TILTING THEORY AND FUNCTOR CATEGORIES I. CLASSICAL TILTING

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Abstract. Tilting theory has been a very important tool in the classification of finite dimensional algebras of finite and tame representation type, as well as, in many other branches of mathematics. Happel [Ha] proved that generalized tilting induces derived equivalences between module categories, and tilting complexes were used by Rickard [Ri] to develop a general Morita theory of derived categories.

In the other hand, functor categories were introduced in representation theory by M. Auslander [A], [AQM] and used in his proof of the first Brauer-Thrall conjecture [A2] and later on, used systematically in his joint work with I. Reiten on stable equivalence [AR], [AR2] and many other applications.

Recently, functor categories were used in [MVS3] to study the Auslander-Reiten components of finite dimensional algebras.

The aim of the paper is to extend tilting theory to arbitrary functor categories, having in mind applications to the functor category $\mathrm{Mod}(\mathrm{mod}_\Lambda)$, with Λ a finite dimensional algebra.

1. Introduction and basic results

Tilting theory traces back his history to the article by Bernstein, Gelfand and Ponomarev ([BGP] 1973), where they defined partial Coxeter functors. These functors were generalized by Auslander Platzeck and Reiten ([APR] 1979), but a major brake through was the article by Brenner and Butler ([BB] 1979). The notion of tilting was further generalized by Miyashita ([Mi] 1986) and Happel ([Ha] 1987). Happel showed that generalized tilting induces derived equivalence between the corresponding module categories. These results inspired Rickard [Ri] to develop his Morita theory of derived categories.

This paper is the first one in a series of articles in which, having in mind applications to functor categories from subcategories of modules over a finite dimensional algebra to the category of abelian groups, we generalize tilting theory from modules to functor categories.

The first paper is dedicated to classical tilting and it consists of four sections:

In the first section we fix the notation and recall some notions from functor categories that will be used through the paper. In the second section, we generalize

Date: December 15, 2010.

1991 *Mathematics Subject Classification.* 2000]Primary 05C38, 15A15; Secondary 05A15, 15A18.

Key words and phrases. Classical Tilting, Functor Categories.

The second author thanks CONACYT for giving him financial support during his graduate studies.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Bongartz´s proof [B] of Brenner-Butler´s theorem [BB] to arbitrary functor categories. In the third section we deal with two special cases: first we extend tilting from subcategories to full categories, and second we apply the theory so far developed, to study tilting for infinite quivers with no relations, and apply it to compute the Auslander-Reiten components of infinite Dynkin quivers. In the last section we recall results from [MVS1], [MVS2], [MVS3] on graded and Koszul categories and we apply our results from the third section, to compute the Auslander-Reiten components of the categories of Koszul functors over a regular Auslander-Reiten component of a finite dimensional algebra. These results generalize previous work by [MV] on the preprojective algebra.

1.1. **The Category** $\mathrm{Mod}(\mathcal{C})$**.** In this section $\mathcal{C}$ will be an arbitrary skeletally small pre additive category, and $\mathrm{Mod}(\mathcal{C})$ will denote the category of contravariant functors from $\mathcal{C}$ to the category of abelian groups. Following the approach by Mitchel [M], we can think of $\mathcal{C}$ as a ring "with several objects" and $\mathrm{Mod}(\mathcal{C})$ as a category of $\mathcal{C}$-modules. The aim of the paper is to show that the notions of tilting theory can be extended to $\mathrm{Mod}(\mathcal{C})$, to obtain generalizations of the main theorems on tilting for rings. To obtain generalizations of some tilting theorems for finite dimensional algebras, we need to add restrictions on our category $\mathcal{C}$, like: existence of pseudo kernels, Krull-Schmidt, Hom-finite, dualizing, etc.. To fix the notation, we recall known results on functors and categories that we use through the paper, referring for the proofs to the papers by Auslander and Reiten [A], [AQM], [AR].

1.2. **Functor Categories.** Let $\mathcal{C}$ be a pre additive skeletally small category. By $\mathrm{Mod}(\mathcal{C})$ we denote the category of additive contravariant functors from $\mathcal{C}$ to the category of abelian groups. Then, $\mathrm{Mod}(\mathcal{C})$ is an abelian category with arbitrary sums and products, in fact it has arbitrary limits and colimits, and filtered limits are exact (Ab5 in Grothendiek terminology). It has enough projective and injective objects. For any object $C \in \mathcal{C}$, the representable functor $\mathcal{C}(\ ,C)$ is projective, arbitrary sums of representable functors are projective, and any object $M \in \mathrm{Mod}(\mathcal{C})$ is covered by an epimorpism $\coprod_i \mathcal{C}(\ ,C_i) \to M \to 0$. We say that an object M in $\mathrm{Mod}(C)$ is finitely generated if there exists an epimorphism $\coprod_{i\in I} \mathcal{C}(\ ,C_i) \to M \to 0$, with I a finite set.

Given a finitely generated functor M and and an arbitrary sum of functors $\coprod_j N_j$, there is a natural isomorphism $\mathrm{Hom}_{\mathcal{C}}(M, \coprod_j N_j) \cong \coprod_j \mathrm{Hom}_{\mathcal{C}}(M, N_j)$ (finitely generated are compact).

An object P in $\mathrm{Mod}(\mathcal{C})$ is projective (finitely generated projective) if and only if P is summand of $\coprod_{i\in I} \mathcal{C}(\ ,C_i)$ for a family (finite) $\{C_i\}_{i\in I}$ of objects in $\mathcal{C}$. The subcategory $\mathfrak{p}(\mathcal{C})$ of $\mathrm{Mod}(\mathcal{C})$, of all finitely generated projective objects, is a skeletally small additive category in which idempotents split, the functor $P : \mathcal{C} \to \mathfrak{p}(\mathcal{C})$, $P(C) = \mathcal{C}(\ ,C)$, is fully faithful and induces by restriction $\mathrm{res} : \mathrm{Mod}(\mathfrak{p}(\mathcal{C})) \to \mathrm{Mod}(\mathcal{C})$, an equivalence of categories [A]. For this reason we may assume that our categories are skeletally small additive categories such that idempotents split, they were called annuli varieties in [A].

Definition 1. [AQM] *Given a preadditive skeletally small category $\mathcal{C}$, we say $\mathcal{C}$ has pseudokernels, if given a map $f : C_1 \to C_0$ there exists a map $g : C_2 \to C_1$ such that the sequence:* $\mathcal{C}(\ C_2) \overset{(\ ,g)}{\to} \mathcal{C}(\ C_1) \overset{(\ ,f)}{\to} \mathcal{C}(\ ,C_0)$ *is exact.*

A functor M is finitely presented if there exists an exact sequence $\mathcal{C}(\ C_1) \to \mathcal{C}(\ ,C_0) \to M \to 0$. We denote by $\mathrm{mod}(\mathcal{C})$ the full subcategory of $\mathrm{Mod}(\mathcal{C})$ consisting of finitely presented functors. It was proved in [AQM] $\mathrm{mod}(\mathcal{C})$ is abelian if and only if $\mathcal{C}$ has pseudokernels.

We will say indistinctly that M is an object of $\mathrm{Mod}(\mathcal{C})$ or that M is a $\mathcal{C}$-module. A representable functor $\mathcal{C}(\ ,C)$ will be sometimes denoted by $(\ ,C)$.

1.3. **Change of Categories.** According to [A], there exists a unique up to isomorphism functor $-\otimes_{\mathcal{C}}- : \mathrm{Mod}(\mathcal{C}^{op}) \times \mathrm{Mod}(\mathcal{C}) \to \mathbf{Ab}$, called the **tensor product**, with the following properties:

(a) (i) For each $\mathcal{C}$-module N, the functor $\otimes_{\mathcal{C}}N : \mathrm{Mod}(\mathcal{C}^{op}) \to \mathbf{Ab}$, given by $(\otimes_{\mathcal{C}}N)(M) = M \otimes_{\mathcal{C}} N$ is right exact.
   (ii) For each $\mathcal{C}^{op}$-module M, the functor $M\otimes_{\mathcal{C}} : \mathrm{Mod}(\mathcal{C}) \to \mathbf{Ab}$, given by $(M\otimes_{\mathcal{C}})(N) = M \otimes_{\mathcal{C}} N$ is right exact.

(b) The functors $M \otimes_{\mathcal{C}} -$ and $-\otimes_{\mathcal{C}} N$ preserve arbitrary sums.

(c) For each object C in $\mathcal{C}$ $M \otimes_{\mathcal{C}} (\ ,C) = M(C)$ and $(C,\ \ ) \otimes_{\mathcal{C}} N = N(C)$.

Given a full subcategory $\mathcal{C}'$ of $\mathcal{C}$. The restriction $\mathrm{res} : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Mod}(\mathcal{C}')$ has a right adjoint, called also the tensor product, and denoted by $\mathcal{C}\otimes_{\mathcal{C}'} : \mathrm{Mod}(\mathcal{C}') \to \mathrm{Mod}(\mathcal{C})$. This functor is defined by: $(\mathcal{C} \otimes_{\mathcal{C}'} M)(C) = (C,\ \ )|_{\mathcal{C}'} \otimes_{\mathcal{C}'} M$, for any M in $\mathrm{Mod}(\mathcal{C}')$ and C in $\mathcal{C}$. The following proposition is proved in [A Prop. 3.1].

Proposition 1. *Let $\mathcal{C}'$ be a full subcategory of $\mathcal{C}$. The functor $\mathcal{C}\otimes_{\mathcal{C}'} : \mathrm{Mod}(\mathcal{C}') \to \mathrm{Mod}(\mathcal{C})$ satisfies the following conditions:*

(a) *$\mathcal{C}\otimes_{\mathcal{C}'}$ is right exact and preserves arbitrary sums.*
(b) *The composition $\mathrm{Mod}(\mathcal{C}') \xrightarrow{\mathcal{C}\otimes_{\mathcal{C}'}} \mathrm{Mod}(\mathcal{C}) \xrightarrow{\mathrm{res}} \mathrm{Mod}(C')$ is the identity in $\mathrm{Mod}(\mathcal{C}')$.*
(c) *For each object C' in $\mathcal{C}'$, $\mathcal{C} \otimes_{\mathcal{C}'} \mathcal{C}'(\ \ ,C') = \mathcal{C}(\ \ ,C')$.*
(d) *$\mathcal{C}\otimes_{\mathcal{C}'}$ is fully faithful.*
(e) *$\mathcal{C}\otimes_{\mathcal{C}'}$ preserves projective objects.*

The functor M in $\mathrm{Mod}(\mathcal{C})$ is called projectively presented over $\mathcal{C}'$, if there exists an exact sequence $\coprod_{i\in I} \mathcal{C}(\ \ ,C'_i) \to \coprod_{j\in J} \mathcal{C}(\ \ ,C'_j) \to M \to 0$, with $C'_i, C'_j \in \mathcal{C}'$. The category $\mathcal{C} \otimes_{\mathcal{C}'} \mathrm{Mod}(\mathcal{C}')$ is the subcategory of $\mathrm{Mod}(\mathcal{C})$ whose objects are the functors projectively presented over $\mathcal{C}'$. The functor res and $\mathcal{C}\otimes_{\mathcal{C}'}$ induces an equivalence between $\mathrm{Mod}(\mathcal{C}')$ and $\mathcal{C} \otimes_{\mathcal{C}'} \mathrm{Mod}(\mathcal{C}')$.

We say that an exact sequence $P_1 \xrightarrow{\alpha} P_0 \xrightarrow{\beta} M \to 0$ is a **minimal projective presentation** of M if and only if the epimorphisms $P_0 \xrightarrow{\beta} M$ and $P_1 \to \mathrm{Im}\alpha$, are minimal projective covers (in the sense of [AF]).

It is of interest to know under what conditions minimal projective presentations exist.

Theorem 1 (A Theor. 4.12)**.** *The following conditions are equivalent:*

(a) *Every object in $\mathrm{mod}(\mathcal{C})$ has a minimal projective presentation.*
(b) *For each object C in $\mathcal{C}$, every finitely presented $\mathrm{End}(C)^{op}$-module has a minimal projective presentation.*

A ring R is **semiperfect**, if every finitely generated R-module has a projective cover.

Corollary 1. *If $\mathcal{C}$ is a category, such that for every C in $\mathcal{C}$, $\mathrm{End}(C)^{op}$ is semiperfect, then every object in $\mathrm{mod}(\mathcal{C})$ has a minimal projective presentation.*

1.4. **Krull-Schmidt Categories.** We start giving some definitions from [AR].

Definition 2. *Let R be a commutative artin ring, an R-category $\mathcal{C}$, is a pre additive category such that $\mathcal{C}(C_1, C_2)$ is an R-module and composition is R-bilinear. Under these conditions $\mathrm{Mod}(\mathcal{C})$ is a R-category which we identify with the category of functors $(\mathcal{C}^{op}, \mathrm{Mod}(R))$.*

An R-category $\mathcal{C}$ is Hom-***finite****, if for each pair of objects C_1, C_2 in $\mathcal{C}$ the R-module $\mathcal{C}(C_1, C_2)$ is finitely generated. We denote by $(\mathcal{C}^{op}, \mathrm{mod}(R))$, the full subcategory of $(\mathcal{C}^{op}, \mathrm{Mod}(R))$ consisting of the $\mathcal{C}$-modules such that; for every C in $\mathcal{C}$ the R-module $M(C)$ is finitely generated. The category $(\mathcal{C}^{op}, \mathrm{mod}(R))$ is abelian and the inclusion $(\mathcal{C}^{op}, \mathrm{mod}(R)) \to (\mathcal{C}^{op}, \mathrm{Mod}(R))$ is exact.*

The category $\mathrm{mod}(C)$ is a full subcategory of $(\mathcal{C}^{op}, \mathrm{mod}(R))$. The functors $D : (\mathcal{C}^{op}, \mathrm{mod}(R)) \to (\mathcal{C}, \mathrm{mod}(R))$, and $D : (\mathcal{C}, \mathrm{mod}(R)) \to (\mathcal{C}^{op}, \mathrm{mod}(R))$, are defined as follows: for any C in $\mathcal{C}$, $D(M)(C) = \mathrm{Hom}_R(M(C), I(R/r))$, with r the Jacobson radical of R, and $I(R/r)$ is the injective envelope of R/r. The functor D defines a duality between $(\mathcal{C}, \mathrm{mod}(R))$ and $(\mathcal{C}^{op}, \mathrm{mod}(R))$. If $\mathcal{C}$ is an Hom-finite R-category and M is in $\mathrm{mod}(\mathcal{C})$, then $M(C)$ is a finitely generated R-module and is therefore in $\mathrm{mod}(R)$.

Definition 3. *An* Hom-*finite R-category $\mathcal{C}$ is* ***dualizing,*** *if the functor $D : (\mathcal{C}^{op}, \mathrm{mod}(R)) \to (\mathcal{C}, \mathrm{mod}(R))$ induces a duality between the categories $\mathrm{mod}(\mathcal{C})$ and $\mathrm{mod}(\mathcal{C}^{op})$.*

It is clear from the definition that for dualizing categories $\mathrm{mod}(\mathcal{C})$ has enough injectives.

To finish, we recall the following definition:

Definition 4. *An additive category $\mathcal{C}$ is* ***Krull-Schmidt****, if every object in $\mathcal{C}$ decomposes in a finite sum of objects whose endomorphism ring is local.*

1.5. **The radical of a category.** The notion of the Jacobson radical of a category was introduced in [M] and [A], it is defined in the following way:

Definition 5. *The Jacobson radical of $\mathcal{C}$, $\mathrm{rad}_{\mathcal{C}}(\ ,\ )$, is a subbifunctor of $\mathrm{Hom}_{\mathcal{C}}(\ ,\ )$ defined in objects as: $\mathrm{rad}_{\mathcal{C}}(X, Y) = \{f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ |for any map $g : Y \to X$, $1 - gf$ is invertible $\}$.*

If M is a $\mathcal{C}$-module , then we denote by $\mathrm{rad}M$ the intersection of all maximal subfunctors of M.

Proposition 2. [A], [BR], [M] *Let $\mathcal{C}$ be an additive category and $\mathrm{rad}_{\mathcal{C}}(\ ,\ )$ the Jacobson radical of $\mathcal{C}$. Then:*

(a) *For every object C in $\mathcal{C}$ $\mathrm{rad}_{\mathcal{C}}(C, C)$ is just the Jacobson radical of $\mathrm{End}_{\mathcal{C}}(C)$.*
(b) *If C and C' are indecomposable objects in $\mathcal{C}$, then the radical $\mathrm{rad}_{\mathcal{C}}(C, C')$ consists of all non isomorphisms from C to C'.*
(c) *For every object C in $\mathcal{C}$, $\mathrm{rad}_{\mathcal{C}}(C,\ ) = \mathrm{rad}\mathcal{C}(C,\ )$ and $\mathrm{rad}_{\mathcal{C}}(\ , C) = \mathrm{rad}\mathcal{C}(\ , C)$.*
(d) *For every pair of objects C and C' in $\mathcal{C}$, $\mathrm{rad}_{\mathcal{C}}(C',\ )(C) = \mathrm{rad}_{\mathcal{C}}(\ , C)(C')$.*

Definition 6. *By an ideal of the additive category $\mathcal{C}$ we understand a sub bifunctor of $\mathrm{Hom}_{\mathcal{C}}(\ ,\ )$.*

Given two ideals of I_1 and I_2 of $\mathcal{C}$ we define I_1I_2 as follows: $f \in I_1I_2(C_1, C_3)$, if and only if, f is a finite sum of morphisms $C_1 \xrightarrow{h} C_2 \xrightarrow{g} C_3$, with $h \in I_1(C_1, C_2)$ and $g \in I_2(C_2, C_3)$.

Given an ideal I of $\mathcal{C}$ and a $\mathcal{C}$-module M, we define a $\mathcal{C}$-sub module IM of M by

$$IM(C) = \Sigma_{f\in I(C,C')}\mathrm{Im}M(f),$$

with C in $\mathcal{C}$. We say a $\mathcal{C}$-module S is **simple**, if it does not have proper $\mathcal{C}$-sub modules.

Lemma 1. [MVS1 Lemma 2.5], [BR] *Let $\mathcal{C}$ be a Krull-Schmidt K-category. Then the following statements are true:*

(a) *Any simple functor in* $\mathrm{Mod}(\mathcal{C})$ *is of the form* $\mathcal{C}(\ \ ,C)/\mathrm{rad}_{\mathcal{C}}(\ \ ,C)$*, for some indecomposable object C in $\mathcal{C}$.*
(b) *For all finitely generated functors F in* $\mathrm{Mod}(\mathcal{C})$*, the radical of F is isomorphic to* $\mathrm{rad}_{\mathcal{C}}F$.
(c) *All finitely generated functors F in* $\mathrm{Mod}(\mathcal{C})$ *have a projective cover.*

1.6. **A Pair of Adjoint Functors.** Let $\mathcal{C}$ be a skeletally small additive category, and $\mathcal{T}$ a skeletally small full subcategory of $\mathrm{Mod}(\mathcal{C})$. Let's define the following functor

$$\phi : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Mod}(\mathcal{T}),\ \ \phi(M) = \mathrm{Hom}(\ \ , M)_{\mathcal{T}}.$$

Our aim in this subsection is to prove ϕ has a left adjoint. Since $\mathrm{Mod}(\mathcal{C})$ is abelian, it has equalizers, and it was remarked above it is Ab5. In this way $\mathrm{Mod}(\mathcal{C})$ is complete [SM V.2 Theo.1]. Since the functor $\mathrm{Hom}_{\mathcal{C}}(T, -)$ preserve limits, for every T in $\mathcal{T}$ it follows that the functor ϕ preserve limits.

Lemma 2. *The category* $\mathrm{Mod}(\mathcal{C})$ *has an injective cogenerator.*

Proof. Denote by D the functor $D : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Mod}(\mathcal{C}^{op})$, $D(F)(C) = (F(C), \mathbb{Q}/\mathbb{Z})$, and let F be a functor in $\mathrm{Mod}(\mathcal{C}^{op})$. Then we have an epimorphism $\coprod_i \mathcal{C}(C_i,\ \ ) \to DF$ and hence, a monomorphism

$$F \xrightarrow{\eta} D^2F \xrightarrow{Df} \prod_{i\in I} D\mathcal{C}(C_i,\ \ ),$$

where $\eta_C(x)(f) = f(x)$. Moreover, $D(C_i,\ \ )$ is injective. Hence $\{D\mathcal{C}(C,\ \ )\}_{C\in\mathcal{C}}$ is a small cogenerator set consisting of injective objects in $\mathrm{Mod}(\mathcal{C})$. □

We need the following

Definition 7. *Let $\mathcal{A}$ be an arbitray category and $\mathbf{X} \to Y = \{X_i \xrightarrow{f_i} Y\}_{i\in I}$ a family of morphisms. A* ***fibered product*** *of $\mathbf{X} \to Y$ is a pair $(\{P \xrightarrow{p_i} X_i\}_{i\in I}, L)$, where $L : P \to Y$ is a morphism such that $f_ip_i = L$, and satisfies the following universal property:*

If $(\{Q \xrightarrow{q_i} X_i\}_{i\in I}, L')$ is a pair such that for each $i \in I$, $f_iq_i = L'$, then there exists a unique morphism $\eta : Q \to P$, such that $L\eta = L'$, and for each $i \in I$, $p_i\eta = q_i$.

Using the fact $\mathrm{Mod}(\mathcal{C})$ is Ab5 and ϕ is left exact, the reader can verify the following

Proposition 3. *In* $\mathrm{Mod}(\mathcal{C})$ *any family of monomorphisms $\mathbf{X} \to Y = \{X_i \xrightarrow{f_i} Y\}_{i\in I}$ has a fibered product. Moreover, ϕ preserves fibered products of monomorphisms.*

Freyd's special adjoint functor theorem [MS V.7 Theo. 2] justifies the following assertion.

Theorem 2. *Let $\mathcal{C}$ be a skeletally small pre additive category. Then the following is true:*

(a) *The category $\mathrm{Mod}(\mathcal{C})$ is complete-small, has a small cogenerator, and any set of subobjects of a functor M has a fibered product.*
(b) *Let $\mathcal{T}$ be a small full subcategory of $\mathrm{Mod}(\mathcal{C})$. Then the functor $\phi : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Mod}(\mathcal{T})$ preserves small limits and fibered products of families of monomorphisms.*
(c) *The functor $\phi : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Mod}(\mathcal{T})$ has a left adjoint.*

Remark 1. *Denote by $-\otimes\mathcal{T}$ the left adjoint of ϕ, it follows from Yoneda's Lemma, that for any pair of objects T in $\mathcal{T}$ and M in $\mathrm{Mod}(\mathcal{C})$, there are natural isomorphisms:*

$$\mathrm{Hom}_{\mathcal{C}}((\ \ ,T)_{\mathcal{T}} \otimes \mathcal{T}, M) \cong \mathrm{Hom}_{\mathcal{T}}((\ \ ,T)_{\mathcal{T}}, \phi(M)) \cong \mathrm{Hom}_{\mathcal{C}}(T, M).$$

By Yoneda's Lemma again, $(\ ,T)_{\mathcal{T}} \otimes \mathcal{T} = T$.

Since there are enough projective and enough injective objects in $\mathrm{Mod}(\mathcal{C})$, and $\mathrm{Mod}(\mathcal{T})$, we can define, for any integer n, the nth right derived functors of the functors $\mathrm{Hom}_{\mathcal{C}}(M,\ \ )$, $\mathrm{Hom}_{\mathcal{C}}(\ \ ,N)$, which will be denoted by $\mathrm{Ext}^n_{\mathcal{C}}(M,\ \ )$ and $\mathrm{Ext}^n_{\mathcal{C}}(\ \ ,N)$ respectively. Analogously, the nth left derived functors of $M \otimes_{\mathcal{C}} -$ and $- \otimes N$, can be defined. They will be denoted by $\mathrm{Tor}^{\mathcal{C}}_n(M,\ \ )$ and $\mathrm{Tor}^{\mathcal{C}}_n(\ \ ,N)$, respectively.

In the same way, the right derived functors of the functor ϕ can be defined, they will be denoted by $\mathrm{Ext}^n_{\mathcal{C}}(\ \ ,-)_{\mathcal{T}} : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Mod}(\mathcal{T})$, and they are defined as $\mathrm{Ext}^n_{\mathcal{C}}(\ \ ,-)_{\mathcal{T}}(M) = \mathrm{Ext}^n_{\mathcal{C}}(\ \ ,M)_{\mathcal{T}}$.

Of course, we can also define the left derived functors of the functor $-\otimes\mathcal{T}$, they will be denoted by $\mathrm{Tor}^{\mathcal{T}}_n(\ \ ,\mathcal{T}) : \mathrm{Mod}(\mathcal{T}) \to \mathrm{Mod}(\mathcal{C})$.

We will see below relations among these functors.

Proposition 4. *If M is finitely presented, then the functors $\mathrm{Ext}^1_{\mathcal{C}}(M,\ \ )$ commute with arbitrary sums.*

Proof. There is an exact sequence

$$0 \to \mathrm{Ker}(\alpha) \to \mathcal{C}(\ \ ,C) \xrightarrow{\alpha} M \to 0 \tag{1.1}$$

with $\mathrm{Ker}(\alpha)$ finitely generated and C an object in $\mathcal{C}$. Let $\{N_i\}_{i\in I}$ be a family of objects in $\mathrm{Mod}(\mathcal{C})$. After applying $\mathrm{Hom}_{\mathcal{C}}(\ \ ,\coprod_{i\in I} N_i)$ to (1.1), it follows the existence of a isomorphism η, such that the following diagram commutes

$$\begin{array}{ccccccc}
(\mathcal{C}(\ \ ,C),\coprod_{i\in I} N_i) & \longrightarrow & (\mathrm{Ker}(\alpha),\coprod_{i\in J} N_i) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{C}}(M,\coprod_{i\in I} N_i) & \longrightarrow & 0 \\
\cong\downarrow & & \cong\downarrow & & \eta\downarrow & & \\
\coprod_{i\in I}(\mathcal{C}(\ \ ,C),N_i) & \longrightarrow & \coprod_{i\in I}(\mathrm{Ker}(\alpha),N_i) & \longrightarrow & \coprod_{i\in I}\mathrm{Ext}^1_{\mathcal{C}}(M,N_i) & \longrightarrow & 0
\end{array}$$

□

Corollary 2. *If $\mathcal{T}$ consists of finitely presented functors, then the functors ϕ and $\mathrm{Ext}^1_{\mathcal{C}}(\ \ ,-)_{\mathcal{T}}$ commute with arbitrary sums.*

2. Brenner-Butler´s Theorem

2.1. **The main theorem.** In this section we introduce the notions of tilting categories and we show, that with slights modifications, Bongartz's proof $[B]$ of Brenner-Butler's theorem $[BB]$, extends to tilting categories over arbitrary skeletally small pre additive categories. We also have the corresponding theorems on the invariance of Grothendieck groups under tilting, and the relations between the global dimension of a category and the global dimension of the tilted category. Then we prove, that under mild conditions, tilting functors restrict to the categories of finitely presented functors. For dualizing categories we have theorems analogous to classical tilting for finite dimensional algebras.

The first definition is a natural generalization of the classical notion of a tilting object.

Definition 8. *Let $\mathcal{C}$ an annuli variety, a subcategory $\mathcal{T}$ of $\mathrm{Mod}(\mathcal{C})$ is a tilting category, if every object in $\mathcal{T}$ is finitely presented and the following conditions are satisfied:*

(i) $\mathrm{pdim}\mathcal{T} \leq 1$.
(ii) *We have $\mathrm{Ext}^1_{\mathcal{C}}(T_i, T_j) = 0$, for every pair of objects T_i, T_j in $\mathcal{T}$.*
(iii) *For every object C in $\mathcal{C}$, the representable functor $\mathcal{C}(\ \ ,C)$ has a resolution*

$$0 \to \mathcal{C}(\ \ ,C) \to T_1 \to T_2 \to 0,$$

with T_1, T_2 in $\mathcal{T}$.

Definition 9. *Given a skeletally small pre additive category $\mathcal{C}$, we introduce the following notion of tensor product: given an abelian group G and an object M in $\mathrm{Mod}(\mathcal{C})$ the tensor product $G \otimes_{\mathbb{Z}} M$, is the functor defined in objects as $(G \otimes_{\mathbb{Z}} M)(C) = G \otimes_{\mathbb{Z}} M(C)$.*

Given a subcategory $\mathcal{T}$ of $\mathrm{Mod}(\mathcal{C})$, we define for every M in $\mathrm{Mod}(\mathcal{C})$ and every object T of $\mathcal{T}$ the evaluation map $e^{(T,M)} : \mathcal{C}(T, M) \otimes_{\mathbb{Z}} M \to M$, $e^{(T,M)}_C(f \otimes m) = f_C(m)$, where, $C \in \mathcal{C}$, $m \in M(C)$ and $f = \{f_C\}_{C \in \mathcal{C}} \in \mathcal{C}(T, M)$. Define the trace of $\mathcal{T}$ in M, as the image of the sum $\coprod e^{(T_i,M)} : \coprod_{i \in I} \mathcal{C}(T_i, M) \otimes_{\mathbb{Z}} T_i \to M$, where $\{T_i\}_{i \in I}$ is a set of representatives of the isomorphism classes of objects in $\mathcal{T}$.

Let T_i be an object in $\{T_i\}_{i \in I}$, and $X_i = \{f^i_j\}_{j \in J_i}$ a set of generators of the abelian group $\mathcal{C}(T_i, M)$, let $\mathbb{Z}^{(X_i)}$ be the free abelian group with basis X_i and $X = \cup X_i$. Then the epimorphism of abelian groups

$$\varphi : \mathbb{Z}^{(X_i)} \to \mathcal{C}(T_i, M) \to 0$$

Induces an epimorphism

$$\psi : \coprod_{i \in I} T_i^{(X_i)} \cong \coprod_{i \in I} \mathbb{Z}^{(X_i)} \otimes_{\mathbb{Z}} T_i \to \coprod_{i \in I} \mathcal{C}(T_i, M) \otimes_{\mathbb{Z}} T_i \quad \to 0$$

composing with the sum of the evaluation maps we obtain a map

$$\coprod e^{(T_i,M)} \psi : \coprod_{i \in I} T_i^{(X_i)} \to M$$

Let's denote $\coprod_{i \in I} T_i^{(X_i)}$ by $T^{(X)}$, then the map: $\Theta_M = \coprod e^{(T_i,M)} \psi : T^{(X)} \to M$ has the following property:

Proposition 5. *Let M be in* $\mathrm{Mod}(\mathcal{C})$*, and $\Theta_M : T^{(X)} \to M$ the map given above. Then for every object $T' \in \mathcal{T}$ and every map $\eta : T' \to M$, there exists a map $\gamma : T' \to T^{(X)}$ such that $\Theta_M\ \gamma = \eta$.*

Given a skeletally small pre additive category $\mathcal{C}$ and $\mathcal{T}$ a tilting subcategory of $\mathrm{Mod}(\mathcal{C})$, we denote by $\mathscr{T}$ the full subcategory of $\mathrm{Mod}(\mathcal{C})$ whose objects are epimorphic images of objects in $\mathcal{T}$. We want to prove that $\mathscr{T}$ is a torsion class of $\mathrm{Mod}(\mathcal{C})$.

The above proposition implies the following:

Proposition 6. *Let M be an object in* $\mathrm{Mod}(\mathcal{C})$*. Then M is in $\mathscr{T}$, if and only if, $\coprod e^{(T_i,M)}$ is an epimorphism.*

We can prove now:

Proposition 7. *If M is in $\mathscr{T}$, then, for every object $T \in \mathcal{T}$,* $\mathrm{Ext}^1_{\mathcal{C}}(T, M) = 0$.

Proof. Let's assume M is in $\mathscr{T}$. Then there exists a short exact sequence

$$0 \to \mathrm{Ker}(\alpha) \to \coprod_{j\in J} T_j \xrightarrow{\alpha} M \to 0, \tag{2.1}$$

where $\{T_j\}_{j\in J}$ is a family of objects in $\mathcal{T}$. Since T is finitely presented, it follows $\mathrm{Ext}^1_{\mathcal{C}}(T, \ \ )$ commutes with arbitrary sums. By hypothesis, $\mathrm{Ext}^1_{\mathcal{C}}(T, T_j) = 0$ for every $j \in J$, and T of projective dimension one, implies for every object K, $\mathrm{Ext}^2_{\mathcal{C}}(T, K) = 0$. Applying $\mathrm{Hom}_{\mathcal{C}}(T, \ \ )$ to the above exact sequence, it follows from the long homology sequence, that the sequence $\mathrm{Ext}^1_{\mathcal{C}}(T, \coprod_{j\in J} T_j) \to \mathrm{Ext}^1_{\mathcal{C}}(T, M) \to \mathrm{Ext}^2_{\mathcal{C}}(T, K)$ is exact. Finally we have $\mathrm{Ext}^1_{\mathcal{C}}(T, M) = 0$. □

Proposition 8. *The category $\mathscr{T}$ is closed under extensions, direct sums and epimorphic images. This is: $\mathscr{T}$ is a torsion class of a torsion theory of* $\mathrm{Mod}(\mathcal{C})$*.*

Proof. We only need to see it is closed under extensions. Consider the exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0,$$

with M_1, M_3 in $\mathscr{T}$. Then $\mathrm{Ext}^1_{\mathcal{C}}(T_i, M_1) = \mathrm{Ext}^1_{\mathcal{C}}(T_i, M_3) = 0$, by Proposition 7.

For every object $T_i \in \{T_i\}_{i\in I}$, we apply the functor $\mathcal{C}(T_i, \ \ )$, to the above exact sequence to obtain, by the long exact sequence, an exact sequence of abelian groups

$$0 \to \mathcal{C}(T_i, M_1) \to \mathcal{C}(T_i, M_2) \to \mathcal{C}(T_i, M_3) \to \mathrm{Ext}^1_{\mathcal{C}}(T_i, M_1) = 0.$$

Applying the tensor product $\otimes_{\mathbb{Z}} T_i$ to the sequence, and adding the exact sequences, we obtain the exact sequence:

$$\coprod_{i\in I} \mathcal{C}(T_i, M_1) \otimes_{\mathbb{Z}} T_i \to \coprod_{i\in I} \mathcal{C}(T_i, M_2) \otimes_{\mathbb{Z}} T_i \to \coprod_{i\in I} \mathcal{C}(T_i, M_3) \otimes_{\mathbb{Z}} T_i \to 0,$$

which induces a commutative diagram:

$$\begin{array}{ccccccc}
 & & \coprod_{i\in I} \mathcal{C}(T_i, M_1) \otimes T_i & \longrightarrow & \coprod_{i\in I} \mathcal{C}(T_i, M_1) \otimes T_i & \to & \coprod_{i\in I} \mathcal{C}(T_i, M_1) \otimes T_i \to 0 \\
 & & \downarrow {\scriptstyle \coprod e^{(T_i,M_1)}} & & \downarrow {\scriptstyle \coprod e^{(M_i,F_2)}} & & \downarrow {\scriptstyle \coprod e^{(T_i,M_3)}} \\
0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 \to 0
\end{array}$$

The morphisms $\coprod e^{(T_i,M_1)}$ and $\coprod e^{(T_i,M_3)}$ are epimorphisms, by Proposition 6. Then it follows, by the Snake lemma, $\coprod e^{(T_i,M_2)}$ is an epimorphism. Again by Proposition 6, M_2 is in $\mathscr{T}$. □

We will prove next that the trace of $\mathcal{T}$ in M, denoted by $\tau_{\mathcal{T}}(M)$, is the idempotent radical corresponding to the torsion theory with torsion class $\mathcal{T}$.

Proposition 9. *For any M in $\mathrm{Mod}(\mathcal{C})$ and T' in $\mathcal{T}$, $\mathrm{Hom}_{\mathcal{C}}(T', M/\tau_{\mathcal{T}}M) = 0$. In particular $\tau_{\mathcal{T}}(M/\tau_{\mathcal{T}}(M)) = 0$.*

Proof. Consider the exact sequence

$$0 \to \tau_{\mathcal{T}}(M) \xrightarrow{j} M \xrightarrow{p} M/\tau_{\mathcal{T}}(M) \to 0 \tag{2.2}$$

and a natural transformation $\eta : T' \to M/\tau_{\mathcal{T}}(M)$. Taking the pull back of the maps η and p we get the following commutative exact diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tau_{\mathcal{T}}(M) & \longrightarrow & W & \longrightarrow & T' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \eta} & & \\
0 & \longrightarrow & \tau_{\mathcal{T}}(M) & \xrightarrow{j} & M & \xrightarrow{p} & M/\tau_{\mathcal{T}}(M) & \longrightarrow & 0
\end{array}$$

By Proposition 7, the top exact sequence splits, hence there exists a map $f : T' \to M$, such that $pf{=}\eta$. By the properties of the trace, $f(T') \subset \tau_{\mathcal{T}}M$ and $\eta = 0$. □

We have the following characterization of the torsion class:

Proposition 10. *Let $\mathcal{T}$ be a tilting category in $\mathrm{Mod}(\mathcal{C})$. Then:*

$$\mathscr{T} = \{M \in \mathrm{Mod}(C) | \mathrm{Ext}^1_{\mathcal{C}}(T, M) = 0,\ T \in \mathcal{T}\}.$$

Proof. Let's assume $M \in \mathrm{Mod}(\mathcal{C})$ and for each T in $\mathcal{T}$, $\mathrm{Ext}^1_{\mathcal{C}}(T, M) = 0$. Then applying the functor $\mathrm{Hom}_{\mathcal{C}}(T, -)$ to the sequence (2.2), it follows from the long homology sequence and the fact $\mathrm{pdim}T = 1$ that

$$0 = \mathrm{Ext}^1_{\mathcal{C}}(T, M) \to \mathrm{Ext}^1_{\mathcal{C}}(T, M/\tau_{\mathcal{T}}(M)) \to \mathrm{Ext}^2_{\mathcal{C}}(T, \tau_{\mathcal{T}}(M)) = 0$$

is exact, and in consequence, $\mathrm{Ext}^1_{\mathcal{C}}(T, M/\tau_{\mathcal{T}}(M)) = 0$.

Let $C \in \mathcal{C}$. Then we have an exact sequence $0 \to \mathcal{C}(\ \ , C) \to T_0 \to T_1 \to 0$, with $T_i \in T$, $i = 0, 1$. After applying $\mathrm{Hom}_{\mathcal{C}}(\ \ , M/\tau_{\mathcal{T}}(M))$ to the sequence, we get, by the long homology sequence, the exact sequence

$$0 = \mathrm{Hom}(T_0, M/\tau_{\mathcal{T}}(M)) \to \mathrm{Hom}((\ \ , C), M/\tau_{\mathcal{T}}(M)) \to \mathrm{Ext}^1(T_1, M/\tau_{\mathcal{T}}(M)) = 0.$$

By Yoneda's Lemma, $0 = \mathrm{Hom}((\ \ , C), M/\tau_{\mathcal{T}}(M)) = M(C)/\tau_{\mathcal{T}}M(C)$, this is: $M = \tau_{\mathcal{T}}(M)$ is in $\mathscr{T}$. The other inclusion is already proved in Proposition 7. □

The torsion class $\mathscr{T}$ induces a torsion pair $(\mathscr{T}, \mathscr{F})$, where

$$\mathscr{F} = \{N \in \mathrm{Mod}(\mathcal{C}) | \mathrm{Hom}(M, N) = 0,\ M \in \mathscr{T}\},$$

[see S].

Proposition 11. $\mathscr{F} = \{N \in \mathrm{Mod}(\mathcal{C})\} | \mathrm{Hom}(T, N) = 0,\ T \in \mathcal{T}\}$.

Proof. Let M be an object in $\mathscr{T}$. Then there is an epimorphism $\coprod_{j \in J} T_j \to M \to 0$, with $\{T_j\}_{i \in J}$ a family of objects in $\mathcal{T}$, and let $N \in \{N \in \mathrm{Mod}(\mathcal{C}) | \mathrm{Hom}(T, N) = 0,\ T \in \mathcal{T}\}$. Then there is a monomorphism

$$0 \to \mathrm{Hom}_{\mathcal{C}}(M, N) \to \mathrm{Hom}_{\mathcal{C}}(\coprod_{j \in J} T_j, N) \cong \prod_{j \in J} \mathrm{Hom}_{\mathcal{C}}(T_j, N) = 0,$$

and M is in $\mathscr{F}$. The other inclusion is clear. □

Let $\mathcal{T}$ be a tilting subcategory of $\mathrm{Mod}(\mathcal{C})$, we define $\mathrm{Add}\mathcal{T}$ as the full subcategory of $\mathrm{Mod}(\mathcal{C})$ whose objects are direct summands of arbitrary sums of objects in $\mathcal{T}$. Let $\phi : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Mod}(\mathcal{T})$ be the functor defined as $\phi(M) = \mathrm{Hom}(\ \ , M)_{\mathcal{T}}$. We adapt Bongartz 's proof of Brenner-Butler's theorem to our situation. We start with a version of [B Prop. 1.4]:

Proposition 12. (i) *If* $M_3 \xrightarrow{h} M_2 \xrightarrow{g} M_1 \xrightarrow{f} M_0$ *is an exact sequence in* $\mathrm{Mod}(\mathcal{C})$*, such that* M_i *belongs to* $\mathscr{T}$*, then the sequence*

$$\phi(M_2) \to \phi(M_1) \to \phi(M_0)$$

is exact.

(ii) *For each* $M \in \mathscr{T}$ *there is an exact sequence*

$$\cdots \xrightarrow{t_{n+1}} T^n \to \cdots \to T^1 \xrightarrow{t_1} T^0 \xrightarrow{t_0} M \to 0$$

with T^i *in* $\mathrm{Add}\mathcal{T}$*, such that the maps* $T^n \xrightarrow{\delta_n} \mathrm{Im}t_n \to 0$*, have the following property: given* T *in* $\mathcal{T}$ *and a map* $f : T \to \mathrm{Im}t_n$*, there exist a map* $h : T \to T^n$*, such that* $\delta_n h = f$*.*

(iii) *For each* M *in* $\mathscr{T}$ *there exists an isomorphism*

$$\phi(M) \otimes \mathcal{T} \to M.$$

(iv) $\mathrm{Tor}_1^{\mathcal{T}}(\phi(M), \mathcal{T}) = 0$.

(v) *For any pair of objects* M, N *in* $\mathscr{T}$ *we have an isomorphism* $\mathrm{Ext}^i_{\mathcal{C}}(M, N) = \mathrm{Ext}^i_{\mathcal{T}}(\phi(M), \phi(N))$*.*

Proof. (i) Since $\mathscr{T}$ is closed under epimorphic images, then $\mathrm{Im}(g)$, $\mathrm{Im}(f)$ and $\mathrm{Coker}(f)$ are in $\mathscr{T}$. From the exact sequences: $0 \to \mathrm{Im}h \to M_2 \to \mathrm{Im}g \to 0$, $0 \to \mathrm{Im}g \to M_1 \to \mathrm{Im}f \to 0$, $0 \to \mathrm{Im}f \to M_0 \to \mathrm{Coker}f \to 0$, we obtain exact sequences in $\mathrm{Mod}(\ \mathcal{T})$:

$$\begin{aligned} 0 &\to (\ , \mathrm{Im}h)_{\mathcal{T}} \to (\ , M_2)_{\mathcal{T}} \to (\ , \mathrm{Im}g)_{\mathcal{T}} \to \mathrm{Ext}^1(\ , \mathrm{Im}h)_{\mathcal{T}} = 0, \\ 0 &\to (\ , \mathrm{Im}g)_{\mathcal{T}} \to (\ , M_1)_{\mathcal{T}} \to (\ , \mathrm{Im}f)_{\mathcal{T}} \to \mathrm{Ext}^1(\ , \mathrm{Im}g)_{\mathcal{T}} = 0, \\ 0 &\to (\ , \mathrm{Im}f)_{\mathcal{T}} \to (\ , M_0)_{\mathcal{T}} \to (\ , \mathrm{Coker}f)_{\mathcal{T}} \to \mathrm{Ext}^1(\ , \mathrm{Im}f)_{\mathcal{T}} = 0. \end{aligned}$$

Gluing the sequences, the result follows.

(ii) By the Propositions 5 and 6, for each $M \in \mathscr{T}$ there is an exact sequence

$$0 \to K_0 \to T^{(X_0)} \xrightarrow{\eta_0} M \to 0 \tag{2.3}$$

such that; every map $\eta : T' \to M$ factors through η_0.

It follows that $\mathrm{Hom}_{\mathcal{C}}(T', T^{(X_0)}) \xrightarrow{(T', \eta_0)} \mathrm{Hom}_{\mathcal{C}}(T', M)$ is an epimorphism. Since $\mathrm{Ext}^1_{\mathcal{C}}(T', T^{(X_0)}) = 0$, after applying $\mathrm{Hom}_{\mathcal{C}}(T', \ \ )$ to (2.3), it follows by the long homology sequence, that $\mathrm{Ext}^1_{\mathcal{C}}(T', K_0) = 0$, and K_0 is in $\mathscr{T}$. The claim follows by induction.

(iii) By part (ii) there exists an exact sequence $\cdots \to T^1 \to T^0 \to M \to 0$. After applying ϕ we obtain an exact sequence $\cdots \to \phi(T^1) \to \phi(T^0) \to \phi(M) \to 0$. Since ϕ and $-\otimes\mathcal{T}$ preserve arbitrary sums, for $i \geq 0$, we have, $\phi(T^i)\otimes\mathcal{T} = (\ , T^i)_{\mathcal{T}}\otimes\mathcal{T} \cong T_i$. It follows, the existence of an isomorphism η, such that the following diagram commutes:

$$\begin{array}{ccccccc}
\phi(T^1)\otimes\mathcal{T} & \longrightarrow & \phi(T^0)\otimes\mathcal{T} & \longrightarrow & \phi(M)\otimes\mathcal{T} & \longrightarrow & 0\\
\cong\downarrow & & \cong\downarrow & & \eta\downarrow & & \\
T^1 & \longrightarrow & T^0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

(iv) It is also clear from the above diagram $\operatorname{Tor}_1^{\mathcal{T}}(\phi(M),\mathcal{T})=0$.

(v) Using the sequence constructed in (ii) and the relation $\operatorname{Ext}^j_{\mathcal{C}}(T^n,N)=0$ for all $j\geq 0$ and $n\geq 0$, we get by dimension-shift $\operatorname{Ext}^j_{\mathcal{C}}(M,N)\cong\operatorname{Ext}^1_{\mathcal{C}}(\operatorname{Im}(t_{j-1}),N)$, for $j\geq 1$. The exact sequences

$$T^{j+1}\to T^j\to\operatorname{Im}(t_j)\to 0,$$
$$0\to\operatorname{Im}(t_j)\to T^{j-1}\to\operatorname{Im}(t_{j-1})\to 0,$$

induce the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
\operatorname{Hom}_{\mathcal{C}}(T^{j-1},N) & \xrightarrow{\operatorname{Hom}_{\mathcal{C}}(q,N)} & \operatorname{Hom}_{\mathcal{C}}(\operatorname{Im}(t_j),N) & \longrightarrow & \operatorname{Ext}^1_{\mathcal{C}}(\operatorname{Im}(t_{j-1}),N) & \longrightarrow & 0\\
 & \searrow^{\operatorname{Hom}_{\mathcal{C}}(t_j,N)} & \downarrow{\scriptstyle \operatorname{Hom}_{\mathcal{C}}(p,N)} & & & & \\
 & & \operatorname{Hom}_{\mathcal{C}}(T^j,N) & & & & \\
 & & \downarrow{\scriptstyle \operatorname{Hom}_{\mathcal{C}}(t_{j+1},N)} & & & & \\
 & & \operatorname{Hom}_{\mathcal{C}}(T^{j+1},N) & & & &
\end{array}$$

Hence, there are isomorphisms:

$$\operatorname{Ext}^j_{\mathcal{C}}(M,N)\cong\operatorname{Ext}^1_{\mathcal{C}}(\operatorname{Im}(t_{j-1}),N)\cong H^j(\operatorname{Hom}_{\mathcal{C}}(T.,N)). \tag{2.4}$$

Applying ϕ to the exact sequence in (ii), we obtain the following projective resolution of $\phi(M)$

$$(\ ,T.)\to(\ ,M)_{\mathcal{T}}:\cdots\to(\ ,T^1)\to(\ ,T^0)\to(\ ,M)_{\mathcal{T}}\to 0.$$

By Yoneda's Lemma, $\operatorname{Hom}_{\mathcal{T}}((\ ,T.),(\ ,N)_{\mathcal{T}})\cong\operatorname{Hom}_{\mathcal{C}}(T.,N)$, therefore:

$$\operatorname{Ext}^j_{\mathcal{T}}(\phi(M),\phi(N))=H^j((\ ,T.),\phi(N))=H^j(\operatorname{Hom}_{\mathcal{C}}(T.,N)). \tag{2.5}$$

The claim follows from (2.5) and (2.4). □

We will see next how to compute $-\otimes\mathcal{T}$, and its left derived functors.

Proposition 13. *For each C in $\mathcal{C}$, and $M\in\operatorname{Mod}(\mathcal{T})$ there is a natural isomorphism in* $\operatorname{Mod}(\ \mathcal{C})$,

$$(M\otimes\mathcal{T})(C)\cong(\mathcal{C}(\ ,C),\ \ )_{\mathcal{T}}\otimes_{\mathcal{T}}M.$$

Proof. Let's consider the following presentation of M

$$\coprod_{i\in I}(\ ,T_i)\overset{((\ ,f_{ij}))}{\to}\coprod_{j\in J}(\ ,T_j)\to M\to 0. \tag{2.6}$$

Applying $\otimes\mathcal{T}$ and evaluating in C, we obtain the following short exact sequence,

$$\coprod_{i\in I}T_i(C)\xrightarrow{(f_{ij})_C}\coprod_{j\in J}T_j(C)\to M\otimes\mathcal{T}\ (C)\to 0. \tag{2.7}$$

By the properties of the tensor product, and Yoneda's Lemma, we have isomorphisms:

$$(\mathcal{C}(\ \ ,C),\ \ )\otimes_{\mathcal{T}}\coprod_{i\in I}(\ \ ,T_i)\cong\coprod_{i\in I}(\mathcal{C}(\ \ ,C),\ \ )\otimes_{\mathcal{T}}(\ \ ,T_i)\cong\coprod_{i\in I}(\mathcal{C}(\ \ ,C),T_i)\cong\coprod_{i\in I}T_i(C)$$

Since $(\mathcal{C}(\ \ ,C),\ \ )\otimes_{\mathcal{T}}$, is right exact, when we apply it to the presentation (2.6), and compare the exact sequence we get, with the exact sequence (2.7), we obtain the desired isomorphism. It is easy to verify that this isomorphism is natural. □

Proposition 14. *For each C in $\mathcal{C}$, and $M\in \mathrm{Mod}(\mathcal{T})$, and any non negative integer n, there exists an isomorphism*

$$\mathrm{Tor}_n^{\mathcal{T}}((\mathcal{C}(\ \ ,C),\ \ )_{\mathcal{T}},M)\cong \mathrm{Tor}_n^{\mathcal{T}}(M,\mathcal{T})(C), \tag{2.8}$$

which is natural in C and M.

Proof. Let's consider an exact sequence

$$0\to\Omega M\to(\ \ ,T)\to M\to 0, \tag{2.9}$$

where T is a sum of objects in $\mathcal{T}$. Applying $\otimes\mathcal{T}$ to (2.9) we get the exact sequence

$$0\to\mathrm{Tor}_1^{\mathcal{T}}(M,\mathcal{T})\to\Omega M\otimes\mathcal{T}\to(\ \ ,T)\otimes\mathcal{T},$$

and applying $(\mathcal{C}(\ \ ,C),\ \ )\otimes_{\mathcal{T}}-$ to (2.9), we get the exact sequence:

$$0\to\mathrm{Tor}_1^{\mathcal{T}}(((\ \ ,C),\ \ ),M)\to((\ \ ,C),\ \ )\otimes\Omega M\to((\ \ ,C),\ \ )\otimes(\ \ ,T).$$

By Proposition 13, there exists an isomorphism γ_C, such that the following diagram commutes

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Tor}_1^{\mathcal{T}}(((\ \ ,C),\ \ ),M) & \longrightarrow & ((\ \ ,C),\ \ )\otimes\Omega M & \longrightarrow & ((\ \ ,C),\ \ )\otimes(\ \ ,T) \\
 & \downarrow{\scriptstyle\gamma_C} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
0 \longrightarrow & \mathrm{Tor}_1^{\mathcal{T}}(M,\mathcal{T})(C) & \longrightarrow & \Omega M\otimes\mathcal{T}(C) & \longrightarrow & T(C)
\end{array}$$

We can easily verify that $\gamma=\{\gamma_C\}_{C\in\mathcal{C}}$ is a natural transformation.

We leave to the reader the proof of the naturality in M. □

Remark 2. *Let C an C' be two objects in $\mathcal{C}$, and $\mathcal{T}$ a tilting subcategory of $\mathrm{Mod}(\mathcal{C})$. Then we have two exact sequences: $0\to\mathcal{C}(\ \ ,C)\xrightarrow{g}T_0\to T_1\to 0$ and $0\to\mathcal{C}(\ \ ,C')\xrightarrow{g'}T_0'\to T_1'\to 0$, with T_i and T_i' in $\mathcal{T}$, for $i=0,1$. Let $f:C\to C'$ be a morphism. Then, taking the push out, we have the following commutative exact diagram:*

$$\begin{array}{ccccccccc}
0 \to & \mathcal{C}(\ \ ,C) & \xrightarrow{g} & T_0 & \longrightarrow & T_1 & \longrightarrow & 0 \\
 & \downarrow{\scriptstyle g'(\mathcal{C}(\ \ ,f))} & & \downarrow & & \| & & \\
0 \longrightarrow & T_0' & \longrightarrow & W & \longrightarrow & T_1 & \longrightarrow & 0
\end{array}$$

Since $\mathrm{Ext}^1_{\mathcal{C}}(T_1,T_0')=0$, the lower exact sequence in the diagram splits. Hence, there exist morphisms u,v, such that the following exact diagram commutes:

$$\begin{array}{ccccccccc}
0 \to & \mathcal{C}(\ \ ,C) & \xrightarrow{g} & T_0 & \longrightarrow & T_1 & \longrightarrow & 0 \\
 & {\scriptstyle\mathcal{C}(\ \ ,f)}\downarrow & & {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & \\
0 \to & \mathcal{C}(\ \ ,C') & \xrightarrow{g'} & T_0' & \longrightarrow & T_1' & \longrightarrow & 0
\end{array} \tag{2.10}$$

Proposition 15. *Let $\mathcal{T}$ be a tilting subcategory of* $\mathrm{Mod}(\mathcal{C})$. *Then for each M in* $\mathrm{Mod}(\mathcal{C})$, *there is an exact sequence*

$$0 \to \phi(M) \otimes \mathcal{T} \to M \to \mathrm{Tor}_1^{\mathcal{T}}(\mathrm{Ext}^1_{\mathcal{C}}(\ \ , M)_{\mathcal{T}}, \mathcal{T}) \to 0.$$

Moreover, $\mathrm{Ext}^1_{\mathcal{C}}(\ \ , M)_{\mathcal{T}} \otimes \mathcal{T} = 0$.

Proof. For each C in $\mathcal{C}$ we have an exact sequence

$$0 \to \mathcal{C}(\ , C) \to T_0 \to T_1 \to 0, \tag{2.11}$$

which induces the following exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(T_1, \ \ )_{\mathcal{T}} \to \mathrm{Hom}_{\mathcal{C}}(T_0, \ \ )_{\mathcal{T}} \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}(\ \ , C), \ \ )_{\mathcal{T}} \to 0. \tag{2.12}$$

After applying $- \otimes_{\mathcal{T}} \mathrm{Ext}^1_{\mathcal{C}}(\ \ , M)_{\mathcal{T}}$ to (2.12), we get the following exact sequence

$$\begin{array}{rcl} 0 & \to & \mathrm{Tor}_1^{\mathcal{T}}((\mathcal{C}(\ \ , C), \ \ )_{\mathcal{T}}, \mathrm{Ext}^1_{\mathcal{C}}(\ \ , M)_{\mathcal{T}}) \to \mathrm{Ext}^1_{\mathcal{C}}(T_1, M) \to \\ & \to & \mathrm{Ext}^1_{\mathcal{C}}(T_0, M) \to \mathrm{Hom}(\mathcal{C}(\ \ , C), \ \ )_{\mathcal{T}} \otimes_{\mathcal{T}} \mathrm{Ext}^1_{\mathcal{C}}(\ \ , M)_{\mathcal{T}} \to 0. \end{array} \tag{2.13}$$

Applying $\mathrm{Hom}_{\mathcal{C}}(\ \ , M)$ to (2.11), we get from the long homology sequence, the exact sequence

$$\begin{array}{rcl} 0 & \to & \mathrm{Hom}_{\mathcal{C}}(T_1, M) \to \mathrm{Hom}_{\mathcal{C}}(T_0, M) \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}(\ \ , C), M) \to \\ & \to & \mathrm{Ext}^1_{\mathcal{C}}(T_1, M) \to \mathrm{Ext}^1_{\mathcal{C}}(T_0, M) \to \mathrm{Ext}^1_{\mathcal{C}}(\mathcal{C}(\ \ , C), M) = 0. \end{array} \tag{2.14}$$

Then, for each C in $\mathcal{C}$ we have the following isomorphisms:

$$\begin{array}{rcl} (\mathrm{Ext}^1_{\mathcal{C}}(\ \ , M)_{\mathcal{T}} \otimes \mathcal{T})(C) & \cong & (\mathcal{C}(\ \ , C), \ \ )_{\mathcal{T}} \otimes_{\mathcal{T}} \mathrm{Ext}^1_{\mathcal{C}}(\ \ , M)_{\mathcal{T}}, \text{ by Proposition 13} \\ & \cong & \mathrm{Ext}^1_{\mathcal{C}}(\mathcal{C}(\ \ , C), M)_{\mathcal{T}} = 0, \text{ by (2.13) and (2.14)}, \end{array}$$

which proves the second part of the Proposition.

Now, by Proposition 14 we know that

$$\mathrm{Tor}_1^{\mathcal{T}}(\mathrm{Ext}^1_{\mathcal{C}}(\ \ , M)_{\mathcal{T}}, \mathcal{T})(C) \cong \mathrm{Tor}_1^{\mathcal{T}}((\mathcal{C}(\ \ , C), \ \ )_{\mathcal{T}}, \mathrm{Ext}^1_{\mathcal{C}}(\ \ , M)_{\mathcal{T}}). \tag{2.15}$$

It follows by 2.15, and the isomorphism $\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}(\ \ , C), M) \cong M(C)$, together with (2.13), and (2.14), that there exist a isomorphisms γ_C such that the following diagram commutes:

$$\begin{array}{ccccccc} & & M(C) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{C}}(T_1, M) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{C}}(T_0, M) \to 0 \\ & & {\scriptstyle \gamma_C}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow \\ 0 & \longrightarrow & \mathrm{Tor}_1^{\mathcal{T}}(\mathrm{Ext}^1_{\mathcal{C}}(\ \ , M)_{\mathcal{T}}, \mathcal{T})(C) & \to & \mathrm{Ext}^1_{\mathcal{C}}(T_1, M) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{C}}(T_0, M) \end{array}$$

Moreover, the Snake Lemma implies γ_C is epimorphisms. Using Remark 2.10, the reader can check that $\gamma = \{\gamma_C\}_{C \in \mathcal{C}}$ is a natural transformation.

After applying $- \otimes_{\mathcal{T}} \phi(M)$ to the exact sequence (2.12) we obtain the following exact sequence

$$\phi(M)(T_1) \to \phi(M)(T_0) \to \mathrm{Hom}(\mathcal{C}(\ \ , C), \ \ ) \otimes_{\mathcal{T}} \phi(M) \to 0.$$

By the isomorphism $\phi(M) \otimes \mathcal{T}(C) \cong \mathrm{Hom}(\mathcal{C}(\ \ , C), \ \ )_{\mathcal{T}} \otimes_{\mathcal{T}} \phi(M)$, and the exact sequence (2.14), there exist a morphisms η_C such that the following diagram commutes:

$$\begin{array}{ccccccccc} & & \phi(M)(T_1) & \longrightarrow & \phi(M)(T_0) & \longrightarrow & \phi(M) \otimes \mathcal{T}(C) & \longrightarrow & 0 \\ & & \cong\downarrow & & \cong\downarrow & & {\scriptstyle \eta_C}\downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}(T_1, M) & \longrightarrow & \mathrm{Hom}(T_0, M) & \to & \mathrm{Hom}(\mathcal{C}(\ C), M) & & \end{array}$$

Using the Snake Lemma again, the map η_C is monomorphism. The reader can check $\eta = \{\eta_C\}_{C\in\mathcal{C}}$ is natural transformation. In this way we have the following exact sequence

$$0 \to \phi(M) \otimes \mathcal{T} \xrightarrow{\eta} M \xrightarrow{\gamma} \mathrm{Tor}_1^{\mathcal{T}}(\mathrm{Ext}^1_{\mathcal{C}}(\ \ , M), \mathcal{T}) \to 0.$$

Proving the proposition. □

Proposition 16. *Let $\mathcal{T}$ be a tilting subcategory of* $\mathrm{Mod}(\mathcal{C})$*. Then for each N in* $\mathrm{Mod}(\mathcal{T})$ *there exists an exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathcal{C}}(-, \mathrm{Tor}_1^{\mathcal{T}}(N, \mathcal{T})) \to N \to \phi(N \otimes \mathcal{T}) \to 0.$$

In addition $\phi(\mathrm{Tor}_1^{\mathcal{T}}(N, \mathcal{T})) = 0$*.*

Proof. We choose the projective resolutions

$$(2.16) \quad L_{.} \to N \quad : \quad \cdots \to (-, T_2) \xrightarrow{(-,h_2)} (-, T_1) \xrightarrow{(-,h_1)} (-, T_0) \to N \to 0$$

$$(2.17) \quad 0 \to P_1 \to P_0 \to T \to 0$$

such that for $i \geq 0$, T_i is a sum of objects in $\mathcal{T}$.

Applying $-\otimes\mathcal{T}$ to the complex $L.$, we obtain the complex $L.\otimes\mathcal{T}$, whose objects are in $\mathrm{Add}\mathcal{T}$, and we have $\mathrm{Ext}^1_{\mathcal{C}}(T, L_{.} \otimes \mathcal{T}) = 0$. In this way, we obtain the following exact sequence of complexes:

$$(2.18) \quad 0 \to \mathrm{Hom}_{\mathcal{C}}(T, L_{.} \otimes \mathcal{T}) \to \mathrm{Hom}_{\mathcal{C}}(P_0, L_{.} \otimes \mathcal{T}) \to \mathrm{Hom}_{\mathcal{C}}(P_1, L_{.} \otimes \mathcal{T}) \to 0.$$

Observe that $L_{.}(T)$ and $\mathrm{Hom}_{\mathcal{C}}(T, L_{.} \otimes \mathcal{T})$ are isomorphic, hence the exact sequence (2.18) becomes

$$0 \to L_{.}(T) \to \mathrm{Hom}_{\mathcal{C}}(P_0, L_{.} \otimes \mathcal{T}) \to \mathrm{Hom}_{\mathcal{C}}(P_1, L_{.} \otimes \mathcal{T}) \to 0.$$

By the above sequence, and the long homology sequence, we get an exact sequence:

$$(2.19) \quad 0 = H_1(L_{.}(T)) \to H_1(\mathrm{Hom}_{\mathcal{C}}(P_0, L_{.} \otimes \mathcal{T})) \to H_1(\mathrm{Hom}_{\mathcal{C}}(P_1, L_{.} \otimes \mathcal{T})) \to$$
$$\to H_0(\mathrm{Hom}_{\mathcal{C}}(L_{.}(T))) \to H_0(\mathrm{Hom}_{\mathcal{C}}(P_0, L_{.} \otimes \mathcal{T})) \to H_0(\mathrm{Hom}_{\mathcal{C}}(P_1, L_{.} \otimes \mathcal{T})) \to 0.$$

Since P_i, $i = 0, 1$ are projective, there exists an isomorphism

$$H_1(\mathrm{Hom}_{\mathcal{C}}(P_i, L_{.} \otimes \mathcal{T})) \cong \mathrm{Hom}_{\mathcal{C}}(P_i, H_1(L_{.} \otimes \mathcal{T})) = \mathrm{Hom}_{\mathcal{C}}(P_i, \mathrm{Tor}_1^{\mathcal{T}}(N, \mathcal{T})).$$

Finally, the exact sequence (2.19) can be written as:

$$(2.20) \quad \begin{aligned} 0 &\to \mathrm{Hom}_{\mathcal{C}}(P_0, \mathrm{Tor}_1^{\mathcal{T}}(N, \mathcal{T})) \to \mathrm{Hom}_{\mathcal{C}}(P_1, \mathrm{Tor}_1^{\mathcal{T}}(N, \mathcal{T})) \to \\ N(T) &\to \mathrm{Hom}_{\mathcal{C}}(P_0, N \otimes \mathcal{T}) \to \mathrm{Hom}_{\mathcal{C}}(P_1, N \otimes \mathcal{T}) \to 0. \end{aligned}$$

After applying $\mathrm{Hom}_{\mathcal{C}}(\ \ , \mathrm{Tor}_1^{\mathcal{T}}(N, \mathcal{T}))$ to the projective resolution of T, we obtain the following exact sequence:

$$(2.21) \quad \begin{aligned} 0 &\to \mathrm{Hom}_{\mathcal{C}}(T, \mathrm{Tor}_1^{\mathcal{T}}(N, \mathcal{T})) \to \mathrm{Hom}_{\mathcal{C}}(P_0, \mathrm{Tor}_1^{\mathcal{T}}(N, \mathcal{T})) \to \\ &\to \mathrm{Hom}_{\mathcal{C}}(P_1, \mathrm{Tor}_1^{\mathcal{T}}(N, \mathcal{T})) \to \mathrm{Ext}^1_{\mathcal{C}}(T, \mathrm{Tor}_1^{\mathcal{T}}(N, \mathcal{T})) \to 0. \end{aligned}$$

Comparing (2.20) and (2.21), we get $\phi(\mathrm{Tor}_1^{\mathcal{T}}(N, \mathcal{T}))(T) = 0$, which proves part of the Proposition. In addition (2.20) and (2.21), imply the existence of a morphism η_T, such that the following diagram commutes:

$$\begin{array}{ccccccc}
 & (P_0, \mathrm{Tor}_1^{\mathcal{T}}(N,\mathcal{T})) & \longrightarrow & (P_1, \mathrm{Tor}_1^{\mathcal{T}}(N,\mathcal{T})) & \to & \mathrm{Ext}^1_{\mathcal{C}}(T, \mathrm{Tor}_1^{\mathcal{T}}(N,\mathcal{T}) & \to 0 \\
 & \cong\downarrow & & \cong\downarrow & & \eta_T\downarrow & \\
0 \to & (P_0, \mathrm{Tor}_1^{\mathcal{T}}(N,\mathcal{T})) & \to & (P_1, \mathrm{Tor}_1^{\mathcal{T}}(N,\mathcal{T})) & \longrightarrow & N(T) &
\end{array}$$

By the Snake Lemma, η_T is mono.

After applying $\mathrm{Hom}_{\mathcal{C}}(\ \ , N\otimes\mathcal{T})$ to the projective resolution of T, we obtain the following exact sequence:

$$\begin{array}{rl}
0 \ \to & \mathrm{Hom}_{\mathcal{C}}(T, N\otimes\mathcal{T}) \to \mathrm{Hom}_{\mathcal{C}}(P_0, N\otimes\mathcal{T}) \to \\
\to & \mathrm{Hom}_{\mathcal{C}}(P_1, N\otimes\mathcal{T}) \to \mathrm{Ext}_{\mathcal{C}}(T, N\otimes\mathcal{T}) \to 0.
\end{array} \tag{2.22}$$

The sequences (2.20), and (2.22), imply the existence of a morphism $\gamma_T : N(T) \to \mathrm{Hom}_{\mathcal{C}}(T, N\otimes\mathcal{T}) = \phi(N\otimes\mathcal{T})$. It follows by the Snake Lemma that γ_T is epimorphism. Moreover,

$$\mathrm{Ext}^1_{\mathcal{C}}(\ \ , N\otimes\mathcal{T}) = 0. \tag{2.23}$$

It's not hard to check that $\eta = \{\eta_T\}_{T\in\mathcal{T}}$ and $\gamma = \{\gamma_T\}_{T\in\mathcal{T}}$ are natural transformations. We leave the details to the reader. In this way the following sequence of functors

$$0 \to \mathrm{Ext}^1_{\mathcal{C}}(-, \mathrm{Tor}_1^{\mathcal{T}}(N,\mathcal{T})) \xrightarrow{\eta} N \xrightarrow{\gamma} \phi(N\otimes\mathcal{T}) \to 0$$

is exact, proving the Proposition. □

We call $F, F' : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Mod}(\mathcal{T})$ to the functors defined by:

$$\begin{array}{rcl}
F(M) & = & \phi(M), \\
F'(M) & = & \mathrm{Ext}^1_{\mathcal{C}}(\ \ , M)_{\mathcal{T}},
\end{array}$$

and, $G, G' : \mathrm{Mod}(\mathcal{T}) \to \mathrm{Mod}(\mathcal{C})$ to the functors:

$$\begin{array}{rcl}
G(N) & = & N\otimes\mathcal{T}, \\
G'(N) & = & \mathrm{Tor}_1^{\mathcal{T}}(N,\mathcal{T}).
\end{array}$$

Let $\mathcal{T}$ be a tilting subcategory of $\mathrm{Mod}(\mathcal{C})$, and $(\mathscr{T}, \mathscr{F})$ the torsion theory considered above. We look to the full subcategories $\mathscr{X}$ and $\mathscr{Y}$ of $\mathrm{Mod}(\mathcal{T})$, defined by:

$$\begin{array}{rcl}
\mathscr{X} & = & \{N \in \mathrm{Mod}(\mathcal{T}) | N\otimes\mathcal{T} = 0\}, \\
\mathscr{Y} & = & \{N \in \mathrm{Mod}(\mathcal{T}) | \mathrm{Tor}_1^{\mathcal{T}}(N,\mathcal{T}) = 0\}.
\end{array}$$

From the previous results, the main theorem of the section follows:

Theorem 3 (Brenner-Butler)**.** *With the above notation, the following statements are true:*

(i) *F and G induce an equivalence between $\mathscr{T}$ and $\mathscr{Y}$.*
(ii) *F' and G' induce an equivalence between $\mathscr{F}$ and $\mathscr{X}$.*
(iii) *The following equations $FG' = F'G = 0$ and $G'F = GF' = 0$ hold..*

Corollary 3. *The pair of subactegories $(\mathscr{X}, \mathscr{Y})$ form a torsion theory in* $\mathrm{Mod}(\mathcal{T})$.

Proof. It is easy to check that for pair of objects X in $\mathscr{X}$, and Y in $\mathscr{Y}$, $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$.

For every N in $\mathrm{Mod}(\mathcal{T})$ there is an exact sequence:

$$0 \to \mathrm{Ext}^1_{\mathcal{C}}(-, \mathrm{Tor}_1^{\mathcal{T}}(N, \mathcal{T})) \xrightarrow{\eta} N \xrightarrow{\gamma} \phi(N \otimes \mathcal{T}) \to 0.$$

By condition (iii) of the Theorem, $\mathrm{Ext}^1_{\mathcal{C}}(-, \mathrm{Tor}_1^{\mathcal{T}}(N, \mathcal{T}))$ is in $\mathscr{X}$ and $\phi(N \otimes \mathcal{T})$ is in $\mathscr{Y}$, which implies the pair $(\mathscr{X}, \mathscr{Y})$ is a torsion theory. □

We want next to generalize the following result on tilting of finite dimensional algebras:

Let A be a finite dimensional K-algebra and ${}_A\mathrm{mod}$ the category of finitely generated left A-modules. Let's suppose that ${}_AT$ is a tilting A-module and $B = \mathrm{End}_A(T)$. Tilting theorem [B] proves T_B is a right tilting B-module and A^{op} is isomorphic to $\mathrm{End}_B(T_B)$.

Proposition 17. *Let $\mathcal{T}$ be a tilting subcategory of* $\mathrm{Mod}(\mathcal{C})$*. Let's assume each T in $\mathcal{T}$ has a projective resolution of finitely generated projectives,*

$$0 \to P_1 \to P_0 \to T \to 0.$$

Then the following statements hold:

(a) *The full subcategory θ of $\mathrm{Mod}(\mathcal{T}^{op})$, with objects $\{(\mathcal{C}(\ ,C),\ )_{\mathcal{T}}\}_{C\in\mathcal{C}}$ is a tilting subcategory in $\mathrm{Mod}(\mathcal{T}^{op})$.*

(b) *The category $\theta = \{(\mathcal{C}(\ ,C),\ )_{\mathcal{T}}\}_{C\in\mathcal{C}}$ is equivalent to $\mathcal{C}^{op}$.*

Proof. (a)(i) For each object C in $\mathcal{C}$ there is a resolution

$$0 \to \mathcal{C}(\ ,C) \to T_0 \xrightarrow{f} T_1 \to 0, \tag{2.24}$$

with $T_i \in \mathcal{T}$, for $i = 0, 1$. By the long homology sequence, there is an exact sequence of objects in $\mathrm{Mod}(\mathcal{C})$,

$$0 \to (T_1,\ )_{\mathcal{T}} \xrightarrow{(f,\ )} (T_0,\ )_{\mathcal{T}} \to (\mathcal{C}(\ ,C),\ )_{\mathcal{T}} \to \mathrm{Ext}(T_1,\ )_{\mathcal{T}} = 0, \tag{2.25}$$

this is; $\mathrm{pdim}(\mathcal{C}(\ ,C),\ )_{\mathcal{T}} \leq 1$.

(ii) Let C' be another object in $\mathcal{C}$. After applying the functor $\mathrm{Hom}_{\mathcal{T}^{op}}(\ ,(\mathcal{C}(\ ,C'),\ )_{\mathcal{T}})$ to (2.25), we get an exact sequence:

$$\begin{aligned} (2.26)\qquad & 0 \to ((\mathcal{C}(\ ,C),\ )_{\mathcal{T}}, (\mathcal{C}(\ ,C'),\ )_{\mathcal{T}}) \to ((T_0,\ ), (\mathcal{C}(\ ,C'),\ )_{\mathcal{T}}) \to \\ & \to ((T_1,\ ), (\mathcal{C}(\ ,C'),\ )_{\mathcal{T}}) \to \mathrm{Ext}^1_{\mathcal{T}^{op}}((\mathcal{C}(\ ,C),\ )_{\mathcal{T}}, (\mathcal{C}(\ ,C'),\ )_{\mathcal{T}}) \to 0. \end{aligned}$$

By Yoneda's Lemma the following diagram:

$$\begin{array}{ccc} ((T_0,\ ), (\mathcal{C}(\ ,C'),\ )_{\mathcal{T}}) & \longrightarrow & ((T_1,\ ), (\mathcal{C}(\ ,C'),\ )_{\mathcal{T}}) \\ \cong \downarrow & & \cong \downarrow \\ (\mathcal{C}(\ ,C'), T_0) & \xrightarrow{(\mathcal{C}(\ ,C'), f)} & (\mathcal{C}(\ ,C'), T_1) \\ \cong \downarrow & & \cong \downarrow \\ T_0(C') & \xrightarrow{f_{C'}} & T_1(C') \end{array}$$

commutes.

Evaluating (2.24) in C', and using Yoneda's Lemma, together with (2.26), there exist the following isomorphisms of abelian groups:

$$\begin{aligned} (2.27)\qquad ((\mathcal{C}(\ ,C),\ )_{\mathcal{T}}, (\mathcal{C}(\ ,C'),\ )_{\mathcal{T}}) &\cong \mathcal{C}(C', C) \\ \mathrm{Ext}^1_{\mathcal{T}^{op}}((\mathcal{C}(\ ,C),\ )_{\mathcal{T}}, (\mathcal{C}(\ ,C'),\ )_{\mathcal{T}}) &\cong 0. \end{aligned}$$

(iii) The sequence

$$0 \to (T,\ \ )_{\mathcal{T}} \to (P_0,\ \ )_{\mathcal{T}} \to (P_1,\ \ )_{\mathcal{T}} \to \mathrm{Ext}^1_{\mathcal{T}^{op}}(T,\ \ )_{\mathcal{T}} = 0$$

is exact, and each $(P_i,\ \ )_{\mathcal{T}}$ is in $\mathrm{add}\{(\mathcal{C}(\ \ ,C),\ \ )_{\mathcal{T}}\}_{C\in\mathcal{C}}$.

(b) We define the functor:

$$\alpha : \mathcal{C}^{op} \to \theta,\ \alpha(C) = (\mathcal{C}(\ \ ,C),\ \ )_{\mathcal{T}}$$

Which by (2.27), is full and faithful, giving the desired equivalence. □

2.2. **The Grothendieck Groups $K_0(\mathcal{C})$ and $K_0(\mathcal{T})$.** Given a ring A and a tilting A-module T it is a classical theorem [B] that the Grotendieck groups of A and $B = \mathrm{End}_A(T)^{op}$ are isomorphic. In this subsection we will prove that there is also an isomorphism between the Grotendieck group, $K_0(\mathcal{C})$, of an arbitrary skeletally small pre additive category $\mathcal{C}$, and the Grotendieck group, $K_0(\mathcal{T})$, of a tilting subcategory $\mathcal{T}$ of $\mathrm{Mod}(\mathcal{C})$. The proof will follow closely [CF].

Definition 10. *Let $\mathcal{C}$ a skeletally small pre additive category $\mathcal{C}$ and let $\mathcal{T}$ be a tilting subcategory of $\mathrm{Mod}(\mathcal{C})$. Let's denote by $|\mathrm{Mod}(\mathcal{C})|$ the set of isomorphism classes of objects in $\mathrm{Mod}(\mathcal{C})$. Let $\mathcal{A}$ be the free abelian group generated by $|\mathrm{Mod}(\mathcal{C})|$ and $\mathcal{R}$ the subgroup of $\mathcal{A}$ generated by relations $M - K - L$ such that $0 \to K \to M \to L \to 0$ is a short exact sequence in $\mathrm{Mod}(\mathcal{C})$. Then, the Groethendiek group of $\mathcal{C}$ is $K_0(\mathcal{C}) = \mathcal{A}/\mathcal{R}$.*

Proposition 18. *The Groethendieck groups $K_0(\mathcal{C})$ and $K_0(\mathcal{T})$ are isomorphic.*

Proof. We define the group homomorphism $\hat{\phi} : \mathcal{A} \to K_0(\mathcal{T})$, sending $\hat{\phi}(M) = |F(M)| - |F'(M)|$, where F and $F'$are the functors given in Brenner-Buttler's theorem. We claim $\mathcal{R}$ is contained in the kernel of $\hat{\phi}$. In fact, let $M - K - L$ be a generator of $\mathcal{R}$. Then there exists an exact sequence in $\mathrm{Mod}(\mathcal{T})$,

$$\begin{aligned} 0 &\to (\ \ ,K)_{\mathcal{T}} \to (\ \ ,M)_{\mathcal{T}} \to (\ \ ,L)_{\mathcal{T}} \to \mathrm{Ext}^1_{\mathcal{C}}(\ \ ,K)_{\mathcal{T}} \to \\ &\to \mathrm{Ext}^1_{\mathcal{C}}(\ \ ,M)_{\mathcal{T}} \to \mathrm{Ext}^1_{\mathcal{C}}(\ \ ,L)_{\mathcal{T}} \to \mathrm{Ext}^2_{\mathcal{C}}(\ \ ,K)_{\mathcal{T}} = 0, \end{aligned} \tag{2.28}$$

hence, the alternating sum,

$$-(|F(K)| - |F'(K)|) - (|F(L)| - |F'(L)|) + (|F(M)| + |F'(M)|) = 0.$$

Which means $\hat{\phi}(M-K-L) = 0$. Then, there is a unique map $\phi : K_0(\mathcal{C}) \to K_0(\mathcal{T})$, given by $\phi(|M|) = |F(M)| - |F'(M)|$.

For each object C in $\mathcal{C}$, there is a short exact sequence $0 \to \mathcal{C}(\ \ ,C) \to T^0 \to T^1 \to 0$ with T^i in $\mathcal{T}$, for $i = 0,1$, which induces the exact sequence in $\mathrm{Mod}(\mathcal{T}^{op})$:

$$0 \to (T_1,\ \ )_{\mathcal{T}} \to (T_0,\ \ )_{\mathcal{T}} \to (\mathcal{C}(\ \ ,C),\ \ )_{\mathcal{T}} \to \mathrm{Ext}^1_{\mathcal{C}}(T_1,\ \ ) = 0.$$

Therefore: for $n > 1$, and any object N in $\mathrm{Mod}(\mathcal{T})$, $\mathrm{pdim}(\mathcal{C}(\ \ ,C),\ \ )_{\mathcal{T}} \leq 1$ implies $\mathrm{Tor}^{\mathcal{T}}_n((\mathcal{C}(\ \ ,C),\ \ )_{\mathcal{T}}, N) = 0$. For an exact sequence $0 \to K \to N \to L \to 0$ in $\mathrm{Mod}(\mathcal{T})$, and any object C in $\mathcal{C}$, there is an exact sequence:

$$\begin{aligned} 0 &\to \mathrm{Tor}^{\mathcal{T}}_1(\mathcal{C}(\ \ ,C),\ \ ),K) \to \mathrm{Tor}^{\mathcal{T}}_1(\mathcal{C}(\ \ ,C),\ \ ),N) \to \mathrm{Tor}^{\mathcal{T}}_1(\mathcal{C}(\ \ ,C),\ \ ),L) \to \\ &\to (\mathcal{C}(\ \ ,C),\ \ ) \otimes K \to (\mathcal{C}(\ \ ,C),\ \ ) \otimes N \to (\mathcal{C}(\ \ ,C),\ \ ) \otimes L \to 0. \end{aligned}$$

Which can be re written as follows:

$$\begin{aligned} 0 &\to \mathrm{Tor}^{\mathcal{T}}_1(K,\mathcal{T}) \to \mathrm{Tor}^{\mathcal{T}}_1(N,\mathcal{T}) \to \mathrm{Tor}^{\mathcal{T}}_1(L,\mathcal{T}) \to \\ &\to K \otimes \mathcal{T} \to N \otimes \mathcal{T} \to L \otimes \mathcal{T} \to 0 \end{aligned} \tag{2.29}$$

In an analogous way, there is a group homomorphism $\psi : K_0(\mathcal{T}) \to K_0(\mathcal{C})$, given by $\psi(|N|) = |G(N)| - |G'(N)|$.

By Brenner-Butler's theorem, there are isomorphisms:

$$\begin{aligned}
\psi\phi(|M|) &= \psi(|F(M)| - |F'(M)|) \\
&= \psi(|F(M)|) - \psi(|F'(M)|) \\
&= (|GF(M)| - |G'F(M)|) - (|GF'(M)| - |G'F'(M)|) \\
&= |GF(M)| - |G'F'(M)|,
\end{aligned}$$

and from the exact sequence

$$0 \to GF(M) \to M \to G'F'(M) \to 0$$

it follows: $|M| = |GF(M)| - |G'F'(M)|$, this is: $\psi\phi = 1_{K_0(\mathcal{C})}$. With a similar argument we prove $\phi\psi = 1_{K_0(\mathcal{T})}$. □

2.3. **Global Dimension and Tilting.** In this subsection we will compare the global dimensions of a category $\mathcal{C}$ and its tilting category $\mathcal{T}$, obtaining results similar to the ring situation. The proof given here uses the same line of arguments as in [ASS].

Let $\mathcal{C}$ be a skeletally small pre additive category and $\mathcal{T}$ a tilting subcategory of $\mathrm{Mod}(\mathcal{C})$. Let $\mathscr{T}$ be the torsion class of $\mathrm{Mod}(\mathcal{C})$, whose objects are epimorphic images of arbitrary sums of objects in $\mathcal{T}$. We proved $\mathscr{T} = \{M \in \mathrm{Mod}(\mathcal{C}) | \mathrm{Ext}^1(\mathcal{T}, M) = 0\}$. We use this fact in the following:

Lemma 3. *Let M be an object in $\mathscr{T}$ and assume $\mathrm{Ext}^1_{\mathcal{C}}(M, \ )_{\mathscr{T}} = 0$. Then M is in $\mathrm{Add}\mathcal{T}$.*

Proof. Let M be an object in $\mathscr{T}$. By Proposition 12 there is a short exact sequence

$$0 \to \mathrm{Ker}(\alpha) \to \coprod_{i\in I} T_i \xrightarrow{\alpha} M \to 0,$$

with $\mathrm{Ker}(\alpha)$ in $\mathscr{T}$. Therefore: the sequence splits and M is in $\mathrm{Add}\mathcal{T}$. □

Proposition 19. *If M is in $\mathscr{T}$, then $\mathrm{pdim}\mathrm{Hom}_{\mathcal{C}}(\ , M)_{\mathcal{T}} \leq \mathrm{pdim}M$.*

Proof. By induction in $\mathrm{pdim}M$. If $\mathrm{pdim}M = 0$, then M is projective, and since M is in $\mathscr{T}$ there is an epimorphism $f : \coprod_{i\in I} T_i \to M \to 0$, with T_i in $\mathcal{T}$, which splits and M is a summand of $\coprod_{i\in I} T_i$. It follows M is in $\mathrm{Add}\mathcal{T}$, and $\mathrm{Hom}_{\mathcal{C}}(\ , M)_{\mathcal{T}}$ is summand of $\coprod_{i\in I}(\ , T_i)$, this is, $\mathrm{Hom}_{\mathcal{C}}(\ , M)_{\mathcal{T}}$ is projective and $\mathrm{pdim}\mathrm{Hom}_{\mathcal{C}}(\ , M) = 0$.

Let's assume $\mathrm{pdim}M = 1$. There is an exact sequence

$$0 \to L \to T_0 \to M \to 0, \tag{2.30}$$

with T_0 in $\mathrm{Add}\mathcal{T}$ and L in $\mathscr{T}$. The sequence (2.30) induces an exact sequence:

$$0 \to (\ , L)_{\mathcal{T}} \to (\ , T_0)_{\mathcal{T}} \to (\ , M)_{\mathcal{T}} \to \mathrm{Ext}^1_{\mathcal{C}}(\ , L)_{\mathcal{T}} = 0.$$

Since $\mathrm{pd}M = 1$, then $\mathrm{Ext}^2_{\mathcal{C}}(M, \ )_{\mathscr{T}} = 0$, and from (2.30) and the long homology sequence, it follows the sequence

$$0 = \mathrm{Ext}^1_{\mathcal{C}}(T_0, \ )_{\mathscr{T}} \to \mathrm{Ext}^1_{\mathcal{C}}(L, \ )_{\mathscr{T}} \to \mathrm{Ext}^2_{\mathcal{C}}(M, \ )_{\mathscr{T}} = 0$$

is exact. In consequence, $\mathrm{Ext}^1_{\mathcal{C}}(L, \ )_{\mathscr{T}} = 0$, and by Lemma 3, L is in $\mathrm{Add}\mathcal{T}$. Therefore $\mathrm{Hom}_{\mathcal{C}}(\ , L)_{\mathcal{T}}$ is projective and $\mathrm{Hom}_{\mathcal{C}}(\ , M)_{\mathcal{T}}$, has projective dimension less or equal to one.

Suppose $n \geq 2$ and the claim is true for all objects in $\mathscr{T}$ with projective dimension less than n. Let M be an object in $\mathscr{T}$ with pdim$M = n$. Then from (2.30) and the long homology sequence we get an exact sequence

$$0 = \mathrm{Ext}^n_{\mathcal{C}}(T_0, \ \ ) \to \mathrm{Ext}^n_{\mathcal{C}}(L, \ \ ) \to \mathrm{Ext}^{n+1}_{\mathcal{C}}(M, \ \ ) = 0.$$

Then pdim$L \leq (n-1)$. By induction hypothesis pdHom$_C(\ \ , L)_{\mathcal{T}} \leq n-1$ and Hom$_{\mathcal{C}}(, \ T_0)_{\mathcal{T}}$ is projective. Applying the contravariant functor Hom$_{\mathcal{T}}(-, \ )$ to the exact sequence (2.30), we obtain by the long homology sequence the inequalities:

$$\mathrm{pdimHom}_{\mathcal{C}}(\ \ , M)_{\mathcal{T}} \leq \mathrm{pdimHom}_{\mathcal{C}}(\ \ , L)_{\mathcal{T}} + 1 \leq 1 + (n-1)$$

□

Theorem 4. *With the same assumptions as above, we have the inequality:*

$$\mathrm{gdim}(\mathcal{T}) \leq 1 + \mathrm{gdim}(\mathcal{C}).$$

Proof. Let X be an object in Mod$(\mathcal{T})$, and cover it with a projective object, to get an exact sequence

$$0 \to Y \to \coprod_{i\in I}(\ \ , T_i) \to X \to 0. \tag{2.31}$$

The functor $\phi : \mathscr{T} \to \mathscr{Y}$ is an equivalence, and $\coprod_{i\in I}(\ \ , T_i)$ is in $\mathscr{Y}$, and $\mathscr{Y}$ is closed under sub-objects, hence Y is in $\mathscr{Y}$. Since ϕ is dense, there exists an object M in $\mathscr{T}$, such that $\phi(M) = \mathrm{Hom}_{\mathcal{C}}(\ \ , M)_{\mathcal{T}} \cong Y$, and pdim$Y \leq$ pdim$M$. From the exact sequence (2.31) we have the following inequalities:

$$\mathrm{pdim}X \leq 1 + \mathrm{pdim}Y \leq 1 + \mathrm{pdim}M \leq 1 + \mathrm{gdim}(\mathcal{C})$$

and $\mathrm{gdim}(\mathcal{T}) \leq 1 + \mathrm{gdim}(\mathcal{C})$. □

2.4. **Brenner-Butler's theorem for categories of finitely presented functors.** In this subsection we will prove that, under mild assumptions on the categories $\mathcal{C}$ and $\mathcal{T}$, Brenner-Butler's theorem holds in the categories of finitely presented functors. To prove it we need to see under which conditions the functor $\phi : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Mod}(\mathcal{T})$ restricts to the categories of finitely presented functors, $\phi : \mathrm{mod}(\mathcal{C}) \to \mathrm{mod}(\mathcal{T})$.

It was recalled in Section 1, that the category of finitely presented functors mod$(\mathcal{C})$ is abelian, if and only if, $\mathcal{C}$ has pseudokerneles [AR]. Hence; it is natural to assume $\mathcal{C}$ and $\mathcal{T}$ have pseudokerneles. Under these conditions we have the following.

Proposition 20. *Let's assume $\mathcal{C}$ and $\mathcal{T}$ have pseudokerneles. Then the functor*

$$\phi|\mathrm{mod}(\mathcal{C}) : \mathrm{mod}(\mathcal{C}) \to \mathrm{Mod}(\mathcal{T}),$$

has image in mod$(\mathcal{T})$.

Proof. (a) For each object $\mathcal{C}$, the functors $(\ \ , \mathcal{C}(\ \ , C))_{\mathcal{T}}$ and $\mathrm{Ext}^1_{\mathcal{C}}(\ \ , \mathcal{C}(\ \ , C))_{\mathcal{T}}$ are in mod$(\mathcal{T})$.

To see this, consider the exact sequence

$$0 \to \mathcal{C}(\ \ , C) \to T_0 \to T_1 \to 0,$$

with T_0, T_1 in $\mathcal{T}$. From the above exact sequence, and the long homology sequence, we get an exact sequence:

$$\begin{array}{rl} 0 & \to \ (\ \ , \mathcal{C}(\ \ , C))_{\mathcal{T}} \to (\ \ , T_0)_{\mathcal{T}} \to (\ \ , T_1)_{\mathcal{T}} \to \\ & \to \ \mathrm{Ext}^1_{\mathcal{C}}(\ \ , \mathcal{C}(\ \ , C))_{\mathcal{T}} \to \mathrm{Ext}^1_{\mathcal{C}}(\ \ , T_0)_{\mathcal{T}} = 0. \end{array}$$

The claim follows from the fact $\mathrm{mod}(\mathcal{T})$ is abelian.

(b) Let M be in $\mathrm{mod}(\mathcal{C})$. Since $\mathcal{C}$ has pseudokernels, then M has a projective resolution

$$\cdots \to \mathcal{C}(\ ,C_3) \xrightarrow{(\ ,f_2)} \mathcal{C}(\ ,C_2) \xrightarrow{(\ ,f_1)} \mathcal{C}(\ ,C_1) \xrightarrow{(\ ,f_0)} \mathcal{C}(\ ,C_0) \to M \to 0$$

Let K_i be $\mathrm{Im}(\ ,f_i)$. Then for all $i \geq 0$, the functors $\mathrm{Ext}^1_{\mathcal{C}}(\ ,K_i)_{\mathcal{T}}$ and $(\ ,K_i)_{\mathcal{T}}$ are finitely presented. Indeed, from the exact sequences,

$$0 \to K_{i+1} \xrightarrow{k_{i+1}} \mathcal{C}(\ ,C_i) \xrightarrow{p_i} K_i \to 0,$$

the long homology sequence, and the fact $\mathrm{pdim}\mathcal{T} \leq 1$, we obtain for each i an exact sequence:

$$0 \to (\ ,K_{i+1})_{\mathcal{T}} \xrightarrow{(\ ,k_{i+1})} (\ ,\mathcal{C}(\ ,C_i))_{\mathcal{T}} \xrightarrow{(\ ,p_i)} (\ ,K_i)_{\mathcal{T}} \xrightarrow{\partial_i} \mathrm{Ext}^1_{\mathcal{C}}(\ ,K_{i+1})_{\mathcal{T}} \to$$
$$\xrightarrow{\mathrm{Ext}^1(\ ,k_{i+1})} \mathrm{Ext}^1(\ ,\mathcal{C}(\ ,C_i))_{\mathcal{T}} \xrightarrow{\mathrm{Ext}^1(\ ,p_i)} \mathrm{Ext}^1(\ ,K_i)_{\mathcal{T}} \to \mathrm{Ext}^2(\ ,K_{i+1})_{\mathcal{T}} = 0.$$

By (a), for all $i \geq 0$, the functor $\mathrm{Ext}^1_{\mathcal{C}}(\ ,\mathcal{C}(\ ,C_i))_{\mathcal{T}}$ is finitely presented. Hence; each $\mathrm{Ext}^1(\ ,K_i)_{\mathcal{T}}$ is finitely generated .

From the exact sequence

$$\mathrm{Ext}^1_{\mathcal{C}}(\ ,K_{i+1})_{\mathcal{T}} \xrightarrow{\mathrm{Ext}^1_{\mathcal{C}}(\ ,k_{i+1})} \mathrm{Ext}^1_{\mathcal{C}}(\ ,\mathcal{C}(\ ,C_i))_{\mathcal{T}} \xrightarrow{\mathrm{Ext}^1_{\mathcal{C}}(\ ,p_i)} \mathrm{Ext}^1_{\mathcal{C}}(\ ,K_i)_{\mathcal{T}} \to 0,$$

it follows $\mathrm{Ext}^1_{\mathcal{C}}(\ ,K_i)_{\mathcal{T}}$ is actually finitely presented. Since $\mathrm{mod}(\mathcal{T})$ is abelian, the kernel of $\mathrm{Ext}^1_{\mathcal{C}}(\ ,K_{i+1})_{\mathcal{T}} \xrightarrow{\mathrm{Ext}^1_{\mathcal{C}}(\ ,k_{i+1})} \mathrm{Ext}^1_{\mathcal{C}}(\ ,\mathcal{C}(\ ,C_i))\ _{\mathcal{T}}$ is finitely presented.

In a similar way, each $(\ ,K_i)_{\mathcal{T}}$ is finitely generated, and it follows that the cokernel of the map $0 \to (\ ,K_{i+1})_{\mathcal{T}} \xrightarrow{(\ ,k_{i+1})} (\ ,\mathcal{C}(\ ,C_i))_{\mathcal{T}}$ is finitely presented.

We have proved each $(\ ,K_i)_{\mathcal{T}}$ is an extension of two finitely presented functors, therefore: it is finitely presented.

(c) From the exact sequence $0 \to K_0 \xrightarrow{k_0} \mathcal{C}(\ ,C_0) \to M \to 0$, and the long homology sequence, we have an exact sequence

$$0 \to (\ ,K_0)_{\mathcal{T}} \to (\ ,\mathcal{C}(\ ,C_0))_{\mathcal{T}} \to (\ ,M)_{\mathcal{T}} \to \mathrm{Ext}^1_{\mathcal{C}}(\ ,K_0)_{\mathcal{T}} \to \mathrm{Ext}^1_{\mathcal{C}}(\ ,\mathcal{C}(\ ,C_0)_{\mathcal{T}}.$$

Using again $\mathrm{mod}(\mathcal{T})$ is abelian, it follows $\phi(M) = (\ ,M)_{\mathcal{T}}$ is finitely presented. □

Corollary 4. *Assume $\mathcal{C}$ and $\mathcal{T}$ have pseudokernels. Then $\mathcal{T}$ is contravariantly finite in* $\mathrm{mod}(\mathcal{C})$.

Proposition 21. *Assume $\mathcal{C}$ and $\mathcal{T}$ have pseudokernels. Then the following statements hold:*

(i) *The functors F, F', G, G' in Brenner-Buter's theorem restrict to the subcategories of finitely presented functors.*

(ii) *Given a functor M in $\mathscr{T} \cap \mathrm{mod}(\mathcal{C})$, there exists a resolution*

$$\to T_n \xrightarrow{t_n} \cdots \to T_2 \xrightarrow{t_2} T_1 \xrightarrow{t_1} T_0 \xrightarrow{t_0} M \to 0$$

such that, each T_i is in $\mathrm{add}\mathcal{T}$, *and* $T_n \xrightarrow{\delta_n} \mathrm{Im}t_n$ *is a $\mathcal{T}$-approximation of* $\mathrm{Im}t_n$.

(iii) *If M is a functor in* $\mathrm{mod}(\mathcal{C})$, *then the the trace $\tau_{\mathcal{T}}(M)$ of $\mathcal{T}$ in M, and $M/\tau_{\mathcal{T}}(M)$ are finitely presented.*

(iv) *Denote by $t_{\mathscr{X}}$ the radical of the torsion theory $(\mathscr{X},\mathscr{Y})$ of* $\mathrm{Mod}(\mathcal{T})$. *Then for any functor N in* $\mathrm{mod}(\mathcal{T})$, *$t_{\mathscr{X}}(N)$ and $N/t_{\mathscr{X}}(N)$ are finitely presented.*

(v) *For any pair of finitely presented functors M, N in $\mathscr{T}$, we have an isomorphism* $\mathrm{Ext}^i_{\mathcal{C}}(M,N) = \mathrm{Ext}^i_{\mathcal{T}}(\phi(M), \phi(N))$.

Proof. (i) We proved F preserves finitely presented functors. If $(\ ,T_1) \to (\ ,T_0) \to N \to 0$ is a presentation of $N \in \mathrm{mod}(\mathcal{T})$, $T_0, T_1 \in \mathrm{add}\mathcal{T}$, then tensoring the exact sequence with $\mathcal{T}$, we obtain an exact sequence $T_0 \to T_1 \to \mathcal{T} \otimes N \to 0$. Since T_0, T_1 are finitely presented, and $\mathrm{mod}(\mathcal{C})$ is abelian, $G(N) = \mathcal{T} \otimes N$ is finitely presented.

We left to the reader to prove that the functors F' and G', preserve finitely presented functors.

(ii) Let M be in $\mathrm{mod}(\mathcal{C})$ and a map $\delta : T^0 \to M$ with T^0 in $\mathrm{Add}(\mathcal{T})$, as in Proposition 5, its image is $\tau_{\mathcal{T}}(M)$. If $f : T_0 \to M$ is a $\mathcal{T}$-approximation, then f factors through δ, and δ factors through f. In consequence, $\mathrm{Im}f = \tau_{\mathcal{T}}(M)$. In particular, if $M \in \mathscr{T}$, then f is an epimorphism.

Using again the fact $\mathrm{mod}(\mathcal{T})$ is abelian, the kernel of f, K_0, is finitely presented. From the exact sequence

$$0 \to K_0 \to T_0 \to M \to 0,$$

the long homology sequence, and the fact $f : T_0 \to M$ is a $\mathcal{T}$-approximation, it follows $\mathrm{Ext}^1_{\mathcal{C}}(\ ,K_0)_{\mathcal{T}} = 0$. Hence, K_0 is in $\mathscr{T}$, and the claim follows by induction.

(iii) Let M be in $\mathrm{mod}(\mathcal{C})$. Then by the proof of (ii), $\tau_{\mathcal{T}}(M)$ finitely generated, and M finitely presented, implies $M/\tau_{\mathcal{T}}(M)$ is finitely presented, and $\mathrm{mod}(\mathcal{C})$ abelian, implies $\tau_{\mathcal{T}}(M)$ is finitely presented.

(iv) Assume N in $\mathrm{mod}(\mathcal{T})$. Since the functors: F, F', G, G' preserve finitely presented functors, all terms in the exact sequence:

$$0 \to \mathrm{Ext}^1_{\mathcal{C}}(-, \mathrm{Tor}^{\mathcal{T}}_1(N, \mathcal{T})) \xrightarrow{\eta} N \xrightarrow{\gamma} \phi(N \otimes \mathcal{T}) \to 0$$

are finitely presented. The claim follows by observing the isomorphisms: $t_{\mathscr{X}}(N) \cong \mathrm{Ext}^1_{\mathcal{C}}(-, \mathrm{Tor}^{\mathcal{T}}_1(N, \mathcal{T}))$ and $N/\ t_{\mathscr{X}}(N) \cong \phi(N \otimes \mathcal{T})$.

(v) Follows as in Proposition 12. □

We denote by $(\tilde{\mathscr{T}}, \tilde{\mathscr{F}})$ and $(\tilde{\mathscr{X}}, \tilde{\mathscr{Y}})$ the intersection of the torsion theories $(\mathscr{T}, \mathscr{F})$ and $(\mathscr{X}, \mathscr{Y})$ with the categories of finitely presented functors, $\mathrm{mod}(\mathcal{C})$ and $\mathrm{mod}(\mathcal{T})$, respectively. From the previous proposition we obtain the following:

Theorem 5 (Brenner-Butler)**.** *Let $\mathcal{T}$ be a tilting subcategory of* $\mathrm{mod}(\mathcal{C})$ *and assume $\mathcal{C}$ and $\mathcal{T}$ have pseudokernls. With the above notation the following statements hold:*

(i) *The functors F and G induce an equivalence between $\tilde{\mathscr{T}}$ and $\tilde{\mathscr{Y}}$.*
(ii) *The functors F' and G' induce an equivalence between $\tilde{\mathscr{F}}$ and $\tilde{\mathscr{X}}$.*
(iii) *We also have: $FG' = F'G = 0$ and $G'F = GF' = 0$.*

Proof. The proof is clear from (i) in the above proposition. □

2.5. **Classical tilting for dualizing varieties.** In order to have a complete analogy with tilting theory for finite dimensional algebras, we need to add more restrictions in our categories, in particular, we need the existence of duality. We will assume in this subsection that $\mathcal{C}$ and $\mathcal{T}$ are dualizing varieties.

It was proved above that the category $\theta = \{\theta_C\}_{C \in \mathcal{C}}$, where $\theta_C = (\mathcal{C}(\ ,C), \ )_{\mathcal{T}}$, is a tilting subcategory of $\mathrm{mod}(\mathcal{T}^{op})$. Then by Brenner-Butler's theorem, there are torsion pairs $(\mathscr{T}(\theta), \mathscr{F}(\theta))$ and $(\mathscr{X}(\theta), \mathscr{Y}(\theta))$ in $\mathrm{mod}(\mathcal{T}^{op})$ and $\mathrm{mod}(\theta)$, respectively, and equivalence of categories

$$\begin{array}{ccc}
\mathscr{T}(\theta) & & \mathscr{F}(\theta) \\
 & \searrow^{\phi_\theta} \quad \swarrow^{\mathrm{Ext}^1_{\mathcal{T}^{op}}(\ ,-)_\theta} & \\
 & \nearrow_{\mathrm{Tor}^\theta_1(\ ,\theta)} \quad \nwarrow_{\otimes\theta} & \\
\mathscr{X}(\theta) & & \mathscr{Y}(\theta)
\end{array}$$

By Proposition 17, there is an equivalence of categories $\alpha(C) : \mathcal{C}^{op} \to \theta$, $\alpha(C) = \theta_C = (\mathcal{C}(\ \ ,C),\ \ )_{\mathcal{T}}$, which induces an equivalence $\alpha* : \mathrm{mod}(\theta) \to \mathrm{mod}(\mathcal{C}^{op})$ given by:

$$\alpha * (H)(C) = H(\alpha(C)) = H(\theta_C) = H((\mathcal{C}(\ \ ,C),\ \ )|\mathcal{T}),$$

for each C in $\mathcal{C}$, and H in $\mathrm{mod}(\theta)$.

Lemma 4. *Let N be an object in $\mathrm{mod}(\mathcal{T})$ and C one object in $\mathcal{C}$. Then the following statements hold:*

(a) *There is an isomorphism $((\mathcal{C}(\ \ ,C),\ \ )_{\mathcal{T}}, DN) \cong D(N \otimes \mathcal{T})(C)$, such that the following square*

$$\begin{array}{ccc}
\mathrm{mod}(\mathcal{C}) & \xleftarrow{\otimes\mathcal{T}} & \mathrm{mod}(\mathcal{T}) \\
\downarrow D & & \downarrow D \\
\mathrm{mod}(\mathcal{C}^{op}) & \xleftarrow{\alpha*\phi_\theta} & \mathrm{mod}(\mathcal{T}^{op})
\end{array}$$

commutes.

(b) *There is an isomorphism $\mathrm{Ext}^1_{\mathcal{T}^{op}}((\mathcal{C}(\ \ ,C),\ \ )_{\mathcal{T}}, DN) \cong D(\mathrm{Tor}^{\mathcal{T}}_1(N,\mathcal{T}))(C)$, such that the following square*

$$\begin{array}{ccc}
\mathrm{mod}(\mathcal{C}) & \xleftarrow{\mathrm{Tor}^{\mathcal{T}}_1(\ \ ,\mathcal{T})} & \mathrm{mod}(\mathcal{T}) \\
\downarrow D & & \downarrow D \\
\mathrm{mod}(\mathcal{C}^{op}) & \xleftarrow{\alpha*\mathrm{Ext}^1(\ \ ,-)|\theta} & \mathrm{mod}(\mathcal{T}^{op})
\end{array}$$

commutes.

(c) *We have the following equivalences of categories:*
   (i) $D(\mathscr{X}(\mathcal{T})) \cong \mathscr{F}(\theta)$, $D(\mathscr{Y}(\mathcal{T})) \cong \mathscr{T}(\theta)$
   (ii) $D(\mathscr{T}(\mathcal{T})) \cong \mathscr{Y}(\theta)$, $D(\mathscr{F}(\mathcal{T})) \cong \mathscr{X}(\theta)$

Proof. Applying the functor $(\ \ , DN)$ to the exact sequence

$$0 \to (T^1,\ \ ) \to (T^0,\ \ ) \to (\mathcal{C}(\ \ ,C),\ \ )_{\mathcal{T}} \to 0. \tag{2.32}$$

We obtain by the long homology sequence, and Yoneda's Lemma, the following exact sequence:

(2.33)

$$0 \to ((\ \ ,C),\ \ )_{\mathcal{T}}, DN) \to DN(T^0) \to DN(T^1) \to \mathrm{Ext}^1((\ \ ,C),\ \ )_{\mathcal{T}}, DN) \to 0,$$

and applying $\otimes N$ to (2.32), by the long homology sequence, we get an exact sequence

$$0 \to \mathrm{Tor}_1^{\mathcal{T}}(N, \mathcal{T})(C) \to N(T^1) \to N(T^0) \to (N \otimes \mathcal{T})(C) \to 0. \tag{2.34}$$

Dualizing (2.34), and comparing it with (2.33), we obtain the isomorphisms in (a) and (b).

To see that the first square commutes, let N be in $\mathrm{mod}(\mathcal{T})$. Then there are equalities:

$$\begin{aligned} \alpha * (\phi_\theta(DN))(C) &= \alpha * ((\ \ , DN)|\theta)(C) \\ &= (((\ \ , C),\ \ ), DN) = D(N \otimes \mathcal{T})(C). \end{aligned}$$

The equalities

$$\begin{aligned} \alpha * (\mathrm{Ext}^1(\ \ , -)|\theta(DN))(C) &= \alpha * (\mathrm{Ext}^1(\ \ , DN)|\theta)(C) \\ &= \mathrm{Ext}_1^{\mathcal{T}}(((\ \ , C),\ \ ), DN) = D(\mathrm{Tor}_1^{\mathcal{T}}(N \otimes \mathcal{T}))(C), \end{aligned}$$

imply, the second square commutes.

It only remains to prove (c). By (a) it follows

$$\mathscr{T}(\theta) = \{N \in \mathrm{mod}(\mathcal{T}^{op}) | \mathrm{Ext}^1(\theta_C, N) = 0\} = D(\mathscr{Y}(\mathcal{T})).$$

By Brenner-Butler's Theorem, there are equivalences of categories:

$$\begin{aligned} \phi_\theta &: \mathscr{F}(\theta) \to \mathscr{X}(\theta) \\ - \otimes \mathcal{T} &: \mathscr{Y}(\mathcal{T}) \to \mathscr{T}(\mathcal{T}). \end{aligned}$$

Then we have a commutative square

$$\begin{array}{ccc} \mathscr{T}(\mathcal{T}) & \xleftarrow{\otimes \mathcal{T}} & \mathscr{Y}(\mathcal{T}) \\ \downarrow D & & \downarrow D \\ \mathscr{Y}(\theta) & \xleftarrow{\alpha * \phi_\theta} & \mathscr{T}(\theta) \end{array}$$

By part (b), it follows

$$\mathscr{F}(\theta) = \{N \in \mathrm{mod}(\mathcal{T}^{op}) | \mathrm{Hom}(\theta_C, N) = 0\} = D(\mathscr{X}(\mathcal{T}))$$

From the equivalence of categories given in Brenner-Butler's Theorem:

$$\begin{aligned} \mathrm{Tor}_1^{\mathcal{T}}(\ \ , \mathcal{T}) &: \mathscr{X}(\mathcal{T}) \to \mathscr{F}(\mathcal{T}) \\ \mathrm{Ext}^1_{\mathcal{T}^{op}}(\ \ , -)_\theta &: \mathscr{T}(\theta) \to \mathscr{Y}(\theta) \end{aligned}$$

We have a commutative square:

$$\begin{array}{ccc} \mathscr{F}(\mathcal{T}) & \xleftarrow{\mathrm{Tor}_1^{\mathcal{T}}(\ \ , \mathcal{T})} & \mathscr{X}(\mathcal{T}) \\ \downarrow D & & \downarrow D \\ \mathscr{X}(\theta) & \xleftarrow{\alpha * \mathrm{Ext}^1(\ \ , -)|\theta} & \mathscr{F}(\theta) \end{array}$$

$\square$

Definition 11. [ASS] *Let $\mathcal{C}$ be Krull-Schmidt. A torsion theory $(\mathscr{T},\mathscr{F})$ in* $\mathrm{mod}(\mathcal{C})$ ***splits,*** *if every indecomposable $M\in \mathrm{mod}(\mathcal{C})$ is, either in $\mathscr{T}$ or in $\mathscr{F}$.*

Proposition 22 (ASS Prop. 1.7)**.** *Let $\mathcal{C}$ be dualizing category and $(\mathscr{T},\mathscr{F})$ a torsion pair in* $\mathrm{mod}(\mathcal{C})$*. Then the following conditions are equivalent:*

(a) *The torsion theory $(\mathscr{T},\mathscr{F})$ splits.*
(b) *Let τ be the radical of the torsion theory. Then for any M en $\mathrm{mod}(\mathcal{C})$, the exact sequence: $0\to \tau(M)\to M\to M/\tau(M)\to 0$, splits.*
(c) *For any N in $\mathscr{F}$ and any M in $\mathscr{T}$, $\mathrm{Ext}^1_{\mathcal{C}}(N,M)=0$.*
(c) *If $M\in\mathscr{T}$, then $\mathrm{TrD}M\in\mathscr{T}$.*
(d) *If $N\in\mathscr{F}$, then $\mathrm{DTr}N\in\mathscr{F}$.*

We say that a tilting subcategory $\mathcal{T}$ of $\mathrm{mod}(\mathcal{C})$ **separates,** if the torsion theory $(\mathscr{T}(\mathcal{T}),\mathscr{F}(\mathcal{T}))$ in $\mathrm{mod}(\mathcal{C})$ splits, and we say it **splits,** if the torsion theory $(\mathscr{X}(\mathcal{T}),\mathscr{Y}(\mathcal{T}))$ in $\mathrm{mod}(\mathcal{T})$ splits.

Lemma 5. *Let $\mathcal{T}$ be a tilting subcategory of* $\mathrm{mod}(\mathcal{C})$ *that splits. Then the following statements hold:*

(a) *Let $0\to M\xrightarrow{f} E\xrightarrow{g}\mathrm{TrD}(M)\to 0$ be an almost split sequence, with M in $\mathscr{T}(\mathcal{T})$. Then the three terms are in $\mathscr{T}(\mathcal{T})$, and $0\to\phi(M)\to\phi(E)\to\phi(\mathrm{TrD}(M))\to 0$ is an almost split sequence, whose terms are in $\mathscr{Y}(\mathcal{T})$.*
(b) *Let $0\to\mathrm{DTr}(M)\to E\to M\to 0$ be an almost split sequence, with M in $\mathscr{F}(\mathcal{T})$. Then the three terms are in $\mathscr{F}(\mathcal{T})$, and $0\to\mathrm{Ext}^1_{\mathcal{C}}(\ ,\mathrm{Dtr}(M))_{\mathcal{T}}\to\mathrm{Ext}^1_{\mathcal{C}}(\ ,E)_{\mathcal{T}}\to\mathrm{Ext}^1_{\mathcal{C}}(\ ,M)_{\mathcal{T}}\to 0$ is an almost split sequence, whose terms are in $\mathscr{X}(\mathcal{T})$.*

Proof. We will prove only (a), being (b) similar. By previous Lemma, $\mathrm{TrD}(M)\in\mathscr{T}(\mathcal{T})$, therefore: $E\in\mathscr{T}(\mathcal{T})$. By the long homology sequence, we get the exact sequence

$$0\to(\ ,M)_{\mathcal{T}}\xrightarrow{(\ ,f)}(\ ,E)_{\mathcal{T}}\xrightarrow{(\ ,g)}(\ ,\mathrm{TrD}(M))_{\mathcal{T}}\to\mathrm{Ext}^1_{\mathcal{C}}(\ ,M)_{\mathcal{T}}=0,\tag{2.35}$$

whose terms are in $\mathscr{Y}(\mathcal{T})$.

The morphism $(\ ,g)_{\mathcal{T}}$ is right minimal. Let $\overline{\eta}:(\ ,E)_{\mathcal{T}}\to(\ ,E)_{\mathcal{T}}$ be an endomorphism, such that $(\ ,g)_{\mathcal{T}}\tilde{\eta}=(\ ,g)_{\mathcal{T}}$. Since $\phi:\mathscr{T}(\mathcal{T})\to\mathscr{Y}(\mathcal{T})$ is an equivalence of categories we have $((\ ,E)_{\mathcal{T}},(\ ,E)_{\mathcal{T}})\cong(E,E)$ and $\tilde{\eta}=(\ ,\eta)_{\mathcal{T}}$, with $\eta:E\to E$. Then $g\eta=g$, and since g is right minimal, it follows η is an isomorphism, hence, $\tilde{\eta}$ is an isomorphism.

Let N be indecomposable and $\tilde{\gamma}:N\to(\ ,\mathrm{TrD}(M))$ a non isomorphism and non zero map. Then, either $N\in\mathscr{X}(\mathcal{T})$ or $N\in\mathscr{Y}(\mathcal{T})$. If $N\in\mathscr{X}(\mathcal{T})$, then $\tilde{\gamma}\in\mathrm{Hom}(\mathscr{X}(\mathcal{T}),\mathscr{Y}(\mathcal{T}))=0$, hence, $N=(\ ,H)_{\mathcal{T}}$, with $H\in\mathscr{T}(\mathcal{T})$ and $\tilde{\gamma}=(\ ,\gamma)_{\mathcal{T}}:(\ ,H)_{\mathcal{T}}\to(\ ,\mathrm{TrD}M)_{\mathcal{T}}$, where γ is non isomorphism and non zero. Then η factors through E, and $\tilde{\eta}$ factors through $(\ ,E)_{\mathcal{T}}$. □

As a consequence of Lemma 5, and Proposition 21, we have:

Lemma 6 (ASS Lemma 5.5)**.** *Let $\mathcal{T}$ be a tilting subcategory of* $\mathrm{mod}(\mathcal{C})$*. If $M\in\mathscr{T}(\mathcal{T})$ and $N\in\mathscr{F}(\mathcal{T})$, then for any $j\geq 1$, there is an isomorphism*

$$\mathrm{Ext}^j_{\mathcal{C}}(M,N)\cong\mathrm{Ext}^{j-1}_{\mathcal{T}}(\phi(M),\mathrm{Ext}^1_{\mathcal{C}}(\ ,N)_{\mathcal{T}}).$$

From the fact $\theta=\{\theta_C=((\ ,C),\ \ )_{\mathcal{T}}\}_{C\in\mathcal{C}}$ is a tilting subcategory in $\mathrm{mod}(\mathcal{T}^{op})$, we obtain the following:

Theorem 6. *Let $\mathcal{T}$ be a tilting subcategory of* $\mathrm{mod}(\mathcal{C})$*. Then:*

(a) *$\mathcal{T}$ separates, if and only if, for any $Y \in \mathscr{Y}(\mathcal{T})$,* $\mathrm{pdim}Y = 1$.

(b) *$\mathcal{T}$ splits, if and only if, for any $N \in \mathscr{F}(\mathcal{T})$,* $\mathrm{idim}N = 1$.

Proof. (b) Is proved in [ASS Theo. 5.6], but, for he benefit of the reader, we repeat the proof here.

First, the sufficiency of the condition. Assume that for every $N \in \mathscr{F}(\mathcal{T})$, we have $\mathrm{idim}N = 1$. Let $X \in \mathscr{X}(\mathcal{T})$ and $Y \in \mathscr{Y}(\mathcal{T})$. Then there exist $M \in \mathscr{T}(\mathcal{T})$ and $N \in \mathscr{F}(\mathcal{T})$ such that $X \cong \mathrm{Ext}^1_{\mathcal{C}}(\ ,N)_{\mathcal{T}}$ and $Y \cong \phi(M)$, by Brenner-Buttler theorem. Hence, by the above Lemma

$$\mathrm{Ext}^1_{\mathcal{T}}(Y,X) \cong \mathrm{Ext}^1_{\mathcal{T}}(\phi(M), \mathrm{Ext}^1_{\mathcal{C}}(\ ,N)_{\mathcal{T}}) \cong \mathrm{Ext}^2_{\mathcal{C}}(M,N) = 0,$$

Conversely, assume that $(\mathscr{X}(\mathcal{T}), \mathscr{Y}(\mathcal{T}))$ is spplitting and let $N \in \mathscr{F}(\mathcal{T})$. Take an injective resolution of N

$$0 \to N \xrightarrow{d^0} I^0 \xrightarrow{d^1} I^1 \xrightarrow{d^2} I^2 \to \cdots$$

Let $L^0 = \mathrm{Im}d^1$ and $L^1 = \mathrm{Im}d^2$. Since $\mathscr{T}(\mathcal{T}) = \mathrm{KerExt}^1_{\mathcal{C}}(\mathcal{T},\ )$ contais the injective objects and it is closed under epimorphic images, and $N \in \mathscr{F}(\mathcal{T})$, it follows $L^1 \in \mathscr{T}(\mathcal{T})$. Then, by the above Lemma, we have:

$$\mathrm{Ext}^1_{\mathcal{C}}(L^1, L^0) \cong \mathrm{Ext}^2_{\mathcal{C}}(L^1, N) \cong \mathrm{Ext}^1_{\mathcal{T}}(\phi(L^1), \mathrm{Ext}^1_{\mathcal{C}}(\ ,N)_{\mathcal{T}})$$

But, $\phi(L^1) \in \mathscr{Y}(\mathcal{T})$, $\mathrm{Ext}^1_{\mathcal{C}}(\ ,N)_{\mathcal{T}} \in \mathscr{X}(\mathcal{T})$ and $(\mathscr{X}(\mathcal{T}), \mathscr{Y}(\mathcal{T}))$ is splitting, by hypothesis. By Proposition 22, this implies $\mathrm{Ext}^1_{\mathcal{T}}(\phi(L^1), \mathrm{Ext}^1_{\mathcal{C}}(\ ,N)_{\mathcal{T}}) = 0$, proving that the exact sequence $0 \to L^0 \to I^1 \to L^1 \to 0$ splits . Therefore, L^0 is injective and $\mathrm{idim}N \leq 1$. Finally, $N \in \mathscr{F}(\mathcal{T})$, implies N is not injective, thus $\mathrm{idim}N = 1$.

We prove that (a) follows from (b):

By definition, $\mathcal{T}$ separates, if and only if, the torsion pair $(\mathscr{T}(\mathcal{T}), \mathscr{F}(\mathcal{T}))$ in $\mathrm{mod}(\mathcal{C})$ splits. By duality, this occurs if and only if, the pair $(\mathscr{X}(\theta), \mathscr{Y}(\theta)) = (D\mathscr{F}(\mathcal{T}), D\mathscr{T}(\mathcal{T}))$, is a splitting torsion theory in $\mathrm{mod}(\mathcal{C}^{op}) \cong \mathrm{mod}(\theta)$. By (b), θ is a tilting subcategory that splits in $\mathrm{mod}(\mathcal{T}^{op})$, if and only if for all $N \in \mathscr{F}(\theta)$, $\mathrm{idim}N = 1$. But $\mathscr{F}(\theta) = D\mathscr{Y}(\mathcal{T})$, and (a) follows. □

Corollary 5. *If* $\mathrm{gdim}\mathcal{C} \leq 1$*, then any tilting subcategory $\mathcal{T} \subset \mathrm{mod}(\mathcal{C})$ splits.*

As a consequence of the previous theorem and Lemma 4, we have the following:

Corollary 6. *If $\mathcal{C}$ and $\mathcal{T}$ are hereditary, then $\mathcal{T}$ splits and separates*

The last theorems in this subsection will have important implications in the hereditary case, that we will consider in next section.

3. Infinite quivers

We begin this section showing that there exist natural examples of the notions discussed in the previous section.

We prove first that for a finite dimensional algebra Λ, having a finitely generated generator M, such that there exists a tilting $\mathrm{End}_{\Lambda}(M)^{op}$-module T, the module T can be extended to a tilting category of mod_{Λ}.

We study next locally finite infinite quivers Q, and consider the quiver algebra -category. We prove that for locally finite infinite quivers, a section on the pre-projective component produces a tilting category. To apply the tilting functor is analogous to apply an infinite sequence of partial Coxeter functors, to change the orientation of the quiver. We next use these results to computate the Auslander-Reiten components of the locally finite infinite Dynkin quivers.

To describe the shape of all Auslander-Reiten components, it is enough to compute them for a fixed orientation, and then to apply tilting. We choose quivers with only sinks and sources.

A simple modification of the arguments given by [ABPRS], proves that regular Auslander-Reiten components of locally finite infinite quivers are of type A_∞.

3.1. **Extending tilting.** In this subsection we will prove that there exist examples of tilting subcategories. The first source of examples is produced in the following way:

Let $\mathcal{C}$ be a K-category, K a field, which is Hom-finite, with an additive generator Λ, such that there is a tilting $\mathrm{End}(\Lambda)^{op}$-module T, $\mathcal{C}' = \{\mathrm{add}\Lambda\}$ is a subcategory of $\mathcal{C}$ containing the generator of $\mathcal{C}$, $\mathrm{Mod}(\mathcal{C}')$ is equivalent to $\mathrm{Mod}(\mathrm{End}(\Lambda)^{op})$ and it has tilting subcategory which corresponds to T under the equivalence. We will extend first the tilting subcategory of $\mathrm{Mod}(\mathcal{C}')$ to a partial tilting subcategory of $\mathrm{Mod}(\mathcal{C})$ and then using Bongartz' argument [B], [CF] we will extend it to a tilting subcategory of $\mathrm{Mod}(\mathcal{C})$.

Lemma 7. *Let $\mathcal{C}'$ be a subcategory of $\mathcal{C}$. If $\mathcal{T}$ is a self orthogonal subcategory of* $\mathrm{Mod}(\mathcal{C}')$*, consisting of finitely presented objects, then $\mathcal{C} \otimes_{\mathcal{C}'} \mathcal{T}$ is a self orthogonal subcategory of* $\mathrm{Mod}(\mathcal{C})$*.*

Proof. Let T_1, T_2 be objects of $\mathcal{T}$, with presentations $\mathcal{C}'(\ \ ,C_1') \to \mathcal{C}'(\ \ ,C_0') \to T_1 \to 0$, $\mathcal{C}'(\ \ ,C_1'') \to \mathcal{C}'(\ \ ,C_0'') \to T_2 \to 0$. Let's consider the functors: $\mathrm{res} : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Mod}(\mathcal{C}')$, $\mathcal{C}\otimes_{\mathcal{C}'} : \mathrm{Mod}(\mathcal{C}') \to \mathrm{Mod}(\mathcal{C})$, as in [A].

We claim $\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{C}\otimes_{\mathcal{C}'} T_1, \mathcal{C}\otimes_{\mathcal{C}'} T_2) = 0$. Indeed, let $\mathbf{e}$ be an element of $\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{C}\otimes_{\mathcal{C}'} T_1, \mathcal{C}\otimes_{\mathcal{C}'} T_2)$, $\mathbf{e} : 0 \to \mathcal{C}\otimes_{\mathcal{C}'} T_2 \to F \to \mathcal{C}\otimes_{\mathcal{C}'} T_1 \to 0$. We apply $\mathcal{C}\otimes_{\mathcal{C}'}$ to the projective presentations of T_1 and T_2, to get projective presentations of the corresponding extension. By the Horse Shoe Lemma, we obtain the following commutative exact diagram:

$$
\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{C}(\ \ ,C_1') & \longrightarrow & \mathcal{C}(\ \ ,C_0') & \longrightarrow & \mathcal{C}\otimes_{\mathcal{C}'} T_1 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{C}(\ \ ,C_1' \coprod C_1'') & \rightarrow & \mathcal{C}(\ \ ,C_0' \coprod C_0'') & \longrightarrow & F & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{C}(\ \ ,C_1'') & \longrightarrow & \mathcal{C}(\ \ ,C_0'') & \longrightarrow & \mathcal{C}\otimes_{\mathcal{C}'} T_2 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}
$$

By [A Prop. 3.2], there exists an isomorphism $\mathcal{C}\otimes_{\mathcal{C}'} \mathrm{res}(F) \cong \mathrm{Id}_{\mathrm{Mod}(\mathcal{C})}(F) = F$.

Since the functor res is exact, we have the following commutative exact diagram:

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{C}'(\ \ ,C_1') & \longrightarrow & \mathcal{C}'(\ \ ,C_0') & \longrightarrow & T_1 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{C}'(\ \ ,C_1'\coprod C_1'') & \rightarrowtail & \mathcal{C}'(\ \ ,C_0'\coprod C_0'') & \longrightarrow & \mathrm{res}F & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{C}'(\ \ ,C_1'') & \longrightarrow & \mathcal{C}'(\ \ ,C_0'') & \longrightarrow & T_2 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

By assumption, the column at the right of the diagram splits, hence tensoring with $\mathcal{C}\otimes_{\mathcal{C}'}$ and using $\mathcal{C}\otimes_{\mathcal{C}'}\mathrm{res}(F)\cong F$, the columns in the first diagram split, hence **e** splits. □

Definition 12. *A full subcategory $\mathcal{T}$ of* $\mathrm{Mod}(C)$ *is a* ***partial tilting****, if its objects satisfy the following two conditions:*

(i) *For each object T in $\mathcal{T}$ there is an exact sequence $0\to P_1\to P_0\to T\to 0$, with P_i finitely generated projective.*

(ii) *For every pair of objects T_i and T_j in $\mathcal{T}$, $\mathrm{Ext}^1_{\mathcal{C}}(T_i,T_j)=0$.*

Proposition 23. *Let $\mathcal{C}'$ be a subcategory of $\mathcal{C}$ containing an additive generator Λ of $\mathcal{C}$. If $\mathcal{T}$ is a partial tilting subcategory of $\mathrm{Mod}(\mathcal{C}')$, then $\mathcal{C}\otimes_{\mathcal{C}'}\mathcal{T}$ is a partial tilting subcategory of $\mathrm{Mod}(\mathcal{C})$.*

Proof. By Lemma 7, $\mathcal{C}\otimes_{\mathcal{C}'}\mathcal{T}$ is self orthogonal. (ii) Let T be an object in $\mathcal{T}$, and $0\to\mathcal{C}'(\ \ ,C_1')\xrightarrow{\mathcal{C}'(\ \ ,f)}\mathcal{C}(\ \ ,C_0')\to T\to 0$, a projective resolution with C_0' and C_1' in $\mathcal{C}'$. Applying $\mathcal{C}\otimes -$, we get the projective presentation $\mathcal{C}(\ \ ,C_1')\xrightarrow{\mathcal{C}(\ \ ,f)}\mathcal{C}(\ \ ,C_0')\to\mathcal{C}\otimes_{\mathcal{C}'}T\to 0$. Since Λ is in $\mathcal{C}'$, then $\mathcal{C}'(\Lambda,C_0')=\mathcal{C}(\Lambda,C_0')$ and $\mathcal{C}'(\Lambda,C_1')=\mathcal{C}(\Lambda,C_1')$. In consequence, $\mathcal{C}\otimes_{\mathcal{C}'}T(\Lambda)=T(\Lambda)$. We need to see $\mathcal{C}(\ \ ,f)$ is monomorphism. Let C be an object in $\mathcal{C}$. Since Λ is an additive generator of $\mathcal{C}$, then there is an epimorphism $g:\Lambda^n\to C\to 0$, and hence a monomorphisms $0\to\mathcal{C}(C,C_i')\to\mathcal{C}(\Lambda,C_i')$, for $i=0,1$.

In this way, we obtain a commutative exact diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{C}(C,C_1') & \xrightarrow{\mathcal{C}(C,f)} & \mathcal{C}(C,C_0') & \longrightarrow & \mathcal{C}\otimes_{\mathcal{C}'}T(C) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathcal{C}(g,C_1')} & & \downarrow{\scriptstyle \mathcal{C}(g,C_0')} & & \downarrow{\scriptstyle \mathcal{C}\otimes_{\mathcal{C}'}T(g)} & & \\
 & & \mathcal{C}(\Lambda^n,C_1') & \xrightarrow{\mathcal{C}(\Lambda^n,f)} & \mathcal{C}(\Lambda^n,C_0') & \longrightarrow & \mathcal{C}\otimes_{\mathcal{C}'}T(\Lambda^n) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \mathcal{C}'(\Lambda^n,C_1') & \xrightarrow{\mathcal{C}'(\Lambda^n,f)} & \mathcal{C}'(\Lambda^n,C_0') & \longrightarrow & T(\Lambda^n) & \longrightarrow & 0
\end{array}$$

and for any object C in $\mathcal{C}$ the map $\mathcal{C}(C,f)$ is a monomorphism. □

Theorem 7. *Let $\mathcal{C}$ be an* Hom-*finite, K-category with additive generator Λ and let R_Λ be the endomorphism ring $R_\Lambda = \operatorname{End}(\Lambda)^{op}$. Assume* $\operatorname{Mod}(R_\Lambda)$ *has a partial tilting module. Then* $\operatorname{Mod}(\mathcal{C})$ *has a tilting subcategory.*

Proof. From the equivalence $\operatorname{Mod}(R_\Lambda)$ and $\operatorname{Mod}(\operatorname{add}\{\Lambda\})$, it follows the category $\operatorname{Mod}(\operatorname{add}\{\Lambda\})$ has a partial tilting object. The category $\mathcal{C}' = \operatorname{add}\{\Lambda\}$, is a subcategory of $\mathcal{C}$ containing Λ. By Proposition 23, there exists an object T in $\operatorname{Mod}(\mathcal{C})$ which is a partial tilting in $\operatorname{Mod}(\mathcal{C})$. Since T is finitely presented, we have an exact sequence

$$0 \to L \to \mathcal{C}(\ \ ,C_0) \to T \to 0, \tag{3.1}$$

with L finitely generated.

Let C be an object in $\mathcal{C}$. Applying $\operatorname{Hom}_{\mathcal{C}}(\ \ ,\mathcal{C}(\ \ ,C))$ to (3.1), and using the long homology sequence, we obtain the following exact sequence:

$$\operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(\ \ ,C_0),\mathcal{C}(\ \ ,C)) \to \operatorname{Hom}_{\mathcal{C}}(L,\mathcal{C}(\ \ ,C)) \to \operatorname{Ext}_{\mathcal{C}}(T,\mathcal{C}(\ \ ,C)) \to 0.$$

Since L is finitely generated, there is an epimorphism $\mathcal{C}(\ \ ,C_1) \to L \to 0$, and hence a monomorphism $0 \to \operatorname{Hom}_{\mathcal{C}}(L,\mathcal{C}(\ \ ,\mathcal{C})) \to \operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(\ \ ,C_1),\mathcal{C}(\ \ ,C)) \cong \operatorname{Hom}_{\mathcal{C}}(C_1,C)$. Since $\mathcal{C}$ is Hom-finite, then $\operatorname{Hom}_{\mathcal{C}}(C_1,C)$ is a finite dimensional K-vector space. It follows $\operatorname{Hom}_{\mathcal{C}}(L,\mathcal{C}(\ \ ,C))$ and $\operatorname{Ext}^1_{\mathcal{C}}(T,\mathcal{C}(\ \ ,C))$ are finite dimensional K-vector spaces.

We proceed now as in [B]:

Let $\mathbf{e}_1,\cdots,\mathbf{e}_d$ be a K-base of $\operatorname{Ext}^1_{\mathcal{C}}(T,\mathcal{C}(\ \ ,C))$. Represent each $\mathbf{e_i}$ as a short exact sequence

$$0 \to \mathcal{C}(\ \ ,C) \xrightarrow{f_i} E_i \xrightarrow{g_i} T \to 0.$$

Consider the following diagram with exact raws:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{C}(\ \ ,C)^d & \xrightarrow{f} & \coprod_{i=1}^{d} E_i & \xrightarrow{g} & T^d & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & \mathcal{C}(\ \ ,C) & \xrightarrow{v} & E_C & \xrightarrow{w} & T^d & \longrightarrow & 0
\end{array}$$

where f and g are sums of morphisms f_i, and g_i respectively and $\nabla = [1,\ldots,1]$. Let's denote by $\mathbf{e}_C$ the element of $\operatorname{Ext}^1_{\mathcal{C}}(T^d,\mathcal{C}(\ \ ,C))$ represented by the exact sequence

$$\mathbf{e}_C : 0 \to \mathcal{C}(\ \ ,C) \xrightarrow{v} E_C \xrightarrow{w} T^d \to 0.$$

Let $u_i : T \to T^d$ be the i-th inclusions. We claim for each $i = 1,\ldots,d$, $\mathbf{e}_i = \operatorname{Ext}^1_{\mathcal{C}}(u_i,\mathcal{C}(\ \ ,C))\mathbf{e}_C$. Indeed, let's consider the following commutative diagram with exact raws:

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{C}(\ \ ,C) & \xrightarrow{f_i} & E_i & \xrightarrow{g_i} & T & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u_i''} & & \downarrow{\scriptstyle u_i'} & & \downarrow{\scriptstyle u_i} & & \\
0 & \to & \mathcal{C}(\ \ ,C)^d & \xrightarrow{f} & \coprod_{i=1}^{d} E_i & \xrightarrow{g} & T^d & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & \mathcal{C}(\ \ ,C) & \xrightarrow{v} & E_C & \xrightarrow{w} & T^d & \longrightarrow & 0
\end{array}$$

where u_i', u_i'' denote the corresponding inclusions. Since $\nabla u_i'' = 1_{\mathcal{C}(\ ,C)}$, we get the following commutative diagram with exact raws:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{C}(\ ,C) & \stackrel{f_i}{\to} & E_i & \stackrel{g_i}{\longrightarrow} & T & \longrightarrow & 0 \\ & & 1\downarrow & & uu_i'\downarrow & & u_i\downarrow & & \\ 0 & \to & \mathcal{C}(\ ,C) & \stackrel{v}{\to} & E_C & \stackrel{w}{\longrightarrow} & T^d & \longrightarrow & 0 \end{array}$$

hence, the claim follows.

Applying $\mathrm{Hom}_{\mathcal{C}}(T,\ )$ to the sequence $\mathbf{e}_C$, we obtain by he long homology sequence, an exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(T,T^d) \stackrel{\delta}{\to} \mathrm{Ext}^1_{\mathcal{C}}(T,\mathcal{C}(\ ,C)) \to \mathrm{Ext}^1_{\mathcal{C}}(T,E_C) \to \mathrm{Ext}^1_{\mathcal{C}}(T,T^d) = 0.$$

Since $\mathbf{e_i} = \mathrm{Ext}^1_{\mathcal{C}}(u_i,\mathcal{C}(\ ,C))\mathbf{e}_C = \delta(u_i)$, it follows each basic element of the vector space $\mathrm{Ext}^1_{\mathcal{C}}(T,\mathcal{C}(\ ,C))$ is in the image of the connecting morphism δ, which is then surjective. Therefore: $\mathrm{Ext}^1_{\mathcal{C}}(T,E_C) = 0$.

We will see that the full subcategory $\mathcal{T}$ of $\mathrm{Mod}(\mathcal{C})$, with $\mathcal{T} = \{T \coprod E_C\}_{C\in\mathcal{C}}$, is a tilting category.

Since E_C is an extension of finitely presented functors, it is finitely presented. Hence, for any object $\mathcal{C}$ in $\mathrm{Mod}(\mathcal{C})$, the sum $T\coprod E_C$ is finitely presented.

(i) $\mathrm{pdim} T\coprod E_C \leq 1$. Since T is a partial tilting object object in $\mathrm{Mod}(\mathcal{C})$, it has $\mathrm{pdim} T \leq 1$, and from the exact sequence $\mathbf{e}_C$ it follows $\mathrm{pdim} E_C \leq 1$. Therefore: $\mathrm{pdim} T\coprod E_C \leq 1$.

(ii) For any pair of objects C and C' in $\mathcal{C}$, $\mathrm{Ext}^1_{\mathcal{C}}(T\coprod E_C, T\coprod E_{C'}) = 0$. Applying $\mathrm{Hom}_{\mathcal{C}}(\ ,T)$ and $\mathrm{Hom}_{\mathcal{C}}(\ ,E_{C'})$ to $\mathbf{e}_C$, we obtain by the long homology sequence, exact sequences:

$$\begin{aligned} 0 &= \mathrm{Ext}^1_{\mathcal{C}}(T^d,T) \to \mathrm{Ext}^1_{\mathcal{C}}(E_C,T) \to \mathrm{Ext}^1_{\mathcal{C}}(\mathcal{C}(\ ,C),T) = 0, \\ 0 &= \mathrm{Ext}^1_{\mathcal{C}}(T^d,E_{C'}) \to \mathrm{Ext}^1_{\mathcal{C}}(E_C,E_{C'}) \to \mathrm{Ext}^1_{\mathcal{C}}(\mathcal{C}(\ ,C),E_{C'}) = 0. \end{aligned}$$

Hence, for any pair of objects C and C' in $\mathrm{Mod}(\mathcal{C})$, we have:

$$\mathrm{Ext}^1_{\mathcal{C}}(T,E_{C'}) = \mathrm{Ext}^1_{\mathcal{C}}(E_C,T) = \mathrm{Ext}^1_{\mathcal{C}}(E_C,E_{C'}) = 0.$$

Using the fact $\mathrm{Ext}^1_{\mathcal{C}}(T,T) = 0$ and the above equalities, the condition (ii) follows. (iii) is immediate. □

3.2. **The Hereditary Case.** In this subsection $Q = (Q_0, Q_1)$ will be a **locally finite,** infinite quiver. We will consider the quiver algebra KQ as a subadditive K-category $\mathcal{C}$, and the finite dimensional representations of the quiver, will be identified with the category of finitely presented contravariant functors $\mathrm{mod}(KQ)$ from KQ to he category of finite dimensional K-vector spaces. The category $\mathcal{C}$ is Hom-finite, dualizing and Krull-Schmidt. By [AR] $\mathrm{mod}(\mathcal{C})$ has almost split sequences. We will describe the Auslander-Reiten components, beginning with he preprojective components.

The preprojective component $\mathcal{K}$ of the Auslander-Reiten $\Gamma(KQ)$, of KQ, can be computed as in the finite quiver situation. It is easy to prove that it is a translation quiver of the form $(-\mathbb{N}\Delta, \tau)$, $\Delta = Q^{op}$. The category $\mathcal{K}$ is the quiver algebra $K(-\mathbb{N}\Delta)$ module the mesh relations $\mathcal{K} = K(-\mathbb{N}\Delta)/ < m_x >$ [R, Lemma 3, Section 2.3].

We will need the following combinatorial lemma, which can be proved as in [BGP].

Lemma 8. *Let Q be a locally finite infinite quiver that is not of type A_∞, A_∞^∞, D_∞. Given a finite subquiver Q'' of Q, there exists a finite full connected subquiver $Q' \subset Q$ such that Q'' is a subquiver of Q' and Q' is not Dynkin.*

Definition 13. [ASS] *Let (Γ, τ) be a connected translation quiver. A connected full subquiver Σ is a **section** of Γ if the following conditions are satisfied:*

(S1) *If $X_0 \to X_1 \to \cdots \to X_t$ is a path in Σ of lenght $l \geq 1$, then $X_0 \neq X_t$.*
(S2) *For each $X \in \Gamma_0$, there exist a unique $n \in \mathbb{Z}$ such that $\tau^n X \in \Sigma_0$.*
(S3) *If $X_0 \to X_1 \to \cdots \to X_t$ is a path in Γ with $X_0, X_t \in \Sigma$, then $X_i \in \Sigma_0$ for all i such that $0 \leq i \leq t$.*

Definition 14. [ARS] *Let I be the integers in one of the intervals $(-\infty, n]$, $[n, \infty)$, $[m, n]$ for $m < n$ or $\{1, \ldots, n\}$ modulo n. Let $\cdots \to X_i \xrightarrow{f_i} X_{i+1} \to \cdots$ be a path in the Auslander-Reiten quiver (Γ, τ) with each index in I. This path is said to be **sectional** if $\tau X_{i+2} \ncong X_i$.*

Lemma 9. *Let $\mathcal{C}$ be a dualizing variety, and $X_1 \overset{f_1}{\to} X_2 \overset{f_2}{\to} \cdots X_{n-1} \overset{f_{n-1}}{\to} X_n$ sectional path in $\mathrm{mod}(\mathcal{C})$. Then the composition $f_{n-1} f_{n-2} ... f_1$ is not zero.*

Proof. The proof is as in [ARS Theo. 2.4]. □

Remark 3. *Let Q be a locally finite infinite quiver, $\mathcal{K}$ the preprojective component of the Auslander-Reiten quiver of KQ, that we identify with the quiver algebra with quiver $(-\mathbb{N}Q, \tau)$ and the mesh relations. Then given an object X in $\mathcal{K}$ and a positive integer n, there exists a finite number of directed paths in $(-\mathbb{N}Q, \tau)$ with starting vertex X and length $\leq n$.*

Theorem 8. *Let Q be a locally finite infinite quiver and Σ a section of $(-\mathbb{N}Q, \tau)$ with no infinite directed paths, then the following is true*

(a) *For any vertex X of $(-\mathbb{N}Q, \tau)$ the number of directed paths from X to the section is finite and there is a non zero path from X to the section.*
(b) *For any vertex X of $(-\mathbb{N}Q, \tau)$ the number of directed paths from the section to X is finite and there is a non zero path from the section to X.*

Proof. (a) The proof will be by induction on the least $n \geq 0$ such that $\tau^{-n} X$ is in Σ.

In the case there is not such an n, then X is a successor of the section and the number of paths to the section is zero.

If $n = 0$, then X is on the section and the claim is true, since the number of directed paths on the section is finite.

Assume $n = 1$ and consider X an element of the preprojective component with almost split sequence:

$$0 \to X \overset{(\sigma\alpha_i)}{\to} \coprod_{i=1}^{n} E_i \overset{(\alpha_i)}{\to} \tau^{-1} X \to 0$$

If all E_i are in the section, then there is nothing to prove.

Let $\alpha_i : E_i \to \tau^{-1} X$ be a map that it is not in the section. Then $\sigma^{-1}\alpha_i : \tau^{-1} X \to \tau^{-1} E_i$ is in the section, assume there is a irreducible map $\beta : E_i \to Y$, different from α_i. Then there is an irreducible map $\sigma^{-1}\beta : Y \to \tau^{-1} E_i$. Let n be the maximum of the lengths of the paths in Σ starting at $\tau^{-1} X$. By the above remark, there exists only a finite number of paths of length $\leq n$ starting at Y and assume $Y \overset{\beta}{\to} Y_1 \overset{\beta_1}{\to} Y_2 \overset{\beta_2}{\to} \ldots \overset{\beta_{n-1}}{\to} Y_n$ is a path that does not meet any of the paths starting at $\tau^{-1}X$ and ending at Σ. Then for $k \leq n$ we have the following diagram of irreducible maps:

$$(3.2)\qquad \begin{array}{ccccccccccccc}
\tau^{-1}Y_{k-1} & \overset{\sigma^{-2}\beta_{k-1}}{\longleftarrow} & \tau^{-1}Y_{k-2} & \cdots\cdots & \tau^{-1}Y_k & \overset{\sigma^{-2}\beta_{k-1}}{\longleftarrow} & \tau^{-1}E_1 & \overset{\sigma^{-1}\alpha}{\longleftarrow} & \tau^{-1}X & \overset{\alpha_2}{\longleftarrow} & E_2 \\
\sigma^{1}\beta_k \uparrow & & \sigma^{-1}\beta_{k-1} \uparrow & & \sigma^{-1}\beta_1 \uparrow & & \sigma^{-1}\beta \uparrow & & \alpha \uparrow & & \sigma\alpha_2 \uparrow \\
Y_k & \overset{\beta_k}{\longleftarrow} & Y_{k-1} & \cdots\cdots & Y_1 & \overset{\beta_1}{\longleftarrow} & Y & \overset{\beta}{\longleftarrow} & E_1 & \overset{\sigma\alpha}{\longleftarrow} & X
\end{array}$$

with $\tau^{-1}Y_{k-1}$ in Σ, and there is no irreducible map $\tau^{-1}\mathrm{Y}_{k-1} \to$Z with Z on Σ. By the definition of section, $\sigma^{-1}\beta_k$:$\mathrm{Y}_k \to \tau^{-1}\mathrm{Y}_{k-1}$ is in Σ.

Assume there is only a finite number of paths for any X in $(-\mathbb{N}Q, \tau)$ with $\tau^{-m+1}X \in \Sigma_0$.

As before, we consider an almost split sequence starting at X and a map α_i : $E_i \to \tau^{-1}X$ that is not in the section. Then, by induction hypothesis, there is a finite number of paths from $\tau^{-1}X$ to the section. Let n be the length of the largest such path. Assume there is a irreducible map $\beta : E_i \to Y$, different from α_i. Arguing as above, we consider a path $Y \overset{\beta}{\to} Y_1 \overset{\beta_1}{\to} Y_2 \overset{\beta_2}{\to} \ldots \overset{\beta_{n-1}}{\to} Y_n$ that does not meet any of the paths starting at $\tau^{-1}X$. Then we obtain a diagram of irreducible maps as in (3.2) and for some integer $k \leq n$, $\tau^{-1}\mathrm{Y}_{k-1}$ in Σ. Then either $\sigma^{-1}\beta_k$:$\mathrm{Y}_k \to \tau^{-1}\mathrm{Y}_{k-1}$ is in Σ, or $\tau^{-1}Y_k$ is in Σ_0 . In any case, it follows from the case $n = 1$, there is only a finite number of paths from X to Σ. Moreover, since sectional paths have non zero composition, there is a non zero path from X to the section.

(b) Is proved using dual arguments or going to the opposite category. □

Proposition 24. *Let $\mathcal{K}$ be a preprojective component $\Gamma(KQ)$ and Σ a section of $\mathcal{K}$ without infinite directed paths. Let P be an indecomposable projective. Then there exist a short exact sequence, $0 \to P \to T^0 \to T^1 \to 0$, with T^0, $T^1 \in \mathrm{add}\Sigma_0$.*

Proof. (1) We will separate the proof in two cases, assuming first Q is not of type A_∞, A_∞^∞ or D_∞. We use the isomorphism $\mathcal{K} \cong K(-\mathbb{N}\Delta)/ < m_X >$, where m_X denotes the set of mesh relations. The projective P is identified with a vertex v_0 in Δ_0. Let $\mathcal{V}_0^- = \{v_i^- \in Q_0 \mid \text{there is a path } v_i^- \to v_0\}$, let $\mathcal{V}_0 = \{v_1, v_2 ..., v_n\}$ be the vertices of Σ that are the ending vertex of a path starting at some v_i^-. Since Σ has no infinite directed paths, the set $\mathcal{V}_1 = \{v_0, v_1 ..., v_m\}$ of all vertices of Σ that are connected with a length zero or directed path to vertices of $\mathcal{V}_0$, is also finite and contains $\mathcal{V}_0$, Σ'' is the full subquiver of Σ with vertices $\mathcal{V}_1$. Denote by $\mathcal{V}_P$ the collection of all paths starting at some vertex of $\mathcal{V}_0^-$ and ending at Σ.

Let $\{T_0, T_1 ..., T_m\}$ be the objects of $\mathcal{K}$ corresponding to the vertices $\mathcal{V}_1$ and denote by $\mathcal{P}$ denote the finite set of indecomposable projective that appear as summands in the minimal projective presentations of any of the T_i for $1 \leq i \leq m$. The objects in $\mathcal{P}$ correspond to vertices of a finite subquiver Q'' of Q, which by Lemma 8 is contained in a finite, full, non Dynkin subquiver $Q'$of Q.

Let $\Delta' = Q'^{op}$, since Δ' is not Dynkin, $(-\mathbb{N}\Delta')$ is a connected, full subquiver of $(-\mathbb{N}\Delta)$ and $\mathcal{V}_P$ is completely contained in $(-\mathbb{N}\Delta')$. We identify the preprojective component $\mathcal{K}'$ of the Auslander-Reiten quiver $\Gamma(KQ')$, with $K(-\mathbb{N}\Delta')/ < m'_X >$, where m'_X is the set of mesh relations in $K(-\mathbb{N}\Delta')$. Let's consider the ideal I of

$K(-\mathbb{N}\Delta)$ defined by:

$f \in I(X,Y)$, if and only if, f factors through $\tau^{-i}Z$, for some $Z \in \Delta_0/\Delta'_0$, $i \in \mathbb{N}$

By the universal properly of quiver algebras, there exists a functor:

$$\bar{F} : K(-\mathbb{N}\Delta') \to \frac{K(-\mathbb{N}\Delta)}{I+ < m_X >}$$

The set of meshes can be separated in three kinds: (i) Meshes m'_{X_j} which are also meshes in $K(-\mathbb{N}\Delta)$, (ii) The mesh m_{X_k} of $K(-\mathbb{N}\Delta)$ that is of the form $\Sigma_{\{\alpha\in\Gamma(\Lambda)|F(\alpha)=X_k\}}\alpha\sigma(\alpha)$, where $\alpha\sigma(\alpha)$ factors through an object in $(-\mathbb{N}\Delta)\setminus(-\mathbb{N}\Delta')$ and (iii) The meshes m_{X_l} of $K(-\mathbb{N}\Delta)$ of the form $m_{X_l} = \rho^1_j + \rho^2_l$, where ρ^1_l is a sum of morphisms factoring through some object in $(-\mathbb{N}\Delta) \setminus (-\mathbb{N}\Delta')$ and ρ^2_l consists of morphims which do not factor through $(-\mathbb{N}\Delta) \setminus (-\mathbb{N}\Delta')$. Then it is clear that the kernel of $\bar{F}$ is $< m'_X >$ and there exists a full, faithful and dense functor:

$$F : \frac{K(-\mathbb{N}\Delta')}{< m'_X >} \to \frac{K(-\mathbb{N}\Delta)}{I+ < m_X >}$$

Let Σ' be the subquiver of Σ consisting of all vertices which correspond to orbits under the inverse Auslander-Reiten translation of projective corresponding to the vertices of Q', since Q' is non Dynkin, Σ' is a section of $\mathcal{K}'$.

(2) According to [HR Theo. 7.2], there exists a short exact sequence of KQ'-modules

$$0 \to P \xrightarrow{(f_i)_i} \coprod_i T_i \xrightarrow{(g_{ji})_{ji}} \coprod_j T_j \to 0$$

with T_i and T_j corresponding to vertices in Σ'_0, and f_i, g_{j_i} are K-linear combinations of paths in $K(-\mathbb{N}\Delta')$.

By the equivalence in part (1), $(f_i)_i$ is a monomorphism and $\sum_i g_{ji}f_i = 0$ in $\mathcal{K}$.

We have the exact sequence of KQ-modules

$$0 \to P \xrightarrow{(f_i)_i} \coprod_i T_i \xrightarrow{(h_{ki})_{ki}} \coprod_k C_k \to 0$$

where $C = \mathrm{Coker}(g_{ji})_{ji}$, is the cokernel of $(g_{ji})_{ji}$ in $\mathcal{K}$ and $C = \coprod_k C_k$ its decomposition in sum of indecomposable KQ-modules.

By the universal property of the cokernel, there exists a morphism $(l_{jk})_{jk}$: $\coprod_k C_k \to \coprod_j T_j$ such that the map g_{ij} is equal to $\sum_k l_{jk}h_{ki} : T_i \to T_j$. By condition (S3) of the definition of section, each $C_k \in \Sigma_0$.

If Q is of type A_∞, A^∞_∞ or D_∞, then we can choose $\Delta' = Q'^{op}$ large enough in order to the injective KQ'-modules of the Auslander-Reiten quiver of $\Gamma(KQ')$ appear as successor of Σ'' and then apply an argument similar to the first case. $\square$

We next prove he main result in this subsection.

Theorem 9. *Let Q be a locally finite infinite quiver, K a field and KQ the quiver algebra consider as a preadditive category. Let $\mathcal{K}$ be a preprojective component of the Auslander-Reiten quiver of KQ and Σ a section of $\mathcal{K}$ without infinite oriented paths. Then,* $\mathrm{add}\Sigma_0$ *is a tilting subcategory of* $\mathrm{mod}(KQ)$.

Proof. Since Q is locally finite, KQ is hereditary and (iii) was proved in Proposition 24, we only need to prove condition (ii). Let T_1 and T_2 be non projective objects in Σ_0, since KQ is a dualizing variety, Auslander-Reiten formula $\mathrm{Ext}^1_{KQ}(T_1,T_2)=\mathrm{D}(\mathrm{Hom}_{KQ}(T_2,\tau T_1))$ holds. Let $\varphi : T_2 \to \tau T_1$ be a non zero morphism, then using the Ausalnder-Reiten sequence we have morphisms between indecomposable objects:

$$T_2 \to \tau T_1 \to E_i \to T_1$$

From condition (S3), $\tau T_1 \in \Sigma$, a contradiction. □

Corollary 7. *If* $\mathcal{T} = \mathrm{add}\Sigma_0$ *is a tilting subcategory, then* $\mathrm{Mod}(\mathcal{T})$ *has global dimension one.*

Proof. Let P be an indecomposable projective in $\mathrm{mod}(\mathcal{T})$. Since $\mathcal{T}$ is Krull-Schmidt $P \cong (\ \ ,T)$ with T in Σ_0. By Brenner-Butler's theorem, the subcategory $\mathscr{Y} = \{N|\mathrm{Tor}^1_{\mathcal{T}}(N,\mathcal{T}) = 0\} \subset \mathrm{mod}(\mathcal{T})$ is a torsion free class containing the projective, and it is equivalent to $\mathscr{T} = \{M|\mathrm{Ext}^1_{\mathcal{C}}(T_i,M) = 0, T_i \in \Sigma_0\} \subset \mathrm{mod}(\mathcal{C})$, via the functor ϕ.

Let Y be a non zero sub-object of P. Then there is a monomorphism $\alpha : Y \to P$. Since P is contained in the subcategory $\mathscr{Y} \subset \mathrm{mod}(\mathcal{T})$, then $Y \in \mathscr{Y}$, since it is closed under sub-objects. By the equivalence $\phi : \mathscr{T} \to \mathscr{Y}$, there exists an indecomposable object M in $\mathscr{T}$ such that $\phi(M) = (\ \ M)_{\mathcal{T}} \cong Y$. Moreover, the inclusion $Y \hookrightarrow P$ is of the form $(\ \ ,f) : (\ \ ,M)_{\mathcal{T}} \to (\ \ ,T)$, with $f : M \to T$ a non zero morphism. Since Y is a non zero functor, there exists an object $T' \in \Sigma_0$ such that $0 \neq Y(T') = \mathrm{Hom}(T',M)$. Let $g \in \mathrm{Hom}(T',M)$ be a non zero morphism, then there is a chain of morphisms

$$T' \xrightarrow{g} M \xrightarrow{f} T$$

Then by Property (S3) of a section, M is in Σ_0 and we conclude Y is projective. □

3.2.1. *Representations of Infinite Dynkin Diagrams.* In this sub section we will apply the results of the previous subsection to compute the Auslander-Reiten quivers of the infinite Dynkin quivers A_∞, A_∞^∞ or D_∞ without infinite paths.

The Auslander Reiten quivers of the infinite Dynkin quivers were computed in [ReVan III. 3] for a fixed orientation. We will apply here the tilting theory so far developed to compute the Auslander Reiten quivers for arbitrary orientations.

We give first the computation of the Auslander Reiten for an infinite Dynkin quiver with only sinks and sources, then we change the orientation by tilting with the objects in a section of the preprojective component with no infinite paths, and we prove that tilting with objects in a section behaves as in the finite dimensional case, it removes a portion from the preprojecive component and glues it on the preinjective component leaving the other components invariant.

Proposition 25. *Let Q be an infinite Dynkin quiver with only sinks and sources, $\Gamma(Q)$ the Auslander-Reiten quiver of Q then:*

(a) $\Gamma(A_\infty)$ *have only two components: the preprojective and the preinjective components.*
(b) $\Gamma(D_\infty)$ *have three components: the preprojective, the preinjective and a regular component, the regular component are of type* A_∞.
(c) $\Gamma(A_\infty^\infty)$ *has 4 components: the preprojective, the preinjective and two regular components, the regular components are of type* A_∞.

Proof. In the case of locally finite infinite quivers, we have almost split sequences and we can compute them, and the preprojective component, as in the finite dimensional case; proceeding by induction starting with the indecomposable projective.

Every indecomposable representation has finite support, hence it can be considered as a representation of a finite Dynkin quiver.

(a) Consider the quiver $A_\infty : 0 \leftarrow 1 \rightarrow 2 \leftarrow 3 \rightarrow \cdots$, for each pair of integers $b \geq a \geq 0$, let $M_a^b \in \text{rep}(A_\infty)$, be the representation defined as $(M_a^b)_i = K$ if $a \leq i \leq b$ and 0 in the remaining vertices. The simple projective objects are $P(2n) = M_{2n}^{2n}$, and the non simple projective are $P(2n+1) = M_{2n}^{2(n+1)}$, with $n \geq 0$. The non projective representation M_a^b with b even, can be written as M_{2k}^{2m}, M_{2k+1}^{2m}, with $m > k+1$, $k \geq 0$ and M_1^{2m}, with $m \geq 1$. A computation shows $\text{DTr}M_{2k}^{2m} = M_{2(k+1)}^{2(m-1)}$, $\text{DTr}(M_{2k+1}^{2m}) = M_{2k-1}^{2(m-1)}$, and $\text{DTr}(M_1^{2m}) = M_0^{2(m-1)}$, hence each M_a^b with b even, is in the preprojective component. By a similar computation M_a^b with b odd is in the preinjective component.

We look now to the quiver D_∞

$$\begin{array}{ccccccccc} 0 & \leftarrow & 2 & \rightarrow & 3 & \leftarrow & 4 & \rightarrow & \cdots \\ & & \downarrow & & & & & & \\ & & 1 & & & & & & \end{array}$$

We use the fact that every indecomposable representation has support in a finite Dynkin quiver whose representations we know, (See [R]).

For each pair of integers m,n, with $m \geq n \geq 2$ we have indecomposable representations, M_n^m, defined as follows: $(M_n^m)_i = K$, if $n \leq i \leq n$, and 0 in any other vertex. For $m \geq 2$ we define the indecomposable representations

$$(N_0^m)_i = \begin{cases} K & \text{if } 1 \leq i \leq m, \\ 0 & \text{if } i = 0 \text{ or } i > m. \end{cases}, \quad (N_1^m)_i = \begin{cases} K & \text{if } i = 0 \text{ or } 1 < i \leq m, \\ 0 & \text{in any other vertex.} \end{cases}$$

For integers m and l, with $m > l \geq 2$, let's define

$$(L_l^m)_i = \begin{cases} K^2 & \text{if } 2 \leq i \leq l, \\ K & \text{if } i \in \{0,1\} \text{ or } l+1 \leq i \leq m, \\ 0 & \text{in the other vertices.} \end{cases}$$

For each integer $m \geq 0$, let's define

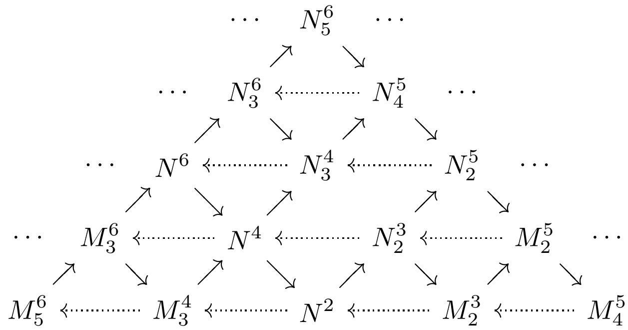
$$(L^m)_i = \begin{cases} K & \text{if } 0 \leq i \leq m, \\ 0 & \text{in the other vertices.} \end{cases}$$

To compute the preprojective components of the Auslander-Reiten quiver $K(D_\infty)$ we compute the orbits of $P(1)$ and $P(0)$ under $\text{Tr}D$ and to compute the preinjective component we take the orbits of $I(1)$ and $I(0)$ under $D\text{Tr}$, to obtain

$$\{P(1), N_1^3, N_0^5, N_1^7, N_0^9, \ldots\}, \ \{P(0), N_0^3, N_1^5, N_0^7, N_1^9, \ldots\}$$
$$\{I(1), N_1^4, N_0^6, N_1^8, N_0^{10}, \ldots\}, \ \{I(0), N_0^4, N_1^6, N_0^8, N_1^{10}, \ldots\}$$

We see that the representations that lies in the preproyective component are: N_1^m, N_0^m, L^m and M_n^m with n and m odd or 0, and L_l^m with l and m odd. The representations that lies in the preinyective component are: N_1^m, N_0^m, M_n^m with n and m odd or 0, and N_l^m with l and m even.

Finally, the representations M_n^m, L_n^m with $m+n$ and $l+m$ odd, and L^m, M_0^m with m even, lies in a component of type $\mathbb{Z}A_\infty$.



(b) We consider next the quiver $A_\infty^\infty : \cdots \to -2 \leftarrow -1 \to 0 \leftarrow 1 \to 2 \leftarrow 3 \to \cdots$ As above, for each pair of integers $b \geq a$, let $M_a^b \in \operatorname{rep}(A_\infty^\infty)$, be the representation defined as $(M_a^b)_i = K$ if $a \leq i \leq b$ and 0 in the remaining vertices. The projective simple are $P(2n) = M_{2n}^{2n}$, and the non simple projective are $P(2n+1) = M_{2n}^{2(n+1)}$, with $n \in \mathbb{Z}$. The non projective representations M_a^b with a and b even can be written as M_{2k}^{2m} with $m > k+1$, and $k \geq 0$, and we have $\mathrm{TrD}(M_{2k}^{2m}) = M_{2(k+1)}^{2(m-1)}$, and by induction M_a^b with a and b even are in the preprojective component. Using the same argument, we can see that in case a and b are odd, then M_a^b is in the preinjective component.

If a is even and b odd, then $\mathrm{DTr}(M_a^b) = M_{a+2}^{b+2}$, and all these representations are in a regular component, if a is odd and b is even, then $\mathrm{DTr}(M_a^b) = M_{a-2}^{b-2}$ and we obtain the elements of the second regular component.

If a is even and b odd. The almost split sequences are of the form

$$0 \to M_{a+2}^{b+2} \to M_a^{b+2} \to M_a^b \to 0$$
$$0 \to M_{a+2}^{b+2} \to M_{a+2}^b \coprod M_a^{b+2} \to M_a^b \to 0$$

the first appears in on the border of the regular component

If a is odd and b even. The almost split sequences are of the form

$$0 \to M_{a-2}^{b-2} \to M_{a-2}^b \to M_a^b \to 0$$
$$0 \to M_{a-2}^{b-2} \to M_a^{b-2} \coprod M_{a-2}^b \to M_a^b \to 0$$

the first appears in on the border of the regular component

Proceeding as in the finite dimensional case, we see that in the three cases the preprojective components are of the form $(-\mathbb{N}Q, \tau)$ and the preinjective components of the form $(\mathbb{N}Q, \tau)$. □

If Q is a locally finite quiver then KQ is a dualizing variety, if Σ is a section without infinite paths, then $\mathcal{T} = \mathrm{add}\Sigma_0$ of $\mathrm{mod}(KQ)$, the functor $\phi : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Mod}(\mathcal{T})$ restricts to the category of finitely presented functors $\phi_{\mathrm{mod}(\mathcal{C})} : \mathrm{mod}(\mathcal{C}) \to \mathrm{mod}(\mathcal{T})$. We also proved $\mathcal{T}$ is a dualizing hereditary category, therefore both KQ and $\mathcal{T}$ splits.

Choosing a section without infinite paths corresponds with changes in the orientation of the quiver Q. We observe next that tilting with the objects of a section without infinite paths in the preprojective component, behave as in the finite quiver situation; The Auslander Reiten components of the tilted category are as follows:

we cut all the predecessors of the section in the preprojective component to get the preprojective component of the tilted category. We glue what we cut as successors of the injective to build the preinjective component. The regular components remain without changes.

Let $\mathcal{P}(\mathcal{C})$ be the preprojective component of the Auslander-Reiten quiver, $\Gamma(KQ)$ and $\mathcal{Q}(\mathcal{T})$ the preinjective component of the Auslander-Reiten quiver de $\Gamma(\mathcal{T})$. By Auslander-Reiten formula we have

$$\begin{aligned}\mathscr{T}(\mathcal{T}) &= \{X \in \text{mod}(KQ) | \text{Ext}^1(T, X) = 0, \, T \in \Sigma_0\} \\ &= \{X \in \text{mod}(KQ) | \text{Hom}(X, \tau T) = 0, \, T \in \Sigma_0\}.\end{aligned}$$

We also know

$$\mathscr{F}(\mathcal{T}) = \{X \in \text{mod}(KQ) | \text{Hom}(T, X) = 0, \, T \in \Sigma_0\}.$$

Definition 15. *The set of predeccessors of Σ (successors, succΣ), preΣ, is the set of indecomposables X such that there is a $T \in \Sigma_0$ and an integer $n > 0$ with $T = \tau^{-n}X$ ($T = \tau^n X$).*

Lemma 10. *Let X be an indecomposable object in* $\text{mod}(KQ)$*. Then, X is in $\mathscr{T}(\mathcal{T})$, if and only if, X is not a predecessor of Σ. Moreover, $\mathscr{F}(\mathcal{T}) = pre\Sigma$.*

Proof. Assume X is in $\mathscr{T}(\mathcal{T})$. If X is a predeccesssor of Σ, then, by Theorem 8, there is a non zero path from X to $\tau\Sigma$, a contradiction.

Assume now X is not a predeccessor of Σ. If there exists, $T \in \Sigma_0$ and a non zero map $f : X \to \tau T$, then for some positive integer k, $\tau^k T$ is projective, hence X is preprojective and it has to be a succesor of Σ. Then there is a $T_0 \in \Sigma_0$ and a non zero map $g : T_0 \to X$.

Hence we have maps: $T_0 \to X \to \tau T \to E \to T$, contradicting the fact, $T_0, T \in \Sigma_0$.

The last claim follows from the fact that the tilting category $\mathcal{T}$ separates. □

We have the following:

Proposition 26. *The follwing statements hold:*

(a) *For each $T \in \Sigma_0$, the $\mathcal{T}$-module* $\text{Ext}^1(\ \ , \tau T)_{\mathcal{T}}$ *is injective.*
(b) *For any positive integer k, and any object T in Σ, there is an isomorphism* $\tau^k \text{Ext}^1(\ \ , \tau T)_{\mathcal{T}} \cong \text{Ext}^1(\ \ , \tau^{k+1}T)_{\mathcal{T}}$.
(c) *Given an indecomposable projective $(-, C)$ there is a natural isomorphism in* $\text{mod}(\mathcal{T}^{op}) : D(\text{Ext}^1(-, (-, C))_{\mathcal{T}}) \cong \tau(((-, C), -)_{\mathcal{T}})$.

Proof. (a) For each $X \in \text{mod}(\mathcal{T})$, and each non projective $T \in \mathcal{T}$ there is an isomorphism

$$(X, \text{Ext}^1(\ \ , \tau T)_{\mathcal{T}}) \cong DX(T).$$

Indeed, let $0 \to (\ \ , T_1) \to (\ \ , T_0) \to X \to 0$ be a projective resolution of X, and T in $\mathcal{T}$ non projective. Applying Auslander-Reiten formula, there is an isomorphism $\eta : (X, \text{Ext}^1(\ \ , \tau T)_{\mathcal{T}}) \to DX(T)$, such that the following diagram commutes

$$\begin{array}{ccccc} 0 \to (X, \text{Ext}^1(\ \ , \tau T)) & \to & ((\ \ , T_0), \text{Ext}^1(\ \ , \tau T)) & \to & ((\ \ , T_1), \text{Ext}^1(\ \ , \tau T)) \\ \eta\downarrow & & \cong\downarrow & & \cong\downarrow \\ 0 \to D(X(T)) & \longrightarrow & D(T, T_0) & \longrightarrow & D(T, T_1) \end{array}$$

(b) Consider an almost split sequence in $\mathscr{F}(\mathcal{T})$:

$$0 \to \tau^2 T \to E \to \tau T \to 0$$

By Lemma 5, it induces an almost split sequence in $\mathscr{X}(\mathcal{T})$:

$$0 \to \mathrm{Ext}^1(-,\tau^2 T)_{\mathcal{T}} \to \mathrm{Ext}^1(-,E)_{\mathcal{T}} \to \mathrm{Ext}^1(-,\tau T)_{\mathcal{T}} \to 0$$

from which it follows $\tau\mathrm{Ext}^1(\ ,\tau T)_{\mathcal{T}} \cong \mathrm{Ext}^1(\ ,\tau^2 T)_{\mathcal{T}}$.
The rest is by induction.

(c) Let $(-,C)$ be an indecomposable projective in $\mathrm{mod}(KQ)$, then there is an exact sequence:

$$0 \to (-,C) \to T_1 \to T_0 \to 0$$

which induces by the long homology sequence exact sequences:

$$0 \to (-,(-,C)) \to (-,T_1) \to (-,T_0) \to \mathrm{Ext}^1(-,(-,C))_{\mathcal{T}} \to \quad 0$$

$$0 \to (T_0 -) \to (T_1,-) \to ((-,C),-)_{\mathcal{T}} \to 0$$

The second exact sequence gives a projective presentation of $((-,C),-)_{\mathcal{T}}$, taking the transpose and dualizing we obtain the exact sequence:

$$0 \to \tau(((-,C),-)_{\mathcal{T}}) \to D((-,T_0)) \to D((-,T_1)) \to 0$$

Dualizing the first exact sequence we obtain an exact sequence:

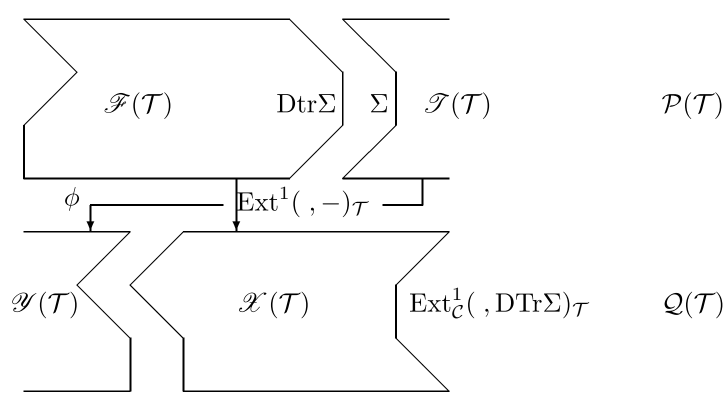
$$0 \to D(\mathrm{Ext}^1(-,(-,C))_{\mathcal{T}}) \to D((-,T_0)) \to D((-,T_1)) \to 0$$

which implies:

$$D(\mathrm{Ext}^1(-,(-,C))_{\mathcal{T}}) \cong \tau(((-,C),-)_{\mathcal{T}})$$

□

The results in the proposition can be interpreted as the construction of the Auslander-Reiten components of $\mathrm{mod}(\mathcal{T})$ by gluing the predecessors of Σ as successors of the injectives and leaving the remaining components unchanged, as illustrated in the following diagram:



As a corollary we obtain the main theorem of the subsection.

Theorem 10. *Let Q be a locally finite infinite Dynkin quiver, $\Gamma(Q)$ the Auslander-Reiten quiver of Q then:*

(a) $\Gamma(A_\infty)$ *have only two components: the preprojective and the preinjective components.*
(b) $\Gamma(D_\infty)$ *have three components: the preprojective, the preinjective and a regular component, the regular component are of type* A_∞.
(c) $\Gamma(A_\infty^\infty)$ *has 4 components: the preprojective, the preinjective and two regular components, the regular components are of type* A_∞.

3.2.2. *The regular components.* In the previous subsection we were concerned with the preprojective components and with all the components of a locally finite infinite quiver of type: A_∞, A_∞^∞ or D_∞. In this subsection we study the regular components of an arbitrary locally finite infinite quiver and prove that the regular components are of type A_∞. To prove this, we will follow very closely the proof given by ABPRS, [see ARS]. Since KQ is hereditary and the radical of $\mathrm{mod}(KQ)$ has properties very similar to the finite dimensional case, we can also use length arguments. We follow the first part of ABPRS's proof to conclude that the number of indecomposable summands, $\alpha(M)$, of a KQ-module M in a regular component, is at most three, and in case $\alpha(M)=3$, there exist chains of irreducible monomorphisms $B_{i,t_i}\to B_{i,t_{i-1}}\to\cdots\to B_{i,1}=B_i\to M$, with $\alpha(B_{i,t_i})=1$ and $\alpha(B_{i,j})=2$ for $j<t_i$, where

$$0\to \mathrm{DTr}M \xrightarrow{\begin{pmatrix} f_1\\ f_2\\ f_3\end{pmatrix}} B_1\coprod B_2\coprod B_3 \xrightarrow{\begin{pmatrix} g_1 & g_2 & g_3\end{pmatrix}} M\to 0$$

is an almost split sequence (see [ARS Prop. 4.11]). To exclude the case $\alpha(M)=3$, we will reduce to a finite quiver situation to obtain a contradiction. We will make use of the following lemma:

Lemma 11. *Let* $\mathcal{C}$ *be a* Hom*-finite dualizing locally finite Krull-Schmidt* K*-category. Let* $\mathcal{B}=\{B_i\}_{i\in I}$ *be a finite family of objects in* $\mathrm{mod}(\mathcal{C})$. *For each* $i\in I$, *consider almost split sequences in* $\mathrm{mod}(\mathcal{C})$

$$0\to \mathrm{DTr}(B_i)\xrightarrow{f_i} E_i\xrightarrow{g_i} B_i\to 0.$$

Then, there is a finite subcategory $\mathcal{C}'\subset\mathcal{C}$, *such that the restriction*

$$0\to \mathrm{DTr}(B_i)|\mathcal{C}'\xrightarrow{g_i|\mathcal{C}'} E_i|\mathcal{C}'\xrightarrow{f_i|\mathcal{C}'} B_i|\mathcal{C}'\to 0$$

is an almost split sequence in $\mathrm{mod}(\mathcal{C}')$

Proof. We leave the proof to the reader. □

Theorem 11. *Let* $Q=(Q_0,Q_1)$ *be a connected* ***locally finite*** *infinite quiver and* M *an indecomposable module in the regular component of the Auslander-Reiten quiver* $\Gamma(KQ)$. *Then* $\alpha(M)\le 2$.

Proof. Since we already know the shape of the Auslander-Reiten components of the infinite Dynkin quivers, we may assume Q is non Dynkin.

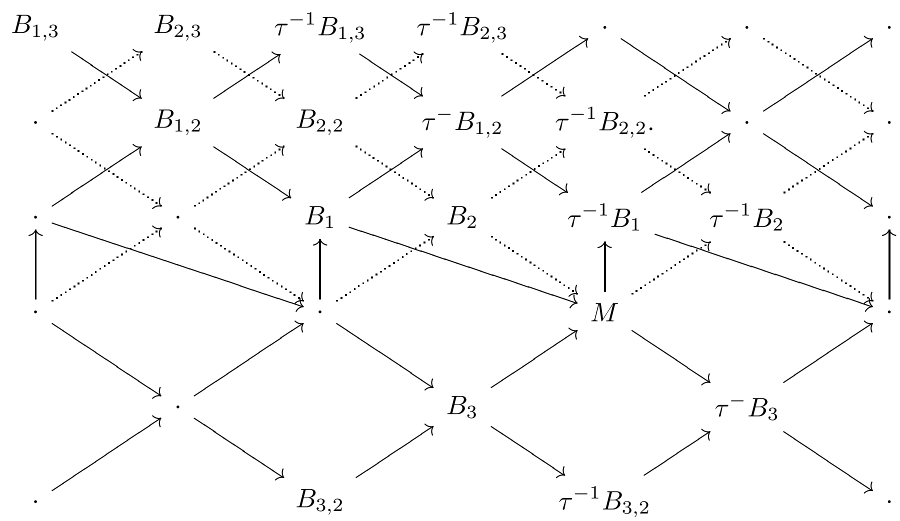
Let's suppose $\alpha(M)=3$, and let $0\to\mathrm{DTr}(M)\to\coprod_{i=1}^3 B_i\to M\to 0$ be an almost split sequence and chains of irreducible monomorphisms $B_{i,t_i}\to\cdots\to B_{i,1}=B_i\to M$, $i=1,2,3$, as above.

(a) We proceed as in the lemma 11, defining $\mathcal{B}=\{M\}\cup\{B_{i,j}\}\cup\{\mathrm{TrD}(B_{i,j})\}$, $i=1,2,3$, $1\le j\le t_i$, to find a subcategory $\mathcal{C}'\subset\mathcal{C}$, such that the almost split sequences of the objects in $\mathcal{B}$, restrict to almost split sequences in $\mathrm{mod}(\mathcal{C}')$

$$
\begin{aligned}
0 &\to \mathrm{DTr}(B_{i,j})|\mathcal{C}' \to E_i|\mathcal{C}' \to B_{i,j}|\mathcal{C}' \to 0, 1 \leq j \leq t_i \\
0 &\to B_{i,j}|\mathcal{C}' \to E_i'|\mathcal{C}' \to \mathrm{TrD}B_{i,j}|\mathcal{C}' \to 0, 1 \leq j \leq t_i \\
0 &\to \mathrm{DTr}M|\mathcal{C}' \to \coprod_{i=1}^{3} B_i|\mathcal{C}' \to M|\mathcal{C}' \to 0,
\end{aligned}
$$

Adding a finite number of objects, if needed, we can identify $\mathcal{C}'$ with KQ', where Q' is a finite connected non Dynkin full subquiver of Q. Observe that these almost split sequences will be almost split sequences for any quiver algebra of a finite subquiver Q'' of Q'containing Q'', since it will contain the support of the objects in the sequences.

Since $\alpha(M) = 3$, the module $M|\mathcal{C}'$ is in a preprojective or in a preinjective component. We assume it is in the preprojective component, the other case will follow by duality.



Since we are assuming Q' is non Dynkin, the preprojective component is of the from $(-\mathbb{N}\Delta', \tau)$, with $Q' = \Delta'$, hence, the section consisting of the three paths $B_{i,t_i} \to \cdots \to B_{i,1} = B_i \to M,\ i = 1, 2, 3$ is isomorphic to Q' after a change of orientation. But for any larger subquiver Q'' the algebra KQ'' will have the same section in its preprojective component as KQ', which implies Q is a finite quiver, contradicting our hypothesis. □

These results are used in the next section.

The shape of the Auslander-Reiten components of infinite quivers was found in an independent way by [BSP].

4. The Auslander Reiten Components of a regular Auslander Reiten Component

In the last section we use the results of the previous section to describe the Auslander Reiten components of a regular Auslander-Reiten component of a finite dimensional algebra, to do this, we need the concepts and results of [MVS1], [MVS2], [MVS3], which we briefly recall here.

Let Λ be a finite dimensional algebra over an algebraically closed field K. We denote by mod_Λ the category of finitely generated left Λ-modules and by $\mathcal{A}_{gr}(\text{mod}_\Lambda)$ the category with the same objects as mod_Λ and whose morphisms between A and B in $\mathcal{A}_{gr}(\text{mod}_\Lambda)$ are given by:

$$\text{Hom}_{\mathcal{A}_{gr}(\text{mod}_\Lambda)}(A,B) = \textstyle\coprod_{i\geq 0} \text{rad}^i_\Lambda(A,B)/\text{rad}^{i+1}_\Lambda(A,B).$$

In [MVS1] the definition of Koszul and weakly Koszul categories was given and it was proved that mod_Λ is weakly Koszul and $\mathcal{A}_{gr}(\text{mod}_\Lambda)$ is Koszul.

Koszul categories are a natural generalization of Koszul algebras and the main results of Koszul theory extend to Koszul categories in particular we have Koszul duality.

Next, recall the construction of the Ext-category of a full subcategory of an abelian category. For an abelian category $\mathcal{D}$ we consider the Ext-category $E(\mathcal{D}')$ of the full subcategory $\mathcal{D}'$of $\mathcal{D}$. The graded category $E(\mathcal{D}')$ has the same objects as $\mathcal{D}'$and maps given by:

$$\text{Hom}_{E(\mathcal{D}')}(A,B) = \coprod_{i\geq 0} \text{Ext}^i_{\mathcal{D}}(A,B),$$

for all objects A and B in $E(\mathcal{D}')$.

We recall the application of this construction to $\text{mod}(\text{mod}_\Lambda)$. Let's denote by $\mathcal{C}$ the category $\text{mod}(\text{mod}_\Lambda)$ and by $\mathcal{S}(\mathcal{C})$ the full additive subcategory of $\mathcal{C}$ generated by the simple functors

$$\text{S}_C\text{=Hom}_\Lambda(\text{-}, C)/\text{rad(-,C)}:(\text{mod}_\Lambda)^{op},$$

for all indecomposable objects C in mod_Λ. The category$\mathcal{A}_{gr}(\text{mod}_\Lambda)$ is graded and according to [IT] the category of graded functors $Gr(\mathcal{A}_{gr}(\text{mod}_\Lambda))$ has global dimension two. Then the Ext-category $E(\mathcal{S}(\mathcal{C}))$ has the same object as $\mathcal{S}(\mathcal{C})$ and maps:

$$\text{Hom}_{E(\mathcal{A}_{gr}(\text{mod}_\Lambda))}(S_A, S_B) = \coprod_{i\geq 0} \text{Ext}^i_{\mathcal{C}}(S_A, S_B)$$

The category $E(\mathcal{S}(\mathcal{C}))$ is locally finite of Loewy length 3. Moreover, the following theorem was proved in [MVS3]:

Theorem 12. *Let C be an indecomposable Λ-module, and consider the indecomposable projective functor $P_C = \text{Hom}_{E(\mathcal{A}_{gr}(\text{mod}_\Lambda))}(-, S_C)$ in $Gr(E(\mathcal{A}_{gr}(\text{mod}_\Lambda)))$. Then one of the following statements is true:*

(i) *P_C is a simple projective, when C is a simple injective Λ-module.*
(ii) *P_C is a projective of Loewy length 2, when C is a non simple injective module.*
(iii) *P_C is a projective injective functor of Loewy length 3, when C is non injective.*

Let $\mathcal{K}$ be a fixed component of the Auslander-Reiten quiver of Λ, *$Agr(\mathcal{K})$* is the full subcategory of $\mathcal{A}_{gr}(\text{mod}_\Lambda)$ generated by objects in $\mathcal{K}$ the corresponding Ext-category *$E(Agr(\mathcal{K}))$* is equivalent to *$E(\mathcal{K})$*. The full subcategory $\text{gr}_{\mathcal{PI}}(E(\mathcal{K})^{op})$ of $\text{gr}(E(\mathcal{K})^{op})$ consisting of objects in $\text{gr}(E(\mathcal{K})^{op})$ without projective-injective summands it is equivalent to a radical square zero category. Radical square zero categories are stably equivalent to hereditary K-categories. The stable equivalence is obtained in a way similar to the radical square zero artin algebras.

Proposition 27. *[MVS3]Let $\mathcal{K}$ be the additive closure of a connected component of the Auslander-Reiten quiver of a finite dimensional algebra Λ over an algebraically closed field K .*

(a) *The category $gr(E(\mathcal{K})^{op}/rad^2) \cong gr(Agr(\mathcal{K})/rad^2)$ is stably equivalent to $gr(\mathcal{H})$, where $\mathcal{H}$ is a disjoint union of hereditary full subcategories corresponding to the sections of the Auslander-Reiten component of $\mathcal{K}$ given by $\mathcal{H}_C$ for an indecomposable module C in $\mathcal{K}$.*

(b) *Each subcategory $gr(\mathcal{H}_C)$ is equivalent to the category of representations of the separated quiver of the component $\mathcal{K}$ of the Ausalnder-Reiten quiver at C.*

For a regular component we have he following:

Theorem 13. *[MVS3] Let $\mathcal{K}$ be a connected regular component of the Auslander-Reiten quiver of a finite dimensional algebra Λ over an algebraically closed field K. Then*

(a) *The category $gr(E(\mathcal{K})^{op}/rad^2) \cong gr(Agr(\mathcal{K})/rad^2)$ is stably equivalent to $gr(\mathcal{H})$, where $\mathcal{H}$ is a disjoint union $\underset{i\in Z}{\cup}\mathcal{H}_i$ of hereditary full subcategories corresponding to the sections of the Auslander-Reiten component of $\mathcal{K}$*

(b) *The categories $\mathcal{H}_i$, $\mathcal{H}_{i+2}$ are equivalent and the categories $\mathcal{H}_i$, $\mathcal{H}_{i+1}$ are opposite categories for all i.*

(c) *If $\mathcal{H}_i$ is finite, then it is non-Dynkin.*

It was also proved in [MVS3] (See also [MVS2]) that a regular component $\mathcal{K}$ the category *$Agr(\mathcal{K})$* is Artin-Schelter regular, its Ext-category *$E(\mathcal{K})$* is locally finite, Frobenius of radical cube zero. We will have a situation very similar to the preprojective algebra and we can use the same line of ideas as in [MV] to prove the following:

Theorem 14. *The following statements hold:*

(1) *$E(\mathcal{K})$ is selfinjective of radical cube zero, for any indecomposable object M in $gr_{\mathcal{PI}}(E(\mathcal{K})^{op})$ generated in degree zero $\Omega(M)$ is either simple generated in degree 2 or it is generated in degree one.*

(2) *The indecomposable non Koszul objects in $gr(E(\mathcal{K})^{op})$ are the objects M such that for some integer n, the n-th syzygy $\Omega^n(M)$ is simple.*

(3) *For any indecomposable non projective object M the Ausalnder-Reiten translation is $\Omega^2(\mathcal{N}(M))$, with $\mathcal{N}$ the Nakayama equivalence. Hence for $n = 2k$ the n-th syzygy $\Omega^n(M)$ is simple, if and only if, $Dtr^k M$ is simple and for $n = 2k+1$, the syzygy $\Omega^n(M)$ is simple, if and only if , $Dtr^{k-1}M$ is $P/socP$, with P an indecomposable projective.*

(4) *$gr_{\mathcal{PI}}(E(\mathcal{K})^{op})$ is radical square zero. It is stably equivalent to the disjoint union $\underset{i\in Z}{\cup}\mathcal{H}_i$ of hereditary full subcategories corresponding to the sections of the Auslander-Reiten component of $\mathcal{K}$ and the categories $\mathcal{H}_i$, $\mathcal{H}_{i+2}$ are equivalent and the categories $\mathcal{H}_i$, $\mathcal{H}_{i+1}$ are opposite categories for all i, the functor producing the stable equivalence sends all the almost split sequences $0 \to DtrM \to E \to M \to 0$ with $DtrM$ non simple, to almost split sequences and nodes (in this case all simple are nodes) correspond to two simple objects, one projective and one injective.*

(5) *The Auslander-Reiten components of $E(\mathcal{K})$ containing the projective objects are built by gluing for each i, a preprojective component of $\mathcal{H}_i$ with a*

preinjective component of $\mathcal{H}_i$ by identifying the simple projective with the corresponding simple injective objects and adding projective injective objects to recover the almost split sequences with a projective injective in the middle term.

(6) *All the regular components are of type A_∞ .*
(7) *The objects in the preinjective and regular components are Koszul.*
(8) *The objects in the preprojective and regular components are co Koszul.*
(9) *The category of indecomposable non projective Koszul object in $E(\mathcal{K})$ has almost split sequences to the left and they are almost split in the whole category $gr(E(\mathcal{K})^{op})$.*

Corollary 8. *The following statements hold:*

(a) *If $\mathcal{K}$ is of type A_∞ it is a tube, then $E(\mathcal{K})$ has a finite number of preprojective components and it has no regular component.*
(b) *If $\mathcal{K}$ is of type A_∞ and it is not a tube, then $E(\mathcal{K})$ has a countable number of preprojective components and it has no regular component.*
(c) *If $\mathcal{K}$ is of type D_∞ , then $E(\mathcal{K})$ has a countable number of preprojective components and it has no regular component.*
(b) *If $\mathcal{K}$ is of type A_∞^∞, then it has a countable number of preprojective components and a countable number of regular components of type A_∞*

If $\mathcal{K}$ is a regular component, then $E(\mathcal{K})$ is Koszul and $E(E(\mathcal{K})) = Agr(\mathcal{K})$. Koszul duality $\Psi : K_{E(\mathcal{K})} \to K_{Agr(\mathcal{K})^{op}}$ sends almost split sequences to almost split sequences, from these facts, we have the following analogous of the preprojective algebra: ([MV] Theorem 2.8 and [MVS3]).

Theorem 15. *Let $\mathcal{K}$ be a regular Auslander-Reiten component of a finite dimensional algebra Λ. The Koszul functors of $Agr(\mathcal{K})$ have the following properties:*

(i) *Every indecomposable, non simple, non projective Koszul functor M has projective dimension one.*
(ii) *For every indecomposable, non simple, Koszul functor M, there exists a non splittable short exact sequence graded objects and maps: $0 \to M \to E \to r^2\sigma M[2] \to 0$*

    where σ is an auto equivalence of $Agr(\mathcal{K})$ and r denotes the radical. The objects E and $r^2\sigma M$ are Koszul and the sequence is almost split in $K_{Agr(\mathcal{K})^{op}}$.
(iii) *The indecomposable Koszul functors of $Agr(\mathcal{K})$ are distributed in components, preprojective components, corresponding to the preprojective components of the sections of $\mathcal{K}$ and regular components, corresponding to the regular components of the sections of $\mathcal{K}$.*
(i') *Every indecomposable, non simple, non injective co Koszul functor M has injective dimension one.*
(ii') *For every indecomposable, non simple, co Koszul functor M, there exists a non splittable short exact sequence graded objects and maps: $0 \to \sigma M/soc^2\sigma M \to E \to M \to 0$ where σ is an auto equivalence of $Agr(\mathcal{K})$ and $soc^2 M$ denotes the second socle. The objects E and $\sigma M/soc^2\sigma M$ are co Koszul and the sequence is almost split in the category of co Koszul functors $coK_{Agr(\mathcal{K})^{op}}$.*

(iii') *The indecomposable co Koszul functors of $Agr(\mathcal{K})$ are distributed in components, preinjective components, corresponding to the preinjective components of the sections of $\mathcal{K}$ and regular components, corresponding to the regular components of the sections of $\mathcal{K}$.*

This theorem has a nicer interpretation in the quotient category module the functors of finite length [MVS3]. We recall this construction. We will obtain results similar to the case considered in [MZ].

Let $\mathcal{K}$ be a regular Auslander-Reiten component of a finite dimensional algebra Λ. To simplify the notation we will denote by $\mathcal{C}$ the category $Agr(\mathcal{K})$ and by $\mathrm{gr}(\mathcal{C})_0$ the finitely presented graded functors and degree zero maps, by [MVS1] $\mathrm{gr}(\mathcal{C})_0$ is abelian.

Denote by tors($\mathcal{C}$) the full subcategory of $\mathrm{gr}(\mathcal{C})_0$ of all functors of finite length. This is a Serre category, we can take the quotient category Qgr($\mathcal{C}$)= $\mathrm{gr}(\mathcal{C})_0$/tors($\mathcal{C}$).

The category $Qgr(\mathcal{C})$ has the same objects as $\mathrm{gr}(\mathcal{C})_0$ and maps:

$\mathrm{Hom}_{Qgr(\mathcal{C})}(\pi M, \pi N) = \varinjlim_{(M',N')\in\mathcal{L}} Hom_{gr(C)_0}(M', N/N')$, where $\mathcal{L}$={(F,G)| F⊂M, G⊆N, and M/F, G in tors($\mathcal{C}$)}.

Let π :$\mathrm{gr}(\mathcal{C})_0 \to Qgr(\mathcal{C})$ be the canonical projection. It is known [P], $Qgr(\mathcal{C})$ is abelian and π is exact. If we denote by $t(M) = \sum\limits_{L\in tors(\mathcal{C})} L$ and L a subfunctor of M, then t is an idempotent radical and we say that a functor M is torsion if $t(M) = M$ and torsion free if $t(M) = 0$.

It was proved im [MVS2] that for a finetely presented functor M the torsion part $t(M)$ is of finite lenght, in particular finetely presented.

Denote by $M_{\geq k}$ the truncation subfunctor of the graded functor M, this is $(M_{\geq k})_j$ =0, if $j < k$ and $(M_{\geq k})_j = M_j$, if $j \geq k$. By definition, $M/M_{\geq k}$ is of finite length and the maps in $Qgr(\mathcal{C})$ can be written as follows:

$\mathrm{Hom}_{Qgr(\mathcal{C})}(\pi M, \pi N) = \varinjlim_k Hom_{gr(C)_0}(M_{\geq k}, N/t(N))$.

Since $\pi(M) \cong \pi(M/t(M))$ we may always assume N is torsion free and in such a case the exact sequence: $0 \to M_{\geq k} \to M_{\geq k-1} \to M_{k-1}/M_{\geq k} \to 0$ induces an exact sequence: $0 \to Hom_{gr(C)_0}(M_{k-1}/M_{\geq k}, N) \to Hom_{gr(C)_0}(M_{\geq k-1}, N) \to Hom_{gr(C)_0}(M_{\geq k}, N)$ whose first therm is zero and $\varinjlim_k Hom_{gr(C)_0}(M_{\geq k}, N) = \bigcup\limits_k Hom_{gr(C)_0}(M_{\geq k}, N)$.

Since $\mathcal{C}$ has global dimension two, any finitely presented functor M has a truncation $M_{\geq k}$ such that $M_{\geq k}[k]$ is Koszul (see [MVS3] or [AE]) and in $Qgr(\mathcal{C})$ to objects πM and πN are isomorphic, if and only if there are truncations $M_{\geq k}$, $N_{\geq k}$ such that $M_{\geq k} \cong N_{\geq k}$ in $\mathrm{gr}(\mathcal{C})_0$.

Hence $Qgr(\mathcal{C}) = \bigcup\limits_{i\in Z} \widetilde{K}_{\mathcal{C}}[i]$ where $\widetilde{K}_{\mathcal{C}}[i]$ is the image under π of the Koszul $\mathcal{C}$-functors, shifted by i. The situation is analogous to the sheaves on projective space studied in [MZ].

From this it follows:

Theorem 16. *If we denote by $\mathcal{C}$ the associated graded category of a regular component of a finite dimensional algebra, then the category $Qgr(\mathcal{C})$ has almost split sequences, they are of the form: $0 \to \pi M[k] \to \pi E[k] \to \pi\sigma M[k+2] \to 0$ with σ an auto equivalence of $\mathcal{C}$. The category $Qgr(\mathcal{C})$ is the union of connected components of the Auslander-Reiten quiver and these components are of the following kind: Proj[k] and Reg[k], this means the k-th shift of $\pi(\mathrm{Pr}\,oj_{\mathcal{C}})$ or $\pi(\mathrm{Reg}_C)$, where $\mathrm{Pr}\,oj_{\mathcal{C}}$*

and Reg $_{\mathcal{C}}$ denote a preprojective component and a regular component of $\mathcal{C}$, respectively. $\pi(\Pr oj_{\mathcal{C}})$ and $\pi(\Pr oj_{\mathcal{C}})$, respectively, $\pi(\mathrm{Reg}_C)$ and Reg_C, have isomorphic translation quivers.

Proof. By the above observation, any indecomposable object in $Qgr(\mathcal{C})$ is of the form $\pi(M)[\ell]$ with M an indecomposable Koszul functor. The endomorphism ring of $\pi(M)[\ell]$ is $\mathrm{End}_{Qgr(\mathcal{C})}(\pi(M)[\ell]\ )=\underset{k}{\cup}Hom_{gr(C)_0}(M_{\geq k},M)$.

The truncations $M_{\geq k}$ [k] are isomorphic to $r^kM[k]$,hence they correspond under Koszul duality to $\Omega^kM[k]$, since E($\mathcal{C}$) is selfinjective, they are indecomposable with local endomorphism ring. A map $f\in\mathrm{End}_{Qgr(\mathcal{C})}(\pi(M)[\ell]\ )$ is represented by a map $\overline{f}:M_{\geq k}\to M$ and restricting to the image a map $M_{\geq k}\to M_{\geq k}$, hence it is either an iso or nilpotent and it follows $\mathrm{End}_{Qgr(\mathcal{C})}(\pi(M)[\ell]\ )$ is local. We have proved $Qgr(\mathcal{C})$ is Krull-Schmidt.

Consider the almost split sequence: $0\to M\overset{j}{\to}E\overset{p}{\to}r^2\sigma M[2]\to 0$ apply π and shift it, to obtain to obtain the exact sequence: $0\to\pi M[k]\overset{\pi(j)[k]}{\to}\pi E[k]\overset{\pi(p)[k]}{\to}\pi\sigma M[k+2]\to 0$ if the sequence splits. there exists a map h:$\pi\sigma M[k+2]\to\pi E[k]$ such that $\pi(p)[k]h=1$.

The map h is of the form $\pi(t)$ with $t:r^\ell\sigma M[k+2]\to r^\ell E[k]$. It follows the exact sequence $0\to r^\ell M[\ell]\overset{j}{\to}r^\ell E[\ell]\overset{p}{\to}r^{\ell+2}\sigma M[\ell+2]\to 0$ splits. Using Koszul duality $\Phi:K_{\mathcal{C}}\to K_{E(\mathcal{C})}$ it follows $0\to\Omega^{2+\ell}\sigma\Phi(M)\to\Omega^\ell\Phi(E))\to\Omega^\ell\Phi(M)\to 0$ splits. Therefore $0\to\Omega^2\sigma\Phi(M)\to\Phi(E))\to\Phi(M)\to 0$ splits, a contradiction.

Let $\pi N[j]$ be an indecomposable object with N Koszul and $f:\pi N[j]\to\pi\sigma M[k+2]$ a non isomorphism. As above, there exists a map $t:N_{\geq\ell}\to\sigma M[k-j+2]$ such that $\pi t=f$, which induces a non isomorphism of Koszul objects $t:N_{\geq\ell}[\ell]\to\sigma M[\text{k+}\ell\text{-j+2}]_{\geq \text{k}+\ell-j+2}$.

The sequence: $0\to M_{\geq\text{k+}\ell\text{-j}}[\text{k+}\ell\text{-j}]\overset{j}{\to}E_{\geq\text{k+}\ell\text{-j}}[\text{k+}\ell\text{-j}]\overset{p}{\to}r^2\sigma M_{\geq\text{k+}\ell\text{-j}}[\text{k+}\ell\text{-j+2}]\to 0$ is almost split. Then there exists a map $s:N_{\geq\ell}[\ell]\to E_{\geq\text{k+}\ell\text{-j}}[\text{k+}\ell\text{-j}]$ with $ps=t$. It follows f lifts to $\pi E[k]$.

In a similar way we prove the map $\pi M[k]\overset{\pi(j)[k]}{\to}\pi E[k]$ is left almost split.

The remaining claims are clear. □

We will end the paper with the following remark:

Remark 4. *Consider a locally finite infinite quiver Q and construct the translation quiver ZQ. For each arrow α of ZQ there exits a unique arrow $\sigma\alpha$ such that the end of $\sigma\alpha$ coincides with the starting of α, and a unique arrow $\sigma^{-1}\alpha$ starting at the end of α. We define the following set ρ of relations in ZQ:*

(i) *If α and β are arrows with the same end, then $\alpha\sigma\alpha-\beta\sigma\beta$ is a relation.*
(ii) *If γ is an arrow ending at the start of α and different from $\sigma\alpha$, then $\alpha\gamma$ is a relation.*
(iii) *If γ is an arrow starting at the end of α and different from $\sigma^{-1}\alpha$, then $\gamma\alpha$ is a relation.*

Consider the category of representations $rep(ZQ,\rho)$ over a field K of the quiver with relations (ZQ,ρ).The category $rep(ZQ,\rho)$ is by construction selfinjective of radical cube zero. It follows from the arguments in [M] it is also Koszul . The theory developed in [MVS1], [MVS2], [MVS3] [MVS4] applies to this situation and we obtain as Ext-category of $E(ZQ,\rho)$,The category of representations of the quiver ZQ with mesh relations η. It follows rep(ZQ,η) is an Artin-Shelter regular Koszul *category of global dimension two. We do not know whether or not it corresponds to an Auslander-Reiten quiver of a finite dimensional algebra, however the above theorems for regular Auslander-Reiten components hold, in particular they also hold for a connected component of the stable non regular Ausalnder-Reiten quiver, as considered in [MVS3].*

References

[A] Auslander M., *Representation Theory of Artin Algebras I.* Communications In Algebra, 1(3), 177-268 (1974).

[A2] Auslander M., *Applications of morphisms determined by modules.* Representation theory of algebras (Proc. Conf., Temple Univ., Philadelphia, Pa.,1976), pp. 245–327. Lecture Notes in Pure Appl. Math., Vol. 37, Dekker, New York, 1978.

[ABPRS] Auslander M., Bautista R., Platzeck M. I., Reiten, I., Smalø S. O., *Almost split sequences whose middle term has at most two indecomposable summands.* Canad. J. Math. 31 (1979), no. 5, 942-960.

[AE] Avramov L., Eisenbud E., *Regularity of modules over a Koszul Algebra,* J. Algebra 153 (1992), 85-90.

[AF] Anderson F. W., Fuller K. R., *Rings and Categories of Modules.* Graduate Texts in Mathematics 13, Springer Verlang 1974.

[APR] Auslander M., Platzek M. I., and Reiten I., *Coxeter functors without diagrams.* Trans. Amer. Math. Soc. 250 (1979), 1-46.

[AQM] Auslander M., *Representation dimension of Artin Algebras I.* Queen Mary College Mathematics Notes, 1971., Selected Works of Maurice Auslander, Part 1 AMS, 505-574.

[AR] Auslander M., and Reiten I., *Stable Equivalence of Dualizing R-Varietes* Adavances in Mathematics Vol.12, No.3, (1974)., 306-366.

[AR2] Auslander M.; Reiten, I. *Stable equivalence of Artin algebras.* Proceedings of the Conference on Orders, Group Rings and Related Topics (Ohio State Univ., Columbus, Ohio, 1972), pp. 8–71. Lecture Notes in Math., Vol. 353, Springer, Berlin, 1973.

[ARS] Auslander M., Reiten I. and Smalø S. O., *Representation Theory of Artin Algebras.* Cambridge University Press. 1995.

[ASS] Assem I., Simson D. and Skowroński A., *Elements of the representation theory of associative algebras, 1: Techniques of representation theory*, London Mathematical Society Student Texts, vol. 65, Cambridge Univ. Press, Cambridge, 2006.

[B] Bongartz K., *Tilted algebras* M. Auslander (ed.) E. Lluis (ed.), Representations of Algebras. Proc. ICRA III, Lecture Notes in Mathematics, 903, Springer (1981) pp. 26-38.

[BB] Brenner S., Butler M. C. R., *Generalisations of the Bernstein-Gelfand-Ponomarev reflection functors* in Proc. ICRA II (Ottawa, 1979), Lecture Notes in Math. No. 832, Springer-Verlag, Berlin, Heidelberg, New York, 1980, pp. 103-169.

[BGP] Bernstein I. N., Gelfand I.M., Ponomarev V. A., *Coxeter functors and Gabriel's theorem.* Russian Math. Surveys, 28 (1973) pp. 17-32 Uspekhi Mat. Nauk, 28 (1973) pp. 19-33.

[BLP] Bautista R., Liu S., and Paquette Ch., *Auslander-Reiten theory over an infinite quiver.* ICRA XIV Conference Abstracts, August 11 - 15, 2010.

[BR] Bautista R., *Irreducible morphisms and the radical of a category.* An. Inst. Mat. Univ. Nac. Autónoma México 22 (1982), 83-135 (1983).

[CF] Colby Robert R. and Fuller Kent R., *Equivalence and Duality for Module Categories (with Tilting and Cotilting for Rings)*, Cambridge Tracts in Mathematics, vol. 161, 2004, Cambridge University Press.

[Ha] Happel D., *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras,* London Math. Soc. Lecture Note Soc. vol. 119 (1988).

[HR] Happel, D. and Ringel C.M., *Tilted algebras.* Trans. Amer. Math. Soc. 274 (1982), no. 2, 399-443.

[IT] K. Igusa, G. Todorov, *Radical layers of representable functors,* Journal of algebra 89 (1)(1984) 105-145.

[M] Mitchell, B., *Rings with several objects.* Advances in Math. 8, 1-161 (1972).

[Mi] Miyashita Y., *Generalized Tilting Modules and Applications on Module Theory,* Math. J. Okayama. Univ. 34 (1992), 75-98.

[MV] Martínez-Villa R., *Applications of Koszul algebras: The preprojective algebra,* Representation Theory of Algebras, CMS Conf. Proc. vol. 18, Amer. Math. Soc., Providence, RI (1996), pp. 487-504.

[MVS1] Martínez-Villa R., Solberg Ø., *Graded and Koszul categories,* Appl. Categ. Structures, doi:10.1007/s10485-009-9191-6, in press.

[MVS2] Martínez-Villa R., Solberg Ø., *Artin and Schelter regular algebras and categories,* Journal of Pure and Applied Algebra, in press.

[MVS3] Martínez-Villa R., Solberg, Ø., *Noetherianity and Gelfand-Kirillov dimension of components.* J. Algebra 323 (2010), no. 5, 1369-1407.

[MVS4] Martínez-Villa R., Solberg Ø., *Serre duality for Artin-Schelter regular K-categories.* Int. J. Algebra 3 (2009), no. 5-8, 355-375.

[MZ] Martínez-Villa R., Zacharia D., *Auslander-Reiten sequences, locally free sheaves and Chebysheff polynomials.* Compos. Math. 142 (2006), no.2, 397–408.

[P] Popescu N., *Abelian categories with applications to rings and modules.* Acad. Press (1973).

[ReVan] Reiten I., Van Den Bergh M., *Noetherian Hereditary Abelian Categories Satisfying Serre Duality.* Journal of the American Mathematical Society, Vol. 15, No. 2, 295-366 (2002).

[R] Ringel C. M., *Tame algebras and integral quadratic forms,* Lecture Notes in Mathematics, 1099, Springer (1984)

[Ri] Rickard J., *Morita Theory for Derived Categories.* J. London Math. Soc. 1989, 39, 436-456.

[S] Stenström Bo., *Rings of Quotients.* 217, Springer-Verlang, New York 1975.

[SM] Sounders Mac Lane. *Categories for the Working Mathematician.* Springer-Verlag,. New York, I971.

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