Robust MPC for linear systems with bounded multiplicative uncertainty

Martin Evans*, Mark Cannon, Basil Kouvaritakis

Abstract—A robust tube-based Model Predictive Control (MPC) strategy is proposed for linear systems with multiplicative parametric uncertainty. The tubes are defined by sequences of polytopic sets for which we propose two methods of construction, respectively employing low-complexity paralleloptopes and polytopes of fixed but arbitrary complexity. A method of computing polytopic terminal sets of arbitrary complexity is also described. An MPC law based on the minimization of an expected quadratic cost is formulated as a quadratic program. An extension to the case of probabilistic constraints requiring the online solution of a mixed-integer program is described.

Keywords: robust control, constrained control, model predictive control, stochastic systems

I. INTRODUCTION

Robust MPC has attracted a considerable amount of research attention but development has been asymmetric in that most results cater for the case where uncertainty in the form of additive disturbances (e.g. [13][14][15][16]) rather than the case where uncertainty is multiplicative and concerns imprecisely known (or varying) system dynamics ([7][8][17]). This is due to computational complexity which in the case of multiplicative uncertainty grows exponentially: given a polytopic uncertainty set with \( \rho \) vertices for the model parameters, then the state prediction \( i \) steps ahead will lie in a polytope with \( \rho^i \) vertices. Of course such polytopic sets can be used to ensure the satisfaction of constraints (including a terminal constraint) but the implementation of such an algorithm would be impractically demanding in the case of large dimensions and/or long prediction horizons. To avoid this difficulty [7] considers an MPC scheme based on Mode 2 predictions only; the dual mode prediction paradigm assumes that in Mode 1 the control moves are degrees of freedom whereas in Mode 2 they are dictated by a predetermined control law, usually taken to be the unconstrained optimal state feedback law. The approach of [7] allows for the online computation of the Mode 2 state feedback law only, however the online computation can still become prohibitive (for anything other than low-order systems and/or short prediction horizons) and can lead to conservative results on account of the restriction of the degrees of freedom that would normally exist in Mode 1. This latter point was circumvented in [17] which extended the ideas in [7] in a way that a Mode 1 horizon could be introduced into the problem but the online computation is intractable.

The aim here is to propose an approach that reduces computation significantly yet can allows for constraints to be met non-conservatively; the approach is of course suboptimal given that use is made of a receding horizon open loop optimization strategy and that use is made of approximations to be described below. The key to this development is the definition, through one-step ahead computations, of fixed complexity polytopic sets that are guaranteed to contain the predicted states and through these, invoke conditions that ensure (explicitly in Mode 1 and implicitly through the use of a terminal set in Mode 2) the satisfaction of system constraints. Such polytopic sets have been used before (e.g. [5][9][10][12][11][14][16]) however these were derived either for the additive disturbance case ([9][14][16]), or the linearisation error case ([12]), or the stochastic case ([5]), or restricted attention to low-complexity polytopic sets ([10][11]).

A novel contribution here is that the polytopes used in Mode 1 need not be low-complexity and although they will have defined orientation (as in [14][16]) they will be individually scaled in each of these orientations. The predicted dynamics are decomposed into nominal and error parts and this allows the definition of tubes in terms a single vector, unlike [10][11] where lower and upper bounding requires the use of two vectors. The technique developed in [6] for the case of stochastic MPC will be adapted for the definition of large volume (and high-complexity) invariant polytopes as terminal sets. The use of uncertainty feedback which, though unknown at current time, would be available to a predicted control law at future prediction time instants, will be deployed to stabilise the plant dynamics for all realizations of the uncertainty. It is noted that, that as expected, the use of polytopic sets restricts the applicability of the approach to small scale systems (e.g. with no more than 10 states).

This work also offers an extension to the case of probabilistic constraints. It is shown that a simple adjustment to the robust MPC algorithm is sufficient to create a stochastic MPC algorithm if the Mode 1 polytopes are low-complexity. Finally, the methods given are compared with respect to cost and computation time by means of a numeric example.

Notation. In this paper \( \mathbb{E} \) denotes expectation and \( \mathbb{E}_k \) denotes expectation given information available at time \( k \) while \( \text{co} \) denotes the convex hull. Absolute values and inequalities involving vectors apply on an element-by-element basis.

II. PROBLEM DESCRIPTION

The system dynamics are assumed to be described by:

\[
 x_{k+1} = (A + \Delta_k)x_k + Bu_k
 \]
where $x \in \mathbb{R}^n_x$, $u \in \mathbb{R}^n_u$ denote the state and input vectors respectively and the uncertainty $\Delta_k$ lies in a polytopic set $D$

$$D = \left\{ \Delta : \Delta = \sum_{j=1}^{o} w_j \Delta^{(j)}, \ w_j \geq 0, \ \sum_{j=1}^{o} \ w_j = 1 \right\}$$

(2)

where the matrices $\Delta^{(j)}$ are known, $w_j$ are random variables and the vector $[w_1 \ldots w_o]^T$ has a known probability distribution such that $E(\Delta_k) = 0$, where $E$ denotes expectation.

The system is subject to the mixed state/input constraint

$$Fx + Gu \leq 1 = [1 1 \ldots 1]^T$$

(3)

and the aim is to minimise over $u$, the cost

$$J_k = E_k \left( \sum_{i=0}^{\infty} T_{k+i|k} Q x_{k+i|k} + u_{k+i|k} \right)$$

(4)

where $E_k(\cdot)$ denotes the expectation given information available at time $k$ and $Q \succeq 0$, $R > 0$. Clearly this is an infinite dimensional optimisation problem and instead, we undertake the repetitive online minimisation over a finite number of decision variables of a cost

$$J_k = E \left( \sum_{i=0}^{\infty} T_{k+i|k} Q x_{k+i|k} + u_{k+i|k} \right)$$

(5)

where $\{ \cdot \}_{k+i|k}$ denotes predictions at time $k + i$ given the information available at time $k$.

We adopt the closed loop paradigm according to which inputs $u_k$ are reparameterised as

$$u_k = Kx_k + c_k$$

(6)

where, as is usual, $K$ can be the unconstrained LQR optimal state feedback gain and $c_k$ are the decision variables. The controller has predictions steps $i = 1, \ldots, N$ for a horizon of length $N$, with $c_{k+i|k} = 0$ for $i \geq N$. After calculating the optimal sequence $e_{k+1|k}, \ldots, e_{k+N-1|k}$, the first element is assigned to $c_k$ to find the input (6) to the plant.

As in [6], the state predictions are decomposed into nominal $z_{k+i|k} = E(x_{k+i|k})$ and uncertain $e_{k+i|k} = x_{k+i|k} - z_{k+i|k}$. For simplicity in presentation (i.e. to avoid the need for observers and Kalman filters), the state at time $k$ is assumed known, so $e_{k|k} = 0$. To avoid instability in the uncertain predictions, we introduce error feedback with linear gain $L$, exploiting future known states, giving the decomposed dynamics:

$$z_{k+i+1|k} = \Phi z_{k+i|k} + Bc_{k+i|k}$$

(7)

$$e_{k+i+1|k} = (\Phi e + \Delta_k) e_{k+i|k} + \Delta_k z_{k+i|k}$$

where $\Phi = A + BK$ and $\Phi e = A + BL$. Accordingly, constraint (3) is reparameterised:

$$F(z + e) + G(Kz + Le + e) \leq 1.$$  

(8)

It is assumed that both $\Phi z$ and $\Phi e + \Delta_k e$ are contractive.

### III. Uncertainty Tubes

Since $e_{k+i|k}$ is uncertain, it must be contained in some set $S_{k+i|k}$ in order to impose (8). In [6] this was a discrete set, calculated from every combination of worst-case disturbances and this led to exponential growth in computation. The present work defines $S_{k+i|k}$ as a polytope, which can be visualised as the cross-section of a tube whose centre is $z_{k+i|k}$. Additional constraints are imposed that guarantee that the predicted state will be robustly contained within the tube under the uncertain dynamics. The complexity and parameterisation of this tube is discussed in Section V.

The choice of $L$ in (7) is a trade-off between tightness of the prediction tube and the amount of control action available. Given matrices $Q_e \succ 0$ and $R_e \succ 0$ defining a stage cost that penalises the error state and error feedback, a generalisation of LQR to polytopic-bounded uncertain dynamics finds the value of $L$ that minimises

$$J_e = E \left( \sum_{i=0}^{\infty} e_{k+i|k}^T Q e_{k+i|k} + e_{k+i|k}^T L^T R_e L e_{k+i|k} \right)$$

(9)

This is determined by

$$\arg \min_{\Theta} \text{tr}(\Theta) \quad \text{s.t.} \quad \Theta - \Phi^T e \Theta - \sum_{j,l} \Delta_k \Delta_k^T \Xi(w_j w_l) \succeq Q_e + L^T R_e L,$$

(10)

which is formulated as a semidefinite program and solved offline. The positive definite penalty weight $R_e$ has been included in the cost above to ensure that $L$ stabilises the error dynamics of (7) whereas $Q_e$ could be chosen to be $(F + GL)^T(F + GL)$ in order to minimise the expected effect of the error on the constraints implied by (8).

### IV. Terminal Set

We now introduce the lifted state space $\xi = [z^T e^T]^T$. In order to guarantee stability beyond the Mode 1 horizon, a terminal set is required that is robustly invariant under an autonomous Mode 2 control law $u_{k+i|k} = K z_{k+i|k} + L e_{k+i|k}$. The resulting Mode 2 dynamics are, for $i \geq N$:

$$\xi_{k+i+1|k} = \Xi(\Delta_k) \xi_{k+i|k}$$

(11)

Our construction of a robustly invariant terminal set begins with the calculation of the maximal robustly invariant ellipsoid $E$:

$$E = \{ \xi : \xi^T S \xi \leq 1, \ \tilde{F} \xi \leq 10 \}$$

(12)

The matrix $S = S^T > 0$ is found as follows:

$$\arg \min_{S} \det S \quad \text{s.t.} \quad \begin{cases} S - E(\Delta_k)^T S E(\Delta_k) > 0 \\ f_h^T S^{-1} f_h \leq 1 \end{cases}$$

(13)

where $E(\Delta_k) = \Xi(\Delta_k)$ and $f^T_h$ are the rows of $\tilde{F}$.  

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The ellipsoid $\mathcal{E}$ constitutes a conservative terminal set. Therefore we present a novel method of finding a polytopic enlargement that is still robustly invariant but has a larger volume than $\mathcal{E}$. The principle is that any state that can be robustly steered into $\mathcal{E}$ in a finite number of steps under the dynamics of (11) can be considered part of the terminal set. Finding such states enlarges the terminal set. Any state that can similarly be steered into the enlarged set also enlarges it, and so on until the rate of growth falls below a threshold.

The procedure exploits the convex nature of the problem by considering convex polytopic regions and by checking the constraints at the vertices of each region. A possible candidate for the shape of these regions is an $n$-dimensional box [2], in which case any box with a vertex that does not satisfy the constraints is partitioned and the constraints are checked at the newly generated vertices. However, the number of such boxes required to attain a suitable resolution grows very quickly with the state dimension. Finding a set of sufficient resolution therefore requires many backward steps from $\mathcal{E}$ to significantly enlarge its volume.

The present work instead represents the terminal set using a polytope $\mathcal{V}(k) = \{ \xi : V_i \xi < 1, \forall i \in I(k) \} = \text{co}\{v_j : j \in J(k)\}$, which can be initialised as any inner polytope of $\mathcal{E}$. Here, for growth iteration $k$, $I(k)$ is the set of facet indices and $J(k)$ is the set of vertex indices. A candidate new vertex $\hat{v}_k$ is selected by generating a random linear combination of the vertices of a random facet of $\mathcal{V}(k)$ such that $\hat{v}_k \notin \mathcal{V}(k)$. If $F \hat{v}_k < 1$ and the candidate vertex is robustly steered into $\mathcal{V}(k)$ under the Mode 2 dynamics, then the new vertex $\hat{v}_k$ is incorporated into $\mathcal{V}(k+1)$. This is done by finding and removing any facet $m \in I(k)$ where $V_m \hat{v}_k > 1$, and by efficiently (locally) computing the convex hull $\text{co}\{\mathcal{V}(k), \hat{v}_k\}$.

This geometric manipulation requires a dual representation of $\mathcal{V}$, whereby both the vertices and hyperplanes are stored and an ownership map $\mathcal{F}_i = \{ j \in J(k) : V_i v_j = 1 \}$, which specifies the vertices for each facet $i \in I(k)$. The complexity of $\mathcal{V}$ does not grow significantly during the expansion because facets are removed as well as added at each step (Figure 1). Polytope reduction is performed by first clustering facet normals according to their cosine metric, then creating new facet normals using each cluster average. Each new facet $i$ is scaled such that $\min_{j \in J(k)} V_i v_j = 1$. This procedure (denoted polyReduce(·) in Algorithm 4.1) reduces the number of facets of the terminal set while checking for invariance at every vertex (Figure 2). The volume can be known exactly before and during the expansion because each facet is an $(n-1)$-simplex in $n$ dimensions, so the removal or addition of such a facet corresponds to the subtraction from $\mathcal{V}$ or addition to $\mathcal{V}$ of a polyhedral cone, whose can be computed easily via coneVolume(·) in Algorithm 4.1.

Assumption 4.1: \{ $\xi : F \xi \leq 1$ \} is compact.

Under Assumption 4.1, Algorithm 4.1 is guaranteed to converge asymptotically. The algorithm terminates successfully when the rate of growth falls below a threshold. The measured rate of growth is low-pass filtered to smooth out the effect of the randomised selection of candidate vertices.

Algorithm 4.1 (Offline terminal set calculation):

Require: $k_{max} > 0$, tol > 0, $\Delta \text{vol} > \text{tol}$, 0 < $\mu$ < 1,
Ensure: $\mathcal{V}_i = \{ j \in J(k) : V_i v_j = 1 \}, \forall i \in I(k), \forall k$.
$\mathcal{V}(k) = \text{co}\{v_j : j \in J(k)\} = \{ \xi : V_i \xi \leq 1, \forall i \in I(k)\}, \forall k$.

Begin: $k \leftarrow 0$
while $k < k_{max}$ and $\Delta \text{vol} > \text{tol}$ do
h ← random element of $\mathcal{V}(k)$
$\hat{v}_k ← \text{random linear combination of } \{ v_j : j \in \mathcal{F}_h \}$
if $\{ \xi^{[j]} \hat{v}_k \in \mathcal{V}(k) : \forall j = 1, \ldots, p_h \}$
$\hat{F} \hat{v}_k \leq 1$
$D_1 ← \{ m \in I(k) : V_m \hat{v}_k > 1 \}$,
$D_j ← \bigcup_{m \in D_1} \mathcal{F}_m$
$\Delta \text{vol} ← \sum_{m \in D_1} \text{coneVolume}\{ v_j : j \in F_m \}$
$(\mathcal{A}, \text{vol}^+ ) = \text{convHull}\{ v_j, 0, \hat{v}_k : j \in D_j \}$
$\mathcal{J}(k+1) ← \mathcal{J}(k) \cup \{ \hat{v}_k \}$
$\mathcal{A} ← \mathcal{A} \setminus \{ i : 0 \in \mathcal{F}_i \}$
$\mathcal{I}(k+1) ← \mathcal{I}(k) \cup \mathcal{A} \setminus D_j$
for each $i \in \mathcal{A}$ find $V_i$ s.t. $V_i v_j = 1, \forall j \in \mathcal{F}_i$ end for
$\Delta \text{vol} ← \mu(\text{vol}^+ - \text{vol}) + (1 - \mu) \Delta \text{vol}$
else
$\mathcal{J}(k+1) ← \mathcal{J}(k)\$
$\Delta \text{vol} ← (1 - \mu) \Delta \text{vol}$ end if
$k ← k + 1$
end while

Figure 1: Growth in volume and number facets of terminal polytopic set by iteration. The marker on the y-axis shows the volume of the robust invariant ellipsoid.

V. FEASIBILITY

Two methods of representing the polytopic cross-section of the uncertainty tube are presented. The parallelepiped method described in Section V-A is simpler and has a stochastic extension to soften the mixed constraints. However, for robust MPC a higher complexity polytope as described in Section V-B can result in a larger region of attraction.
define the tube cross-sections as in (15) except that guarantee the predictions remain in the tube. Firstly we run online. We instead use set inclusion methods to however, computing vertices when in the uncertainty set definition than a parallelotope allows. B. High complexity polytope method In this method, we form an outer approximation to the set of possible values of $e_{k+i|k}$ of the form

$$V \in \{ e : |Ve| \leq \alpha_{k+i|k} \}$$

(15)

where $V$ is an invertible matrix defining face normals of the parallelotope and $\alpha_{k+i|k}$ is a vector whose elements define the scaling of each face. Using such a representation of the bounds on uncertainty in the predictions gives $2^n$ constraints per prediction step, i.e. one per vertex of the parallelotope where a vertex $v^{(r)}_{k+i|k}, r = 1, \ldots, 2^n$ satisfies $|Vv^{(r)}_{k+i|k}| = \alpha_{k+i|k}$. We now form the required constraints as in (8), where the values for $e$ are defined by these vertices.

$$\tilde{F} \begin{bmatrix} z_{k+i|k} \\ p^{(r)}_{k+i|k} \end{bmatrix} \leq 1 \quad (16)$$

We also require that the uncertain part of a prediction at time $k+i$ in the tube will remain in the tube in the successor prediction step. This requires constraints of the form

$$V(\Phi + \Delta^{(j)})u^{(r)}_{k+i|k} + \Delta^{(j)}z_{k+i|k} \leq \alpha_{k+i+1|k}.$$  

(17)

Finally, the terminal constraint is enforced by

$$X_f \begin{bmatrix} z_{k+N|k} \\ p^{(r)}_{k+N|k} \end{bmatrix} \leq 1, \quad r = 1, \ldots, 2^n. \quad (18)$$

The selection of $V$ can have an impact on optimality and region of attraction. It is obvious that hyperplanes defining $V$ should be aligned with the active constraints in order to maximally relax those constraints. However, since the active constraints are not known offline and $V$ cannot be parameterised by online decision variables, the most general solution is to use a higher complexity polytope.

B. High complexity polytope method

Relaxing the constraints requires more degrees of freedom in the uncertainty set definition than a parallelotope allows. However, computing vertices when $V$ is non-square is costly to run online. We instead use set inclusion methods to guarantee the predictions remain in the tube. Firstly we define the tube cross-sections as in (15) except that $V$ can have more rows than columns. Then, under the dynamics of (7), we have the recursion requirement for all vertices of $D$:

$$V \left( (\Phi + \Delta^{(j)})c_{k+i|k} + \Delta^{(j)}z_{k+i|k} \right) \leq \alpha_{k+i+1|k}. \quad (19)$$

If we can find $H^{(j)}$ whose elements are non-negative s.t.

$$H^{(j)}V = V(\Phi + \Delta^{(j)})$$

(20)

then we can enforce (19) with

$$H^{(j)}\alpha_{k+i|k} \leq \alpha_{k+i+1|k} - V\Delta^{(j)}z_{k+i|k}. \quad (21)$$

Proof for this is given in [1].

There may be many values of $H^{(j)}$ that satisfy (20) so we minimise the row sum of $H^{(j)}$ so as to relax (21) as much as possible. The same technique reduces the problem of ensuring the mixed constraints of (8) to finding a value of $H_m$ such that $H_mV = F + GL$ and enforcing

$$H_m\alpha_{k+i|k} \leq 1 - (F + GK)z_{k+i|k} - Gc_{k+i|k}. \quad (22)$$

Finally, the terminal constraint is enforced by finding an $H_t$ such that $H_tV = X_{f,e}$ and imposing

$$H_t\alpha_{k+N|k} \leq 1 - X_{f,z}z_{k+N|k} \quad (23)$$

where $[X_{f,z} | X_{f,e}] = X_f$.

Using the parameters given in Section VIII, one step of the optimisation was performed to illustrate the prediction tubes under this parameterisation. The result is shown in Figure 3.

![Figure 3: High complexity prediction tube, showing initial state $x_0$, tube cross-sections, terminal set and state constraint.](image-url)
In this lifted state space description, the stage cost of (5) becomes $\mathbb{E}(\chi^TQ\chi)$, with
\[
\bar{Q} = \begin{bmatrix}
    Q + K^TRK & K^TRE & K^TRL \\
    * & E^TRE & E^TRL \\
    * & * & Q + L^TRL \\
\end{bmatrix},
\]
where use has been made of the fact that the expected value of the error is zero. To evaluate the predicted cost (5), we define $P = P^T > 0$ such that
\[
P - \mathbb{E}(\Psi^TP\Psi) = \bar{Q}.
\]

**Theorem 6.1:** The cost of (5) for the prediction dynamics of (24) is quadratic in $\bar{c}_k$ and is given by $J_k = \bar{c}_k^T P \bar{c}_k$.

**Proof:** Let $V_{k+i|k} = \chi_{k+i|k}^T P \chi_{k+i|k}$, then, since $\chi_{k+i|k}$ is independent of $\Delta_{k+i}$, we have
\[
V_{k+i|k} - \mathbb{E}_{k+i}(V_{k+i+1|k}) = x^T_{k+i|k}Qx_{k+i|k} + u^T_{k+i|k}Ru_{k+i|k}.
\]
Summing (28) over $i = 0, 1, \ldots$ yields
\[
V_{k|k} = \sum_{i=0}^{\infty} \mathbb{E}_k(x^T_{k+i|k}Qx_{k+i|k} + u^T_{k+i|k}Ru_{k+i|k}),
\]
and it follows that $V_{k|k} = J_k$.

**Remark 6.1:** The cost matrix $P$ can be obtained from the solution of the semidefinite program:
\[
\arg\min_P \text{tr}(P) \quad \text{s.t.} \quad P - \mathbb{E}(\Psi)^T P \mathbb{E}(\Psi) - \sum_{j,l} \bar{\Delta}(j)^T P \bar{\Delta}(l)^T \mathbb{E}(w_j w_l) > \bar{Q}
\]
where $\bar{\Delta}(j) = [0 \ 0 \ I]^T \Delta(j) [I \ 0 \ 0]$.  

**VII. Stochastic Extension**

If the constraint (8) need only be satisfied with a given probability, namely
\[
\Pr(\tilde{F}\xi \leq \bar{1}) \geq p
\]
where the inequality applies elementwise and $p$ is a vector constraining the probability of satisfaction of each row of $\tilde{F}$, then a lower cost can be found with stochastic MPC. As discussed in [6], recursive feasibility requires that for each probabilistic constraint formed, all preceding uncertain predictions must be evaluated robustly with the worst-case uncertainty sequences. The worst-case uncertainty is the vertices of the set of possible values of $\bar{c}_{k+i|k}$, and since the vertices of a high complexity polytope are too computationally demanding to find online, the high complexity polytope method is unsuitable for stochastic MPC. This section describes an SMPC algorithm based on the parallelotope method of Section V-A.

Stochastic constraints are formed by sampling the effect of the uncertainty in (7) on the predicted state over a single prediction time-step, starting from each vertex of the robust tube. These constraints are imposed in place of (16) but with (17) and (18) remaining. Algorithm 4.1 is also modified to soften constraints by changing the line $\tilde{F}\tilde{v}_k \leq 1$ to
\[
\Pr(\tilde{F}\tilde{v}_k \leq 1) \geq p.
\]

The probabilistic constraints are imposed by Monte Carlo sampling. For a given constraint $\Pr(\gamma \leq 1) \geq p$, a sufficiently large (in the sense of [3]) number $(n_s)$ of scenarios $\gamma^{(i)}$ are created by sampling the probability distribution of $D$. The additional constraint is formed:
\[
\sum_{i=1}^{n_s} \beta(i) \geq n_s, \quad \beta = \begin{cases} 1 & \text{if } \gamma^{(i)} \leq 1 \\ 0 & \text{otherwise.} \end{cases}
\]

Our aim is to devise a recursively feasible optimization that imposes a constraint violation probability of approximately $1-p$; however we note that, using results from Random Convex Optimization (e.g. [4]), it is possible to determine bounds on the number of samples required to ensure a given level of reliability of probabilistic constraints. Constraint (33) leads to an online mixed integer programming problem (MIQP) which is handled by a purpose built branch-and-bound algorithm. Finally, by construction our SMPC has guaranteed feasibility, given feasibility at initial time, and it renders the origin of (1) asymptotically stable; the proof for this runs along the lines presented in [6].

**VIII. Numerical Comparison**

The plant model is defined by the parameters:
\[
A = \begin{bmatrix} 1 & -0.25 \\ 0.25 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
\Delta^{(1)} = \begin{bmatrix} 0 & 0.02 \\ 0.05 & 0 \end{bmatrix}, \quad \Delta^{(2)} = \begin{bmatrix} -0.01 & 0 \\ 0.05 & -0.01 \end{bmatrix}, \\
\Delta^{(3)} = \begin{bmatrix} 0.01 & -0.02 \\ 0 & 0.01 \end{bmatrix}
\]
and random coefficients $w_j = \psi_j / \sum_{i=1}^{p} \psi_i$ where $\psi_j$ are selected from a uniform distribution with support equal to the interval $[0, 1]$. The constraints and cost are defined by:
\[
G = \begin{bmatrix} 0 \\ 7 \\ -7 \end{bmatrix}, \quad F = \begin{bmatrix} -1 & 5 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad R = 8
\]
\[
n_v = 8, \quad Q_e = Q, \quad R_e = 3
\]
An initial condition that activates the mixed constraints (3) was chosen to be $x_0 = [0.40 \ 0.22]^T$.

**A. Robust control**

The two methods of parameterising the tubes were compared, using random realisations of $\Delta_0, \ldots, \Delta_4$ and a receding horizon of $N = 5$. One thousand trials were run for each method. Figure 4 shows the first fifty state trajectories for the high complexity method. The state trajectories for the parallelotope method appear identical to these.

<table>
<thead>
<tr>
<th>Method</th>
<th>RH cost</th>
<th>CPU (ms)</th>
<th>Constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parallelogone</td>
<td>1.146</td>
<td>286</td>
<td>1174</td>
</tr>
<tr>
<td>High complexity</td>
<td>1.143</td>
<td>134</td>
<td>383</td>
</tr>
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</table>

Table I: Mean cost/computation time and no. of constraints with robust MPC for low/high complexity polytopes.
B. Stochastic control

The stochastic MPC method of Section VII was simulated with the additional parameters $p = 0.6$, $n_s = 50$. Offline, the terminal set $\mathcal{X}_f$ was re-calculated with the modification (32). Figure 5 shows states that violate the state constraint yet are included in the terminal set because their successor states respect the state constraint with sufficient probability.

Due to the enlarged region of attraction, it was possible to choose an initial condition further from the origin, at $x_0 = \begin{bmatrix} 0.52 & 0.24 \end{bmatrix}^T$. The closed-loop trajectories under fifty uncertainty realisations are shown in Figure 6.

![Figure 4: The closed loop evolution of the state $x_k$ under RMPC with high complexity polytopic tube cross-section for $k = 0, \ldots, 5$, for 50 uncertainty sequence realisations.](image)

![Figure 5: Intersection with $e = 0$ subspace of the robust invariant ellipsoid $\mathcal{E}$, probabilistic state constraint $F$, stochastic enlarged polytope $V$ and reduced complexity polytope $\mathcal{X}_f$.](image)

![Figure 6: The closed loop evolution of the state under SMPC for 50 uncertainty sequence realisations, from an initial condition infeasible under RMPC.](image)

In order to compare the stochastic MPC algorithm to the robust methods described in Section V, the original initial condition $x_0 = \begin{bmatrix} 0.40 & 0.22 \end{bmatrix}^T$ was chosen for another 1000 trials that show a significant cost reduction.

<table>
<thead>
<tr>
<th>Method</th>
<th>RH cost</th>
<th>CPU (ms)</th>
<th>Constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stochastic</td>
<td>1.138</td>
<td>526</td>
<td>3470</td>
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</table>

Table II: Mean cost/computation time and number of constraints for SMPC.

IX. CONCLUSIONS

An efficient offline method for determining an enlarged terminal set is proposed for a system decomposed into nominal and uncertain parts. Two methods of parameterising uncertainty tubes for dual-mode robust MPC are compared. The high complexity polytope method is less computationally demanding than the vertex-based parallelepiped method but the latter is presented with an extension to allow probabilistic constraints while guaranteeing recursive feasibility. Robust MPC numerical results show that the reduced computational demand does not result in loss of closed-loop performance. Stochastic MPC numerical results show a significant enlargement of the region of attraction and reduced closed-loop cost, for a reasonable increase in computational demand.

REFERENCES