Theory and Methodology

Myopic and stationary solutions for stochastic cash balance problems

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Abstract: This paper presents constructive solutions to stochastic cash balance problems. The model studied is in a periodic review framework with convex holding and shortage costs levied both in the beginning and at the end of each period. Transaction costs are linear, with the possibility of unequal fixed and proportional parts for orders and disposals. Optimal myopic solutions are presented for convex and linear costs. The application of an integral of the cumulative demand distribution yields simple formulae for solving the exact values of the policy parameters. The logistic distribution is used to obtain results readily applicable in day-to-day decision-making. Approximations of the Wilson formula type are developed and compared with earlier contributions. Conditions for the optimality of myopic policies are also presented. Stationary solutions are developed, and compared with myopic results in order to evaluate the amount of suboptimality. The stationary distribution and the optimal policy are given for the bilateral exponential demand distribution case. All results have been programmed and tested against the local finance market data.

Keywords: Financial planning, cash management, ordering policy, myopic

1. Introduction

Financial planning has become increasingly important in recent years because of high interest rates and turbulent changes in the business world. Cash balance plan formulation is now widely used in practical financial management. Many of the financial models have been derived from inventory theory and have been extensively analyzed in the literature (cf. stochastic inventory framework in cash management of a bank [1]). Deterministic cash balance problems are typically formulated as linear programs (LP) (see review of deterministic cash flow management [22]), while stochastic ones often employ a control limit approach [6]. Stochastic models have been formulated and solved as dynamic programs (DP), or have been described using Markovian processes, which are subsequently solved by means of LP. Neave [18], Vickson [23] and others [20] have studied the continuous state space DP problem. [7,13] among others have used the discrete state space Markovian approach.

The stochastic steady state problem has been analyzed in the framework of continuous time with the demand generated by a Wiener process by Constantinides [4,5] and also by [19] using...
discrete time periods with the bilateral exponential distributions. The cash transaction flow in the classical Miller–Orr contribution [16] is a symmetric Bernoulli process, i.e. both the inflow and the outflow are exactly of size \( m \) and have the probability \( \frac{1}{2} \). The optimality of simple policies and the respective solutions have been analyzed in the framework of continuous time by Constantinides [4,5] and also in the discrete time framework with unimodal demand distributions by Vickson [23] and in [20].

The crux of the multi-period problem with a finite planning horizon is to prove the optimality of a simple policy (see equation (6)) and to derive formulae for the parameters of such a policy. Linear programming can be used to solve optimal policies. In practical multi-period applications, however, the number of discrete states tends to be large enough to explode the pivot matrix of the linear program to an impracticable size. On the other hand, continuous state space dynamic programming techniques and their solutions suffer from the curse of dimensionality when the number of time periods is large. Myopic one-period solutions have been suggested for avoiding these analytical and computational difficulties. The suboptimality of the myopic solutions have proved beneficial in many day-to-day decision situations. Conditions for the optimality of myopic strategies in the countable and finite state space case have been studied e.g. by [24] and [12].

The framework of discrete time periods is chosen here for several reasons: (i) organizations typically plan and control their finances in discrete intervals. Monthly or weekly cash balance planning, for example, is in common use, (ii) data for forecasting demand are usually stored in periodic aggregates and are readily available as such. Reorganizing these data for the purposes of forecasting a non-stationary demand process involves extra effort, which is often quite difficult, (iii) there is no acceptable risk theory for non-stationary demand for a single period within the planning horizon, as the cumulative demand for all periods is used in this case, and (iv) when changing demand trajectories are allowed, calculation of optimal control becomes a veritable burden.

The prime objective of this study is to develop accurate closed-form myopic and stationary solutions and to apply them in decision situations. A new idea propounded here is integration of the cumulative demand distribution. Simple solutions are derived with this approach. The demand distribution used here is logistic distribution. It can be construed as an approximation of normal distribution [17], the applicability of which, in the case of demand for cash, has been established using the central limit theorem [14], and analyzed empirically [3]. This approximation should be quite reasonable in cash balance applications in which the quantity of available demand data is often insufficient for distinguishing between logistic and normal distributions. Actually, the logistic distribution leads to more cautious policies. This is because the tail of the former is heavier than that of the latter. The main results of the study are given by the solutions of the reorder and disposal point equations (9) and (10), and the stationary solutions (20), (21). These are derived for the general case of convex cost functions. Special cases are elaborated on, and programmed for solving empirical problems.

Stationary problems are studied. Here the basic idea is to analyze the amount of suboptimality in the myopia assumption. It turns out that the stationary distribution cannot be solved with conventional tools whenever the demand distribution is not one-sided, or a sum of such. Use of the bilateral exponential distribution here permits novel results for the comparisons and also for day-to-day planning.

Finance market data are then applied to evaluate the results and to test the convergence of the derived solutions. With these data only two iterations were needed to produce solutions within 1% of optimal.

2. Problem formulation

Consider a stochastic cash balance model with a periodic review. Demand \( \delta \) is a random variable, formed as the difference between cash inflows \( \delta^+ \) and cash outflows \( \delta^- \). Orders for an disposals of cash are controlled variables, the control of which is made at the beginning of each period. The amount \( x \) of cash is available at the beginning of period \( n \), \( n = 1, \ldots, N \). Let \( y \) denote the amount of cash available after the control decision of the period \( n \), that is, after an order or a disposal has been made or no control action taken.
The transaction cost \( a_n(y-x) \) is defined as:

\[
a_n(y-x) = \begin{cases} 
K_n + k_n \cdot (y-x) & \text{if } y-x > 0, \\
0 & \text{if } y = x, \\
Q_n + q_n \cdot (x-y) & \text{if } y-x < 0.
\end{cases}
\]

where \( K_n, Q_n, k_n, q_n \geq 0 \). The transaction cost of a firm can be explained as follows: An order for cash is handled in the usual manner by a bank, which charges a fixed fee \( K_n \) for the transaction, and a sum \( k_n \), including e.g. the stamp fee and a commission such as the credit reservation provision, which is proportional to the amount of the transaction. Similar costs apply to the disposal of cash to stocks or earning asset accounts.

(ii) The retain and penalty cost \( m_n(y) \) levies the cash level \( y \) at the beginning of each period. Cash level \( y \) is thus subject to a retain (alternative business opportunity) cost \( r_n(y) \), when \( y > 0 \), and a penalty cost \( p_n(-y) \), when \( y \leq 0 \), and \( m_n(y) \) is defined by:

\[
m_n(y) = \begin{cases} 
r_n(y) & \text{if } y > 0, \\
p_n(-y) & \text{if } y \leq 0.
\end{cases}
\]

(iii) The holding and shortage cost \( l_n(z) \) charges the cash level \( z \) at the end of each period. The amount of cash remaining at the end of period \( n \) is \( z = (y - \delta) \), in which \( \delta \) is a value of the period's demand. The firm faces a minimum (or zero) compensating-balance requirement. Here the optimal balance at this point is nil. Any positive balance is subject to a holding cost \( h_n(z) \), and any negative to a shortage cost \( s_n(-z) \):

\[
l_n(z) = \begin{cases} 
h_n(z) & \text{if } z > 0, \\
s_n(-z) & \text{if } z \leq 0.
\end{cases}
\]

(iv) Random demand variables \( \delta_1, \delta_2, \ldots, \delta_n, \ldots, \delta_N \) are independent, have density functions \( \phi_1, \phi_2, \ldots, \phi_n, \ldots, \phi_N \) which possess moments of all orders, and have cumulative distributions \( \Phi_1, \Phi_2, \ldots, \Phi_n, \ldots, \Phi_N \).

The expected holding and shortage costs are called the loss function, denoted by \( L_n(y) \), and defined by:

\[
L_n(y) = \int_{-\infty}^{\infty} l_n(y-\delta) \phi_n(\delta) \, d\delta.
\]

where \( r_n(y) \), \( p_n(-y) \geq 0 \). If the cost functions \( r_n(\cdot) \) and \( p_n(\cdot) \) are linear, the retain \( r_n \) and penalty \( p_n \) cost ratios are used, i.e. \( r_n(y) = r_n \cdot y \) and \( p_n(y) = p_n \cdot (-y) \), where \( r_n \), \( p_n \geq 0 \). The retain cost \( r_n(y) \) indicates the cost of losing the best alternative business use for cash in the amount \( y \). The penalty cost \( p_n(-y) \) results from an intentional negative cash balance such as an overdraft in a current account.
mal discounted value of future costs in the begin-
ning of period \( n \) is \[18\]:
\[
C_n(x) = \inf_{y} \{ a_n(y-x) + m_n(y) + L_n(y) \\
+ \alpha \phi_n \ast C_{n+1}(y) \}, \tag{5}
\]
in which \( \alpha \) is a discount factor, \( 0 \leq \alpha \leq 1 \), and \( C_{N+1}(x) \geq 0 \) is the terminal cost function, and \( \ast \) stands for the convolution.

A transaction rule \( y_n(x) \) such that
\[
y_n(x) = \begin{cases} 
T_n & \text{if } x < t_n, \\
x & \text{if } t_n \leq x \leq u_n, \\
U_n & \text{if } x > u_n,
\end{cases} \tag{6}
\]
where \( t_n \leq T_n \leq U_n \leq u_n \), is called a simple policy [5]. This rule is also commonly called the two-sided \((s, S)\)-policy, or the \((t, T, U, u)\)-policy [18]. The optimality of a simple policy has been studied in continuous-time [5] and discrete-time frameworks [18,20,23]. It can be based on convexity in the myopic case (cf. Appendix), and it has been proved with convex costs \( m_n(y) \) also in a multi-period case [15].

3. Optimal myopic solutions

When the discount factor \( \alpha \) in (5) is set to zero\(^1\), the dynamic model is reduced to one-period sub-models (cf. the myopic selection principle [21]). These models are called myopic [12,24]. Their solutions are discussed below. Subscripts \( n \) referring to time periods are subsequently ignored.

3.1. General convex model

Let \( G(x, y) \) be the sum of costs in (5) for \( \alpha = 0 \). Then the policy is called myopic and defined by
\[
C(x) = \inf_{y} \{ ( -kx + K ) \sigma(y-x) \\
+ (qx + Q) \sigma(x-y) + G(x, y) \}, \tag{5M}
\]
in which
\[
G(x, y) = \{ k \sigma(y-x) - q \sigma(x-y) \} y \\
+ m(y) + L(y),
\]
and where \( \sigma(\cdot) \) is a unit function of the Heaviside type:
\[
\sigma(y-x) = \begin{cases} 
1 & \text{if } y-x > 0, \\
0 & \text{if } y-x \leq 0.
\end{cases}
\]

The Appendix shows that the convexity of \( m(y) \) and \( l(y-\delta) \) implies that of \( G(x, y) \) (in \( x < y \) and in \( x > y \), respectively). Consequently, the optimal transaction rule of (5M) is a simple policy.

The optimal policy of the general convex model in (5M) is set forth in (7)–(10) below. The order level \( T \) satisfies
\[
L'(T) = -k - m'(T). \tag{7}
\]
The disposal level \( U \) satisfies
\[
L'(U) = q - m'(U), \tag{8}
\]
Similarly, the reorder point \( t (\leq T) \) is defined by
\[
L(t) - L(T) = K + k(T-t) + m(T) - m(t), \tag{9}
\]
and the disposal point \( u (\geq U) \) by
\[
L(u) - L(U) = Q + q(u-U) + m(U) - m(u). \tag{10}
\]

The estimation of a convex holding and shortage cost function \( l(y-\delta) \) can be based on the observations of the pairs \( (l_j, y_j-\delta_j) \), \( j = 1, 2, 3, \ldots, J \), applying the method of least squares (cf. [10]). The solution of the optimal policy (7)–(10) can then be obtained by numerical methods.

3.2. The linear model

Many times available data do not permit an estimation of a general convex holding and shortage cost \( l(y-\delta) \); linear holding \( h \) and shortage \( s \)

\( ^1 \) Costs that are levied at the end of the period, and should therefore formally be discounted. Here the substitution \( \alpha = 0 \), nevertheless, leaves \( L_n(y) \) intact and yields a model without discounting.
cost ratios can then be used as approximations where \( h(y - \delta) = h \cdot (y - \delta) \), \( s(\delta - y) = s \cdot (\delta - y) \), and \( h, s \geq 0 \). Then the loss function \( L(y) \) in (4) has the derivative
\[
L'(y) = -s + (h + s) \Phi(y),
\]
where \( \Phi(y) = \text{Prob}[\delta \leq y] \).

Provided that the retain \( r(y) \) and penalty \( p(-y) \) costs are also linear, i.e. \( r(y) = r \cdot y \), \( p(-y) = p \cdot (-y) \) and \( r, p \geq 0 \) the order and disposal levels are defined simply by
\[
\Phi(T) = \frac{(-k - r + s)}{(h + s)},
\]
if \( T > 0 \), and \( r \) is replaced by \(-p\) if \( T < 0 \), and
\[
\Phi(U) = \frac{(q - r + s)}{(h + s)},
\]
where \( r \) is replaced as above with \( U \) as the argument.

Integrating (11) one obtains
\[
L(T) - L(t) = h(T - t) - (h + s) \{ \Gamma^*(t) - \Gamma^*(T) \},
\]
where \( \Gamma^*(x) \) is defined by
\[
\Gamma^*(x) = \int_x^{\infty} (1 - \Phi(y)) \ dy.
\]
Substituting this into (9) and (10) yields
\[
\Gamma^*(t) - \Gamma^*(T) = \left\{ K + (k + r + h)(T - t) \right\} / (h + s),
\]
for \( t, T \geq 0 \), and \( r \) is replaced by \(-p\) for \( t, T < 0 \), as well as by \( (rT + pt)/(T - t) \) otherwise, and
\[
\Gamma^*(u) - \Gamma^*(U) = \left\{ Q + (q - r - h)(u - U) \right\} / (h + s),
\]
where \( r \) is replaced similarly with \( U \) and \( u \) were arguments. It is of interest to note that cost parameters and demand terms have been separated in (9\textsuperscript{LG}) and (10\textsuperscript{LG}).

4. Optimal myopic policy in the logistic distribution case

It has been shown in [14] with the central limit theorem that normal distribution applies to discounted cash flows. The use of normal distribution is difficult because in (9\textsuperscript{L}) and (10\textsuperscript{L}) it would require two numerical integrations. Logistic distribution, therefore, replaces the normal distribution here.

4.1. Exact solution

Unimodal and symmetric logistic distribution has been proposed as a substitution for normal distribution [2]. Its cumulative form is simple:
\[
\Phi(y) = \text{Prob}[\delta \leq y] = 1/(1 + \exp[-d(y - \mu)]),
\]
where \( \mu \) is the mean and \( d \) is formed from the standard deviation \( \sigma \) by \( d = \pi/(\sigma\sqrt{3}) \). Logistic distribution is close to normal distribution and even closer to \( t \)-distribution with nine degrees of freedom [17].

There is no clear empirical evidence as to which distribution one should use in models involving cash flows [8]. Logistic distribution is used here especially because it is easy to calculate. Use of this distribution leads to more cautious policies than normal distribution would. This is due to the convexity of \( l_n(y - \delta) \) and to the fact that the tails of logistic distribution with their form \( \exp(-|\delta|) \) are fatter than those of normal distribution with their form \( \exp(-\delta^2) \).

Substituting (14) into (7\textsuperscript{LG}) and (8\textsuperscript{LG}), and discarding \( n \):
\[
T = \mu + \{ \ln[(k + r - s)/(k + r + h)] \} / d,
\]
where the retain cost ratio \( r \) is replaced by \(-p\), if \( T < 0 \), and
\[
U = \mu + \{ \ln[(q - r - s)/(q + r + h)] \} / d,
\]
where \( r \) is also replaced by \(-p\), if \( U < 0 \).

The 'cumulative-cumulative' distribution \( \Gamma^*(x) \) in (13) is here simply
\[
\Gamma^*(x) = -\{ \ln\Phi(x) \} / d,
\]
where \( \Phi(x) \) is the cumulative distribution in (14). Now, \( t \) and \( u \) can be numerically computed from
\[
\ln\left\{ \Phi(T)/\Phi(t) \right\} = d \{ K + (k + r + h)(T - t) \} / (h + s),
\]
or (9\textsuperscript{LG})
\[ \ln \left( \frac{\Phi(U)}{\Phi(u)} \right) = d \left\{ Q + (q - r - h) \right\} \frac{(u - U)}{(h + s)}. \]  
(10LG)

Whenever \( t, T < 0 \) the retain cost ratio \( r \) is replaced by the penalty cost ratio \( -p \), and if \( t < 0 < T \) then \( r(T - t) \) is replaced by \( (rT - pt)/(T - t) \) in (9LG). The replacement of \( r \) in (10LG) is analogous with \( U \) and \( u \) as arguments.

If \( T \) and \( U \) are constant, (9LG) and (10LG) become:

\[ 1/\Phi(t) = 1 + \exp\left( -d(t - \mu) \right) \]
\[ = \exp(-\beta t + \gamma), \]  
(9LGE)

where \( \beta = d(k + r + h)/(h + s) \) and \( \gamma = dK/(h + s) + \beta T - \ln \Phi(T) \), and

\[ 1/\Phi(u) = 1 + \exp\left( -d(u - \mu) \right) \]
\[ = \exp(-\beta u + \gamma), \]  
(10LGE)

where \( \beta = d(-q + r + h)/(h + s) \) and \( \gamma = dQ/(h + s) + \beta U - \ln \Phi(U) \).

Calculation of \( t \) and \( u \) is now made by first computing \( T \) and \( U \), solving (9LGE) and (10LGE) with simple numerical methods. This computing requires only a couple of program statements in a high level computer language.

4.2. Approximate solutions

The economic order quantity of a stochastic model can be approximated with a Wilson formula type solution derived by using the first two terms of a Taylor expansion. When the demand distribution is logistic, the approximate solutions become:

\[ T - t = \left\{ \frac{2K(h + s)}{d(k + r + h)(-k - r + s)} \right\}^{1/2} \]  
(15G)

and

\[ u - U = \left\{ \frac{2Q(h + s)}{d(-q + r + h)(q - r + s)} \right\}^{1/2}, \]  
(16G)

where \( d \) is the shape parameter as defined above in (14). If the demand distribution is a bilateral exponential, one obtains

\[ T - t = \left\{ \frac{2K/b(k + r + h)}{b(k + r + h)} \right\}^{1/2} \]  
(15E)

and

\[ u - U = \left\{ \frac{2Q/b(-q + r + h)}{b(-q + r + h)} \right\}^{1/2}, \]  
(16E)

where \( b \) is the mean of both the inflows and the outflows.

When the retain and penalty cost, or \( m(\gamma) \), is nil, solutions \(^2\) (15E) and (16E) become identical with those of [9] and [19]. If the correction terms \((h + s)/(-k + s)\) and \((h + s)/(q + s)\) in (15G) and (16G), respectively, are also ignored and the standard deviation is \( \sigma \), the shape parameter \( b \) above would be \( \sqrt{2}/\sigma \), [9]. Thus the \( d \) in (15G) and (16G) would be \( b \) multiplied by \( \sqrt{\pi}/\sqrt{6} \) or about 1.13.

5. Stationary results and the amount of suboptimality

Stationary results are presented for comparison purposes and to evaluate the amount of suboptimality of the myopia assumption. These novel solutions are also available as such for day to day decisionmaking. The stationary or steady state approach is based on the assumption that each period \( n, n = 1, 2, 3, \ldots \), possesses the same cost functions \( a(y - x), m(y) \) and \( l(y - \delta) \) and that the random variables \( \delta, \delta, \delta, \ldots \) of the demand are independent and identically distributed with a density function \( \Phi(\delta) \). Now the optimal value of future costs \( C(x) \) (in (5)) tends to increase over the computational opportunities of a microcomputer when \( n \to \infty \) even with a discount factor \( \alpha < 1 \). If the future costs are discounted, and supposing that no computational limitation emerge, e.g. the method of successive approximations can be applied (cf. [11], p. 4). The expected one-period costs \( C^{(1)}(t, T, U, u, \phi) \) have thus been minimized instead of the cumulative future costs. The aim of these studies has been to find the stationary density distribution \( f(x) \), \( f(x) = \)

\(^2\) The formulae in [9] are erroneously introduced as steady state solutions, although they are actually one-period results as shown in [19].
f(x; t, T, U, u, φ), which is defined in the cash balance case by

\[ f(x) = \phi(T - x)F(t) \]

\[ + \int_{0}^{x} \phi(x - 1) f(x - 1) dx - \phi(U - x)[1 - F(u)], \quad (17) \]

where \( F(\cdot) \) is the respective stationary cumulative distribution, \( x \) the amount of cash in the beginning of a period, say \( n \), and \( x_{-1} \) is the amount of cash in the beginning of the previous period \( n - 1 \) [19].

The corresponding functional equation of the inventory theory has been solved by the renewal theory transform [11]. A cornerstone of the solution is use of the zero point provided by the one-sided demand distribution. The zero point is used when tackling the anomalies caused by the reorder point \( t \). The solution of (17) with a general demand distribution is based on random walks. Unfortunately, one can then hardly give the general solution, i.e. tackle the anomalies caused by the reorder point \( t \) and the disposal point \( u \). On the other hand, by assuming that the demand consists of the inflows \( \delta^+ \) and the outflows \( \delta^- \) the renewal theory can be used to provide the general solution of the stationary distribution \( f(x) \).

Here we consider a special case in which the inflows and the outflows are independent and exponentially distributed both with parameter \( b \). The steady state distribution \( f(x) \) is then [19]

\[ f(x) = \begin{cases} 
  f(t) \exp(b(x - t)), & x \leq t, \\
  f(t)\{1 + b(x - t)\}, & t < x \leq T, \\
  f(t)\{1 + b(T - t)\}, & T < x \leq U, \\
  f(u)\{1 + b(u - x)\}, & U < x \leq u, \\
  f(u) \exp(b(u - x)), & u < x.
\end{cases} \quad (18) \]

where \( f(t)\{1 + b(T - t)\} = f(u)\{1 + b(u - U)\} \) (the non-symmetric distribution is also given in [19]).

The one-period expected costs \( C^{(1)} \) are then (cf. [11])

\[ C^{(1)}(t, T, U, u, \phi) = \int_{-\infty}^{\infty} C(x) f(x) dx, \quad (19) \]

where \( C(x) \) is defined in (5M) and the simple policy assumption is used [19]. When the control transaction sums are bigger the fixed transaction costs are of less importance (cf. [4]), i.e. here we assume that \( K = Q = 0 \). Assume also that the service of the period cannot be started with a negative balance, i.e. \( T, U \geq 0 \), which is implied by high penalty costs \( p \gg 0 \). Then the steady state \( (T, U) \)-policy with linear cost functions is defined by [19]:

\[ U - T = \left\{ \left[ \left( 1 + 2(k + q) \right)/\left( r + h \right) \right]^{1/2} - 1 \right\}/b, \quad (20) \]

and

\[ T = \ln\left\{ \left[ h + s \right]/\left[ 2(r + h) + b(r + h)(U - T) \right] \right\}/b. \quad (21) \]

The corresponding myopic solution with a symmetric bilateral exponential distribution is [19]

\[ T = \ln\left\{ \left[ h + s \right]/\left[ 2(k + r + h) \right] \right\}/b \]

and

\[ U = \ln\left\{ \left[ h + s \right]/\left[ 2(-q + r + h) \right] \right\}/b. \quad (23) \]

If the fixed transaction costs are included, \( K, Q > 0 \), both [16] and [4] give the same formula for \( u - U \), which is essentially similar to that of the bilateral exponential demand case [19]. The contribution of Constantinides [4] yields systematically negative reorder points. It provides neither a relevant basis for comparison nor is it a realistic method in practice. The actual measurements of the amount of suboptimality and the comparisons between various solutions are based on empirical data from a bank and are presented in the following section.

6. Empirical analysis

6.1. Coefficients in local finance markets

Consider a rolling monthly liquidity budget with deterministic and stochastic cash flow components, the former of which is included in the mean of the random demand. The local cost coefficients of the analysis are currently as follows (given in marks and percent per month):

When one month's bills of exchange are used for raising cash, the order costs are: (i) \( K = 400 \)
The fixed clerical costs when buying stocks are say (iii) \( Q = 150 \) marks, and the proportional cost ratio is (iv) \( q = 1.5\% \), where the stamp tax is 0.5 and the bank commission 1\%. Stocks and bonds are not, however, used as instruments especially by smaller firms, because management lack sufficient expertise. Short-term firm-to-firm notes, locally called market money, or community deposits are the typical earning accounts used via a bank for divesting. Then the fixed transaction cost equals the clerical costs, i.e. (iii') \( Q = 150 \) marks, and the proportional transaction cost ratio is (iv') \( q = 0.0\% \).

The retain (opportunity) cost is taken as one twelfth of a typical firm's internal cost of capital, 18 percent, or (v) \( r = 1.5\% \). The penalty cost is at least double the standard two percent two week cash discount, i.e. (vi) \( p > 4\% \). Here we assume \( p = 4\% \). Intentionally negative balances at the beginning of the period might, however, be a non-feasible planning assumption. The interest rate is the basic interest, which is roughly 10.0\% and the bank's marginal 1.5\%, i.e. (vii) \( h = 0.96\% \) as a monthly ratio. The holding cost ratio \( h \) varies on the market from 0.9\% to 1.3\% depending on the firm and also on the bank, although \( h = 1\% \) could also be used as a fairly good estimate.

The four percent monthly interest rate for two-week cash discounts is also taken as the minimal shortage cost ratio, \( s \). Since \( s \) is typically difficult to estimate, solutions will be given for values (viii) \( s \geq 4\% \). Actually, provided that the possible goodwill loss could be ignored, the charge for delayed payments permitted by law is only 1.333\% per month, which would mean a very cheap shortage cost ratio. If delayed payments were applicable, this opportunity would support the realism of the shortage cost ratio \( s = 4\% \).

Two cases are analyzed by different means and standard deviations of the demand distribution as well as with different earning assets: (a) \( \mu = 100\,000, \sigma = 450\,000 \) marks, and stocks are bought, as well as (b) \( \mu = 0 \) and \( \sigma = 800\,000 \) marks, and short term notes are used. Both cases are on the scale typical of smaller firms, which deal with local bank branches and also require assistance and expertise.

6.2. Optimal policy

The optimal policy was calculated with the data given above and a simple procedure for solving formulas (9LGE) and (10LGE).

**Procedure ReorderDisposalPoint(Order_Case; \( d, \mu, k_q, r, s, K_Q, T_U; t_u \); \{ in the order case: Order_Case := true; \( k_q := k; K_Q := K; T_U := T \} \{ in the disposal case: Order_Case := false; \( k_q := -q; K_Q := Q; T_U := U \})**

\begin{verbatim}
begin
  compute \( \beta, \gamma \); \{ \( \beta \) and \( \gamma \) defined by formulas (9LGE) and (10LGE) \}
  compute \( \varepsilon \); \{ \( \varepsilon = T_t - t \) or \( u - U \) defined by formulas (15G) and (16c) \}
  initialize IterMax := 9; Iter := 1, Precision := 0.001; OldFnc := 999 999.9;
  if Order_Case
    then \( t_u := T_U - \varepsilon \)
    else \( t_u := T_t + \varepsilon \)
  end;
  repeat
    \( Fnc := 1 + \exp(-d * (t_u - \mu)) - \exp(-\beta * t_u + \gamma) \);
    \( \text{Der := } -d * \exp(-d * (t_u - \mu)) + \beta * \exp(-\beta * t_u + \gamma) \);
    if abs(Fnc) - abs(OldFnc) \leq 0
      then begin
        \( T_U := t_u \);
        \( t_u := T_U - Fnc/\text{Der} \);
        OldFnc := Fnc;
        increment Iter;
      end
    else error
  end;
  until (abs[(t_u - T_U)/t_u] \leq \text{Precision}) or (Iter > IterMax);
end;
\end{verbatim}

Two iterations were enough to produce solutions within 1\% of optimal. This performance is

3 Microcomputer programs are available upon request. They are written in Turbo Pascal and also provide pictures on screen in colour or on printer.
believed to be superior to earlier results, which are also based on Wilson type approximations as the initial value of the calculations. Optimal control parameters were computed for various values of the shortage costs, as shown in Figures 2 and 3.

6.3. Comparison of the exact, approximate and stationary solutions

The optimal policy of the exact myopic solution derived from \((9^\text{LGE})\) and \((10^\text{LGE})\) was then compared with the approximate solution defined by \((15)\) and \((16)\). The exact and approximate values of the reorder point \(t\) are presented in Figure 4.

Whenever, the shortage cost \(s\) is very high in proportion to the other costs, \(s \gg h + k\), the different myopic solutions are almost identical. When \(s\) is set at a more realistic level, the two solutions become markedly different. The exact one tends to hold a positive cash balance, which is in keeping with common business practice. The approximate one, however, will consistently suggest a negative balance. This is clearly not realistic. As the condition of a very high \(s\) is seldom met, the exact solution is believed to be much more applicable to practical financial planning. On the other hand, the fast convergence and the reasonable size of the program permits use of the procedure of the exact solution, even in microcomputers.

The amount of suboptimality is analyzed in the case of no fixed costs by comparing the developed
stationary and myopic solutions and presented in Figure 5.

The stationary model leads to slightly more cautious ordering policies, although the difference is negligible. The more conservative disposal policy of the myopic model costs more, provided that the steady state assumptions hold.

Loosely speaking, myopic policies are optimal provided that the opening balance of the next period does not depend on the closing balance of the present period [12, 24]. If a transaction is performed at the beginning of every period this is trivially true, although the solutions formulas would be slightly modified [20].

7. Conclusions

This study analyzes stochastic cash balance problems with respect to cash management decision-making. First, new solutions for the myopic model are derived. A key element in the calculation is the notion of the integral of cumulative demand distribution. Second, steady state results are given and the amount of suboptimality caused by the myopia is analyzed. Third, exact myopic solutions are programmed on a microcomputer as are two approximate ones. The steady state results are also programmed as well as compared with the different myopic approaches. Fourth, the empirical cases are construed using the characteristics of the local finance markets as a base.

It is evident that the derived contributions are easy to use and would benefit day-to-day business. Use of the integral of cumulative distribution turns out to work in practice. The developed iterative methods for computing exact solutions converge very rapidly and thus provide both myopic and steady state policy parameters which can be easily calculated and presented both numerically and graphically with microcomputers.

The empirical analysis emphasizes the relation between the various interest rates. When the firm's cost of capital is fixed, any change in the external interest rates will enhance the change in the optimal policy. When the intangible negative image costs are ignored, delaying payments is the cheapest source of financing whenever the penalty interest rate is that of the local law.

Appendix. The optimality of simple and myopic policies

The optimality of a simple policy presented above is based on the convexity of $G_n(x, y)$.

**Lemma A1.** Let $m_n(y)$ be convex, $l_n(y)$ be convex and bounded by polynomials, $a_n(y - x)$ be of form (1), and $\phi_n(\delta)$ possess moments of all orders. Then $G_n(x, y)$ is convex for $x < y$ and for $x > y$, respectively.

**Proof.** As $l_n(\cdot)$ is convex it is continuous. Since $l_n(\cdot)$ is bounded between polynomials, and because $\phi_n(\cdot)$ possesses moments of all orders $L_n = \phi_n \ast l_n$ exists by the dominated convergence theorem. $L_n(\cdot)$ is convex because $l_n(\cdot)$ is convex [18], and so is $G_n(x, y)$ in $x < y$ and $x > y$, respectively, as a sum of convex functions. □

Note that piecewise continuous densities $\phi_n(\cdot)$ are considered. For discrete distributions the results have to be reformulated. The discontinuities of $\phi_n(\cdot)$ do not generate any additional terms in (11) but those of $l_n(\cdot)$ do emerge additional functions [20].
The conditions of the optimality of a myopic policy are given in the multi-period decision situation.

**Lemma A2.** Let the statement of Lemma A1 hold. Suppose that for all \( x \) and \( n \in \{1, 2, \ldots, N\} \)
\[
\inf_y \left\{ a_n(y - x) + L_n(y) + m_n(y) + \alpha \phi_n \cdot C_{n+1}(y) \right\} \geq C_{n+1}(x). \tag{A1}
\]
If there is a myopic policy \( \lambda \) such that
\[
a_n(y - x) + L_n(y) + m_n(y) + \alpha \phi_n \cdot C_{n+1}(y) \leq C_{n+1}(x), \tag{A2}
\]
then that myopic policy \( \lambda(y(x)) \) is optimal \(^5\), and \( C_n(x) = C_{n+1}(x) \).

**Proof.** If \( C_n \) is the optimal policy, \( C_n \leq C_n^* \), where \( C_n^* \) is the optimal accrued cost of any myopic policy. One obtains \( C_{n+1} = C_{n+1}^* \) by definition. Assume that \( C_{n+1}(x) = C_{n+1}(x) = C_{n+1}^*(x) \). Then \( C_n(x) \geq C_{n+1}(x) \) because of (A1) (cf. (5) and [24]). In addition, for every \( x \) there exists a myopic policy such that \( C_n^*(x) \leq C_{n+1}(x) \). Consequently, \( C_n(x) = C_{n+1}(x) = C_n^*(x) \), and the myopic policy is optimal. \( \square \)

It is of interest to note here that if Lemma A1 holds, and if \( C_{n+1}(x) \) is convex, the myopic policy \( \lambda_n(x) \) is simple, because the cost functions are all convex.

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4 White (see [24], p. 704) also "assumes that there exists an admissible myopic strategy" without giving any consideration to the existence of such a strategy. This is a very strong assumption as one can easily construct examples which do not necessarily have a myopic solution.

5 It appears in [24] as if the Bayesian approach were relevant for proving the optimality of a myopic policy, which is actually not needed as shown in the proof of Lemma A2.

**References**


