Octagon kite systems

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Abstract

The spectrum of octagon kite system (OKS) which is nesting strongly balanced 4-kite-designs is determined.

1 Introduction

Let \( \lambda \cdot K_v \) be the complete multigraph defined on a vertex set \( X \). Let \( G \) be a subgraph of \( \lambda \cdot K_v \). A \( G \)-decomposition of \( \lambda \cdot K_v \) is a pair \( \Sigma = (X, \mathcal{B}) \), where \( \mathcal{B} \) is a partition of the edge set of \( \lambda \cdot K_v \) into subsets all of which yield subgraphs that are isomorphic to \( G \). A \( G \)-decomposition is also called a a \( G \)-design of order \( v \) and index \( \lambda \); the classes of the partition \( \mathcal{B} \) are said to be the blocks of \( \Sigma \). Thus, \( \mathcal{B} \) is a collection of graphs all isomorphic to \( G \) such that every pair of distinct elements of \( X \) is contained in \( \lambda \) blocks of \( \Sigma \). A 4-kite is a graph \( G=C_4+e \), formed by a cycle \( C_4=(x, y_1, y_2, y_3) \), where the vertices are written in cyclic order, with an additional edge \( \{x, z\} \). In what follows, we will denote such a graph by \( [(y_1, y_2, y_3), (x), z] \). We will say that \( x \) is the centre of the kite, \( z \) the terminal point, \( y_1, y_3 \) the lateral points and \( y_2 \) the median point. A \( (C_4+e) \)-design will also called a 4-kite-design. It is

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known that a 4-kite-design of order \( v \) exists if and only if: \( v \equiv 0 \) or \( 1 \mod 5 \), \( v \geq 10 \). An \textit{octagon quadrangle} is the graph consisting of a cycle \( C_8 = (x_1, x_2, ..., x_8) \), where the vertices are written in cyclic order, with two additional chords \( \{x_1, x_4\} \) and \( \{x_5, x_8\} \). In what follows, we will denote such a graph by \( [(x_1), x_2, x_3, (x_4), (x_5), x_6, x_7, (x_8)] \). An octagon quadrangle system \( \text{OQS} \) is a \( G \)-design, where \( G \) is an octagon quadrangle. \( \text{OQS}(v) \)’s have been studied in [2][3][5]. Problems about \( G \)-designs can be found in [8][9][10][13][16].

Let \( \Sigma = (X, \mathcal{B}) \) be an \( \text{OQS} \) of order \( v \) and index \( \lambda \). We say that \( \Sigma \) is \textit{4-kite nesting}, if for every octagon quadrangle \( Q \in \mathcal{B} \) there exists at least a 4-kite \( K(Q) \in \{K_1(Q), K_2(Q)\} \) such that the collection \( \mathcal{K} \) of all these 4-kites \( K(Q) \) form a 4-kite-design of order \( \mu \). This kite system is said to be \textit{nested} in \( \Sigma \). We will call it an \textit{octagon 4-kite system} of order \( v \) and indices \((\lambda, \mu)\), briefly also \textit{OKS} or \textit{OKS}_{\lambda,\mu}, \textit{OKS}_{\lambda,\mu}(v).

If \( \Omega \) is the family of all the 4-kites \( \{K_1(Q), K_2(Q)\} \) contained in the octagon quadrangles \( Q \in \mathcal{B} \), we observe that also the family \( \mathcal{K}^c=\Omega-\mathcal{K} \) forms a 4-kite-design of index \( \mu' = \lambda - \mu \). If, for every octagon quadrangle \( Q \in \mathcal{B} \), both families of 4-kites \( \Omega_1=\{K_1(Q) : Q \in \mathcal{B}\} \), \( \Omega_2=\{K_2(Q) : Q \in \mathcal{B}\} \) form a 4-kite design of index \( \mu \), we will say that the \( \text{OQS} \) is an \textit{octagon bi-kite system}. For these systems, interesting open problems are: 1) to verify ”Berge’s conjecture” [12][15]; 2) to study the ”intersection problem” [6][7][12]; 3) to study the ”balance” [4][11].

\textbf{G-designs with two equi-indices}

Let \( G \) be a graph and let \( v = 4h + 1 \) an integer. A \( G \)-design of equi-indices \( \lambda, \mu \) is a pair \( \Sigma = (X, \mathcal{B}) \) be a pair where \( X = Z_v \) and \( \mathcal{B} \) is a collection of graphs, all isomorphic to \( G \), called blocks and defined in a subset of \( Z_v \), such that for every pair of distinct element \( x, y \in Z_v \):

1) if the distance (difference) between \( x, y \) is equal to \( 1, 2, ..., h \), then the pair \( x, y \) is contained in exactly \( \lambda \) blocks of \( \Sigma \);

2) if the distance (difference) between \( x, y \) is equal to \( h+1, h+2, ..., 2h \), then the pair \( x, y \) is contained in exactly \( \mu \) blocks of \( \Sigma \).

Example: Let \( v = 13 \). In \( Z_{13} \) the set of all the possible differences is \( \Delta = \{1, 2, 3, 4, 5, 6\} \). Partition \( \Delta \) into the following two classes: \( A = \{1, 2, 3\} \), \( B = \{4, 5, 6\} \). It is possible to define a \( K_3 \)-design (\textit{Steiner Triple System}) of order \( v = 13 \) and \textit{equi-indices} \((\lambda, \mu) = (1, 2)\), as follows:

\[ \forall \{x, y\} \subseteq Z_{13}, \ x \neq y, \ |x - y| = 1, 2, 3 \implies \lambda = 1, \]

\[ \forall \{x, y\} \subseteq Z_{13}, \ x \neq y, \ |x - y| = 4, 5, 6 \implies \mu = 2, \]
where $\lambda = 1$ and $\mu = 2$ mean respectively that the pair $x, y$ is contained in exactly one or two blocks of the system.

The blocks $\{i, i + 1, i + 5\}, \{i, i + 2, i + 7\}, \{i, i + 3, i + 7\}$, for every $i \in \mathbb{Z}_{13}$, define such a system.

**Strongly Balanced 4-kite-Designs**

It is known that a $G$-design $\Sigma$ is said to be *balanced* if the degree of each vertex $x \in X$ is a constant: in other words, the number of blocks of $\Sigma$ containing $x$ is a constant. The following concept has been introduced in [4].

Let $G$ be a graph and let $A_1, A_2, ..., A_h$ be the orbits of the automorphism group of $G$ on its vertex-set. Let $\Sigma = (X, \mathcal{B})$ be a $G$-design. We define the degree $d_{A_i}(x)$ of a vertex $x \in X$ as the number of blocks of $\Sigma$ containing $x$ as an element of $A_i$. We say that: $\Sigma = (X, \mathcal{B})$ is a strongly balanced $G$-design if, for every $i = 1, 2, ..., h$, there exists a constant $C_i$ such that $d_{A_i}(x) = C_i$, for every $x \in X$.

It is clear that: A strongly balanced $G$-design is a balanced $G$-design. Further, it is possible to prove that:

If $\Sigma = (X, \mathcal{B})$ in a balanced OQS of order $v$ and index $\lambda$, then $\Sigma$ is strongly balanced.

If $\Omega$ is a balanced 4-kite-design, it is possible that $\Omega$ is not strongly balanced. We say that a $G$-design is *simply balanced* if it is balanced, but not strongly balanced.

Starting from the remark that it is possible to partition an octagon quadrangle $Q = [(x_1), x_2, x_3, (x_4), (x_5), x_6, x_7, (x_8)]$, into two 4-kites $K_1(Q) = [(x_1, x_2, x_3), (x_4), x_5]$, $K_2(Q) = [(x_5, x_6, x_7), (x_8), x_1]$, we will give some results about OQSs which can be partitioned into two strongly balanced $(C_4 + e)$-designs.

## 2 Main Existence Theorems

It is possible to prove that:

**Theorem 2.1** Let $\Omega = (X, \mathcal{B})$ be an OKS$_{\lambda, \mu}(v)$. Then

i) $\lambda = 2 \cdot \mu$;

ii) $(\lambda, \mu) = (2, 1)$ or $(4, 2)$ or $(6, 3)$ or $(8, 4)$ implies $v \equiv 0, 1 \pmod{5}$, $v \geq 8$;
iii) \((\lambda, \mu) = (10, 5)\) implies \(v \equiv 0, 1 \text{ mod } 2, v \geq 8.\)

**Theorem 2.2** Let \(\Sigma = (Z_v, \mathcal{B})\) be a 4-kite-design of equi-indices \((\lambda, \mu)\), with \(v\) odd. Then

i) \(|\mathcal{B}| = (\lambda + \mu) \cdot v \cdot (v - 1)/20 \in N;\)

ii) \((\lambda, \mu) = (2, 3)\) implies \(v \equiv 1 \text{ mod } 4, v \geq 5.\)

Now we prove the conclusive Theorems. In what follows, if \(B = [(a), b, c, (d), (\alpha), \beta, \gamma, (\delta)]\) in a block of a system \(\Sigma\) defined in \(Z_v\), then the translates of \(B\) are all the blocks of type \(B_j = [(a + j), b + j, c + j, (d + j), (\alpha + j), \beta + j, \gamma + j, (\delta + j)]\), for every \(j \in Z_v\). \(B\) is called a base block of \(\Sigma\).

**Theorem 2.3** There exists an OKS of order \(v\) and equi-indices \((2, 3)\), with \(v\) odd, if and only if: \(v \equiv 1 \text{ mod } 4, v \geq 9.\)

**Proof** \(\Rightarrow\) Let \(\Sigma = (Z_v, \mathcal{B})\) be an OQS of order \(v\) and equi-indices \((2, 3)\), with \(v\) odd. Since every block contains eight vertices, from Theorem 5.2.ii), it follows \(v \equiv 1 \text{ mod } 4, v \geq 9.\)

\(\Leftarrow\) Let \(v = 4h + 1, h \in N, h \geq 2.\)

Consider the following octagon quadrangles:

\[
\begin{align*}
B_1 &= [(0), h, 3h + 1, (1), (2h + 1), 3h, h + 1, (2)], \\
B_2 &= [(0), 1, 3h + 1, (2), (2h + 1), 3h - 1, h + 1, (3)], \\
B_3 &= [(0), 2, 3h + 1, (3), (2h + 1), 3h - 2, h + 1, (4)], \\
& \quad \ldots \ldots \\
B_{i-1} &= [(0), h - 2, 3h + 1, (h - 1), (2h + 1), 3h - (h - 2), h + 1, (h)], \\
B_i &= [(0), h - 1, 3h + 1, (h), (2h + 1), h + 1, 3h + 2, (1)].
\end{align*}
\]

Consider the system \(\Sigma = (X, \mathcal{B})\), defined in \(X = Z_v\), having \(B_1, \ldots, B_i, \ldots, B_h\) as base blocks. This means that \(B_1, B_2, \ldots, B_i, \ldots, B_h\) belong to \(\mathcal{B}\) and also all the translates.

It is possible to verify that \(\Sigma\) is an OKS of order \(v = 4h + 1\) and index \(\lambda = 5\). Further, if we divide every block \(Q = [(x_1), x_2, x_3, (x_4), (x_5), x_6, x_7, (x_8)]\), into the two 4-kites

\[
K_1(Q) = [(x_1, x_2, x_3), (x_4), (x_5)], \quad K_2(Q) = [(x_5, x_6, x_7), (x_8), x_1],
\]

we can verify that the collection of all the upper 4-kites form a 4-kite-design \(\Sigma_1 = (Z_v, \mathcal{B}_1)\) of equi-indices \((\lambda = 2, \mu = 3)\), while the collection of all the lower 4-kites form a 4-kite-design \(\Sigma_2 = (Z_v, \mathcal{B}_2)\) of equi-indices \((\lambda = 3, \mu = 2)\).

Observe that: \(i)\) in \(\Sigma_1\) the pair \(x, y \in Z_v\) associated with the index \(\lambda = 2\) have difference \(|x - y|\) belongs to \(A = \{1, 2, \ldots, h\}\), while those associated with \(\mu = 3\) have difference belongs to \(B = \{h + 1, h + 2, \ldots, 2h\}; \ ii)\) in \(\Sigma_2\) the pair
x, y ∈ Z_v associated with the index λ = 3 have difference |x − y| belongs to A, while those associated with µ = 2 have difference belongs to B.

This prove that Σ is an OKS of order v = 4h + 1, h ≥ 2, where the two 4-kite-designs nested in it have equi-indices (2, 3) and (3, 2) respectively. □

This Theorem permits to prove that:

**Theorem 2.4** There exists a 4-kite-design of order v and equi-indices (2, 3), with v odd, if and only if:

\[ v \equiv 1 \mod 4, \ v \geq 5. \]

**Proof** The statement follows directly from Theorem 2.3, considering also that the design Σ_5, defined in Z_5 and having for blocks all the translates of the base 4-kite: [(2, 1, 3), (0), 4], is a 4-kite-design of order 5 and equi-indices (2,3). □

**Theorem 2.5** For every \( v \equiv 1 \mod 4, \ v \geq 5 \), there exists a strongly balanced 4-kite-design of order v.

**Proof** See Theorems 2.3 and 2.4. □

**References**


