as the composition of three parts: the disjoint union of \(m\) outer \(n\)-cycles, the disjoint union of \(m\) inner \(n\)-cycles and the middle part which consists of the 4-sided faces and the spokes. The middle part can be treated as the disjoint union of \(n\)-cycles for the labeling purposes. To simplify the proofs, we denote by

- \(C_O\): the disjoint union of \(m\) outer cycles of all \(D_n\) in \(mD_n\), where \(V(C_O) = \{y^j_i : 1 \leq i \leq n, 1 \leq j \leq m\}\) and \(E(C_O) = \{y^j_i y^j_{i+1} : 1 \leq i \leq n, 1 \leq j \leq m\}\),

- \(C_I\): the disjoint union of \(m\) inner cycles of all \(D_n\) in \(mD_n\), where \(V(C_I) = \{x^j_i : 1 \leq i \leq n, 1 \leq j \leq m\}\) and \(E(C_I) = \{x^j_i x^j_{i+1} : 1 \leq i \leq n, 1 \leq j \leq m\}\),

- \(C_M\): the disjoint union of \(m\) middle cycles of all \(D_n\) in \(mD_n\), where the vertices of the cycles will be the 4-sided faces of \(mD_n\), i.e. \(V(C_M) = \{z^j_i : 1 \leq i \leq n, 1 \leq j \leq m\}\) and the edges of the cycles will be the spokes of \(mD_n\), i.e. \(E(C_M) = \{z^j_i z^j_{i+1} = x^j_{i+1}y^j_{i+1} : 1 \leq i \leq n, 1 \leq j \leq m\}\). Note, that the indices are taken modulo \(n\).

Thus, the labeling denoted by \((\alpha(C_O), \beta(C_I), \gamma(C_M))\) means a combination of the labeling \(\alpha\) for the disjoint union of \(m\) outer cycles in \(mD_n\), \(\beta\) for the disjoint union of \(m\) inner cycles in \(mD_n\), and \(\gamma\) for the disjoint union of \(m\) middle cycles in \(mD_n\).

To label \(C_O, C_I\) and \(C_M\), we will use some known results on \((a, d)\)-edge-antimagic total labeling and an \((a, d)\)-vertex-antimagic total labeling on cycles. Let us recall the definition of these labelings.

Let \(G = (V, E)\) be a graph of order \(v\) and size \(e\). Under a total labeling \(g\) (labeling of type \((1, 1, 0)\)), the associated \textit{edge-weight} is

\[
w_g(xy) = g(x) + g(xy) + g(y)
\]
weights form an arithmetic sequence starting from $x$ for $a > 0$ where $g$ is the $(2, m)$-antimagic total labeling is equivalent to an $(a, d)$-vertex-magic total labeling defined by Kotzig and Rosa in [11]. The definition of the $(a, d)$-vertex-magic total labeling was introduced in [4] and [15] as a natural extension of a notion of the vertex-magic total labeling defined by MacDougall et al. in [14]. More comprehensive information on edge-antimagic total labelings and vertex-antimagic total labelings can be found in [7] and [10].

We consider $mC_n$, as a disjoint union of $m$ copies of cycle $C_n$ with the vertex set $V(mC_n) = \{u_i : 1 \leq i \leq n, 1 \leq j \leq m\}$ and edge set $E(mC_n) = \{u_i^j u_{i+1}^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ with the indices taken modulo $n$.

Dafik et al. [9] showed that the labeling
\[
g_1(u_i^j) = (i - 1)m + j, \quad \text{for} \quad 1 \leq i \leq n \quad \text{and} \quad 1 \leq j \leq m
\]
\[
g_1(u_i^j u_{i+1}^j) = (2n - i + 1)m + 1 - j, \quad \text{for} \quad 1 \leq i \leq n \quad \text{and} \quad 1 \leq j \leq m
\]
is the $(2mn + 2, 1)$-edge-antimagic total labeling of $mC_n$, for every $m \geq 2$ and $n \geq 3$.

It is not difficult to see that, for every $m \geq 2$ and $n \geq 3$, the labeling
\[
g_2(u_i^j) = 2g_1(u_i^j) - 1, \quad \text{for} \quad 1 \leq i \leq n \quad \text{and} \quad 1 \leq j \leq m
\]
\[
g_2(u_i^j u_{i+1}^j) = 2g_1(u_i^j u_{i+1}^j) - 2mn, \quad \text{for} \quad 1 \leq i \leq n \quad \text{and} \quad 1 \leq j \leq m
\]
is the $(2mn + 2, 2)$-edge-antimagic total labeling of $mC_n$.

Observe that for the 2-regular graphs (and only for the 2-regular graphs) an $(a, d)$-edge-antimagic total labeling is equivalent to an $(a, d)$-vertex-antimagic total labeling (for $d = 0$ see [17]). Evidently, for every $m \geq 2$ and $n \geq 3$, the following labeling
\[
g_3(u_i^j) = 2g_1(u_i^j u_{i+1}^j) - 2mn, \quad \text{for} \quad i = 1 \quad \text{and} \quad 1 \leq j \leq m
\]
\[
g_3(u_i^j) = 2g_1(u_i^{i-1} u_i^j) - 2mn, \quad \text{for} \quad 2 \leq i \leq n \quad \text{and} \quad 1 \leq j \leq m
\]
\[
g_3(u_i^j u_{i+1}^j) = 2g_1(u_i^j) - 1, \quad \text{for} \quad 1 \leq i \leq n \quad \text{and} \quad 1 \leq j \leq m
\]
is the $(2mn + 2, 2)$-vertex-antimagic total labeling of $mC_n$, since all verifications are trivial.
In [9], it is proved that for \( m \geq 2, n \geq 3 \), the graph \( mC_n \) admits a \((m(n + 1) + 3, 3)\-vertex-antimagic total labeling \( g_4 \), where

\[
\begin{align*}
g_4(u_1^i) &= j, & \text{for } i = 1 \text{ and } 1 \leq j \leq m \\
g_4(u_n^i) &= m(n - 1 + i) + j, & \text{for } 2 \leq i \leq n \text{ and } 1 \leq j \leq m \\
g_4(u_n^i + 1) &= im + j, & \text{for } 1 \leq i \leq n \text{ and } 1 \leq j \leq m.
\end{align*}
\]

Using the labeling \( g_2 \) we define the labeling \( \alpha_1 \) for \( C_O \) and the labeling \( \beta_1 \) for \( C_I \) in the following way

\[
\begin{align*}
\alpha_1(y_i^j) &= g_2(u_i^j) + 1, \\
\beta_1(x_i^j) &= g_2(u_i^j), \\
\alpha_1(y_i^j + 1) &= g_2(u_i^j + 1) + 4mn, \\
\beta_1(x_i^j + 1) &= g_2(u_i^j + 1) + 4mn - 1,
\end{align*}
\]

for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \).

Using the labeling \( g_4 \), we construct the labeling \( \gamma_1 \) for \( M_D \) as follows

\[
\gamma_1(x_i^j + 1) = \begin{cases} 
 g_4(u_i^j + 1) + 2mn, & \text{for } 1 \leq i \leq n - 1, 1 \leq j \leq m \\
 g_4(u_i^j) + 2mn, & \text{for } i = n, 1 \leq j \leq m.
\end{cases}
\]

It is easy to verify that the labeling \( \alpha_1, \beta_1 \) and \( \gamma_1 \) use each integer from the set \{1, 2, \ldots, 6mn\} exactly once. The following lemma gives a super 7-antimagic labeling of type \((1, 1, 1)\) of \( mD_n \) for a special case, when \( n = 4 \).

**Lemma 2.1.** For every \( m \geq 2 \), the graph \( mD_4 \) admits a super 7-antimagic labeling of type \((1, 1, 1)\).

**Proof.** We consider the labeling \((\alpha_1(C_O), \beta_1(C_I), \gamma_1(C_M))\) for all \( m \geq 2 \) and \( n = 4 \). All the weights of the 4-sided faces \( z_{i,j}^4 \), \( 1 \leq i \leq 4, 1 \leq j \leq m \), form an arithmetic progression of the difference 7 with the values \( 77m + 8, 77m + 15, \ldots, 105m + 1 \). On the other hand, all weights of the 4-sided faces \( z_i^4 \), \( 1 \leq j \leq m \), have the same value 96m.

Now, we swap the vertex label \( \beta_1(x_i^j) = 2(i - 1)m + 2j - 1 \) with the vertex label \( \alpha_1(y_i^j) = 2(i - 1)m + 2j \), for every \( 1 \leq i \leq 3 \) and \( 1 \leq j \leq m \), and the edge label \( \beta_1(x_i^jx_i^j) = 20m + 1 - 2j \) (respectively, \( \beta_1(x_i^jx_i^j) = 18m + 1 - 2j \)) with the face label \( \gamma_1(z_{i,j}^3) = 15m + j \) (respectively, \( \gamma_1(z_{i,j}^3) = 18m + j \)), for \( 1 \leq j \leq m \). The swapping process does not have any impact on the weights of the 4-sided faces \( z_{i,j}^4 \), however the weights of 4-sided faces \( z_i^4 \), \( 1 \leq j \leq m \), constitute the set \{81m + 1 + 6j : 1 \leq j \leq m\}. 


If we complete the face labels

\[ \sigma_1(z^j_4) = 24m + j, \quad \text{for} \quad 1 \leq j \leq m \]

\[ \sigma_1(\text{external face}) = 25m + 1, \]

then the weights of all 4-sided faces in \( mD_n \) form the arithmetic progression \( 77m + 8, 77m + 15, \ldots, 105m + 1, 105m + 8, \ldots, 112m + 1 \). Hence, the resulting labeling is super 7-antimagic of type \((1,1,1)\).

\[ \square \]

3. Main Results

In this section we will present \( d \)-antimagic labeling of type \((1,1,1)\) of the disjoint union of \( m \) copies of prism, for \( d \in \{6,7\} \).

**Theorem 3.1.** For every \( m \geq 2 \) and \( n \geq 3 \), the graph \( mD_n \) has a super 7-antimagic labeling of type \((1,1,1)\).

**Proof.** For \( m \geq 2 \) and \( n = 4 \) the assertion follows from previous Lemma 2.1. Now, we consider the labeling \((\alpha_1(C_0), \beta_1(C_1), \gamma_1(C_M))\) for all \( m \geq 2 \) and \( n \geq 3 \), \( n \neq 4 \). The weights of the 4-sided faces \( z^j_{4,i}, 1 \leq i \leq n, 1 \leq j \leq m \), constitute an arithmetic progression of the difference 7, namely \( m(19n + 1) + 8, m(19n + 1) + 15, \ldots, m(26n + 1) + 1 \) and the common weight for all \( n \)-sided faces is equal to \( 6mn^2 \).

If we swap the edge label \( \beta_1(x^j_{i,x}x^j_{i+1}) = 2m(3n - i + 1) + 1 - 2j \) with the face label \( \gamma_1(z^j_{4,i}) = m(3n + i) + j, \) for \( 1 \leq i \leq 2 \) and \( 1 \leq j \leq m \), then the weights of the 4-sided faces will not be changed, but the weights of the \( n \)-sided faces will form the set \( \{6mn(n - 1) + 5m - 2 + 6j : 1 \leq j \leq m\} \). It suffices to complete the face labeling

\[ \sigma_2(z^j_n) = 6mn + j, \quad \text{for} \quad 1 \leq j \leq m \]

\[ \sigma_2(\text{external face}) = m(6n + 1) + 1, \]

and the resulting labeling is the super 7-antimagic labeling of type \((1,1,1)\).

\[ \square \]

**Theorem 3.2.** The graph \( mD_n \) admits a super 6-antimagic labeling of type \((1,1,1)\) for \( 2 \leq m \leq 2n \) and \( n \geq 3, n \neq 4 \).

**Proof.** Using the labeling \( g_3 \), we construct the labeling \( \gamma_2 \) for the 4-sided faces and spokes of the middle part of the \( mD_n \) as follows

\[ \gamma_2(x^j_{i,y}y^j_{i}) = 2mn + g_3(u^j_{i}u^j_{i+1}), \quad \text{for} \quad 1 \leq i \leq n, 1 \leq j \leq m \]

\[ \gamma_2(z^j_{4,i}) = \begin{cases} 2mn + g_3(u^j_{i+1}), & \text{for} \quad 1 \leq i \leq n - 1, 1 \leq j \leq m \\ 2mn + g_3(u^j_{1}), & \text{for} \quad i = n, 1 \leq j \leq m. \end{cases} \]
Under the labeling \((\alpha_1(C_O), \beta_1(C_I), \gamma_2(C_M))\), the weights of the 4-sided faces \(z_{4,i}^{j}; 1 \leq i \leq n, 1 \leq j \leq m\), constitute the set \(\{20mn + 7, 20mn + 13, \ldots, 26mn + 1\}\) and all the weights of the \(n\)-sided faces have the same value \(6mn^2\).

If we swap the vertex label \(\beta_1(x_{i}^{j}) = 2j - 1\) with the vertex label \(\beta_1(x_{1}^{m-j+1}) = (2m - j) + 1\) and the face label \(\gamma_2(z_{4,n}^{j}) = 2(n + 1)m + 2 - 2j\) (respectively, \(\gamma_2(z_{4,1}^{j}) = 4mn + 2 - 2j\)) with the face label \(\gamma_2(z_{4,n}^{m-j+1}) = 2mn + 2j\) (respectively, \(\gamma_2(z_{4,1}^{m-j+1}) = 2m(2n - 1) + 2j\)), for every \(1 \leq j \leq m\), then the weights of 4-sided faces will not be changed, but the weights of \(n\)-sided faces will constitute an arithmetic progression of the difference 4, namely \(\{6mn^2 + 2m + 2 - 4j : 1 \leq j \leq m\}\).

Only if \(m \leq 2n\) then for every \(k, 1 \leq k \leq m - 1\), we swap the edge label \(\alpha_1(y_{i}^{m-k}, y_{i+1}^{m-k}) = 6mn - 2(im - k) + 2\) with the edge label \(\beta_1(x_{i}^{m-k}, x_{i+1}^{m-k}) = 6mn - 2(im - k) + 1\), for \(1 \leq i \leq \lfloor \frac{k+1}{2}\rfloor\), and the vertex label \(\alpha_1(y_{i}^{m-k}) = 2(im - k)\) with the vertex label \(\beta_1(x_{i}^{m-k}) = 2(im - k) - 1\), for \(2 \leq i \leq \lfloor \frac{k+1}{2}\rfloor\). Applying the swapping will result in changes of the face weights of \(z_{n}^{j}\) to obtain an arithmetic sequence of the difference 5 for the weights of the \(n\)-sided faces.

If we complete the face labels
\[
\sigma_3(z_{n}^{j}) = m(6n + 1) + 1 - j, \quad \text{for} \quad 1 \leq j \leq m
\]
\[
\sigma_3(\text{external face}) = m(6n + 1) + 1,
\]
then the resulting labeling is a super 6-antimagic labeling of type \((1, 1, 1)\) as requires. \(\square\)

4. Conclusion

We have shown that for \(mD_n\) there exists the super 7-antimagic labeling of type \((1, 1, 1)\) for \(m \geq 2, n \geq 3\), and the super 6-antimagic labeling of type \((1, 1, 1)\) for \(n \geq 3 n \neq 4\) and \(2 \leq m \leq 2n\). We have tried to find a super 6-antimagic labeling of type \((1, 1, 1)\) for \(n \geq 3\) and for every \(m \geq 2\), but so far without success. So, we propose the following

**Problem 4.1.** For the graph \(mD_n\) determine if there is a super 6-antimagic labeling of type \((1, 1, 1)\), for every \(m \geq 2\) and \(n \geq 3\).

In Introduction we have shown that the upper bound for the feasible values of the difference \(d\) is 29. Let us recall the open problem from the paper [1].

**Problem 4.2.** Find other possible values of the difference \(d\) and the corresponding super \(d\)-antimagic labelings of type \((1, 1, 1)\) for \(mD_n\), \(m \geq 2, n \geq 3\) and \(d > 7\).

**Acknowledgement**

Support of Slovak VEGA Grant 1/4005/07 and Higher Education Commission Pakistan Grant HEC(FD)/2007/555 is acknowledged.
On super $d$-antimagic labelings of disjoint union of prisms

References


