A TORSION THEORY IN THE CATEGORY OF COCOMMUTATIVE HOPF ALGEBRAS

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Abstract. The purpose of this article is to prove that the category of cocommutative Hopf K-algebras, over a field K of characteristic zero, is a semi-abelian category. Moreover, we show that this category contains a torsion theory whose torsion-free and torsion parts are given by the category of groups and by the category of Lie K-algebras, respectively.

1. Introduction

The starting point of this article on Hopf algebras is a well-known result due to A. Grothendieck, as outlined in [Swe69], saying that the category of finite-dimensional, commutative and cocommutative Hopf K-algebras over a field K is abelian. This result was extended by M. Takeuchi to the category of all commutative and cocommutative Hopf K-algebras, not necessarily finite-dimensional [Tak72]. The category $\mathbf{Hopf}_{K,coc}$ of cocommutative Hopf K-algebras is not additive, thus it can not be abelian. In the present paper we investigate some of its fundamental exactness properties, showing that it is a homological category (Section 3), and that it is Barr-exact (Section 5), leading to the conclusion that the category $\mathbf{Hopf}_{K,coc}$ is semi-abelian [JMT02] when the field K is of characteristic zero (Theorem 5.1) This result establishes a new link between the theory of Hopf algebras and the more recent one of semi-abelian categories, both of which can be viewed as wide generalizations of group theory. Since a category $\mathbf{C}$ is abelian if and only both $\mathbf{C}$ and its dual $\mathbf{C}^{op}$ are semi-abelian, this observation can be seen as a "non-commutative" version of Takeuchi's theorem mentioned above. This result was independently obtained by Clemens Berger and Stephen Lack.

We furthermore prove the existence of a non-abelian torsion theory $(\mathbf{T}, \mathbf{F})$ in $\mathbf{Hopf}_{K,coc}$, where the torsion subcategory $\mathbf{T}$ is the category of *primitive Hopf K-algebras*, which is equivalent to the category of Lie K-algebras, and the torsion-free subcategory $\mathbf{F}$ is the category of *group Hopf K-algebras*, which is equivalent to the category of groups.

The categories of groups and of Lie K-algebras are two typical examples of semi-abelian categories: this shows again that the theories of cocommutative Hopf algebras and of semi-abelian categories are strongly intertwined. The category $\mathbf{Hopf}_{K,coc}$ inherits some fundamental exactness properties from groups and Lie algebras thanks to the well-known canonical decomposition of a cocommutative Hopf algebra into a semi-direct product of a group Hopf algebra and a primitive Hopf algebra (a result associated with the names Cartier-Gabriel-Kostant-Milnor-Moore). The present work opens the way to some new applications of categorical Galois theory [Jan91] in the category of cocommutative Hopf K-algebras, since the reflection from this category to the torsion-free subcategory of group Hopf algebras enjoys all the properties needed for this kind of investigations, as we briefly explain in Section 4.

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2. Preliminaries

2.1. **Semi-abelian categories.**

Semi-abelian categories [JMT02] are finitely complete, pointed, exact in the sense of M. Barr [Bar71], protomodular in the sense of D. Bourn [Bou91], with finite coproducts. These categories have been introduced to capture some typical algebraic properties valid for non-abelian algebraic structures such as groups, Lie algebras, rings, crossed modules, varieties of Ω-groups in the sense of P. Higgins [Hig56] and compact groups. As already mentioned in the introduction, every abelian category is in particular semi-abelian.

Although protomodularity is a property that can be expressed in any category with finite limits, in the pointed context, i.e. when there is a zero object 0 in $\mathbf{C}$, protomodularity amounts to the fact that the following formulation of the *Split Short Five Lemma* holds: given a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \xrightarrow{k} & A & \underset{s}{\overset{f}{\rightleftarrows}} & B \\ & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ 0 & \longrightarrow & K' & \xrightarrow[k']{} & A' & \underset{s'}{\overset{f'}{\rightleftarrows}} & B' \end{array}$$

where $k = ker(f)$, $k' = ker(f')$, $f \circ s = 1_B$, and $f' \circ s' = 1_{B'}$ (i.e. f are f' are split epimorphisms with sections s and s'), if both κ and β are isomorphisms, then so is α.

A useful lemma which holds in protomodular categories is the following ([Bou91], Proposition 11):

Lemma 2.1. *Given a split short exact sequence in a pointed protomodular category*

$$0 \longrightarrow K \xrightarrow{k} A \underset{s}{\overset{f}{\rightleftarrows}} B \longrightarrow 0$$

the pair of morphisms (k, s) is jointly epimorphic.

Any protomodular category $\mathbf{C}$ is a *Mal'tsev* category [CLP91]: this means that every (internal) reflexive relation $\mathbf{C}$ is an (internal) equivalence relation. Recall that a reflexive relation on an object X is a diagram of the form

$$R \; \underset{p_2}{\overset{p_1}{\rightrightarrows}} \; X, \quad \delta: X \to R \tag{2.1}$$

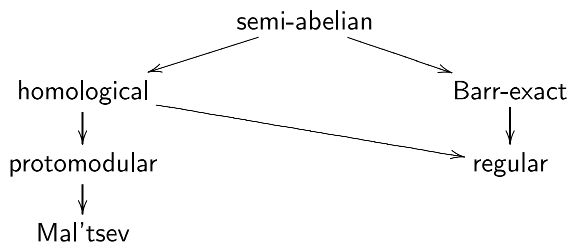
where p_1 and p_2 are jointly monic, and $p_1 \circ \delta = 1_X = p_2 \circ \delta$; such a reflexive relation R is an equivalence relation when, moreover, there exist $\sigma : R \longrightarrow R$ and $\tau : R \times_X R \longrightarrow R$ as in the diagram

$$R \times_X R \xrightarrow{\tau} R \; \underset{p_2}{\overset{p_1}{\rightrightarrows}} \; X, \quad \delta: X \to R, \quad \sigma: R \to R$$

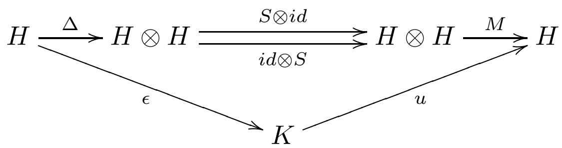
such that $p_1 \circ \sigma = p_2$ and $p_2 \circ \sigma = p_1$ (symmetry), and $p_1 \circ \tau = p_1 \circ \pi_1$ and $p_2 \circ \tau = p_2 \circ \pi_2$ (transitivity).

In the present article, by a *regular* category is meant a finitely complete category where every morphism can be factorized as a regular epimorphism followed by a monomorphism, and where regular epimorphisms are pullback stable. A regular category $\mathbf{C}$ is said to be *Barr-exact* if, moreover, every equivalence relation is effective, i.e. every equivalence relation is the kernel pair of a morphism in $\mathbf{C}$. A category which is pointed, protomodular and regular is said to be *homological* [BB04]. In this context several basic diagram lemmas of homological algebra hold true (such as the snake lemma, the 3-by-3-Lemma, etc.).

We end these preliminaries with the following diagram indicating some implications between the different contexts recalled above:

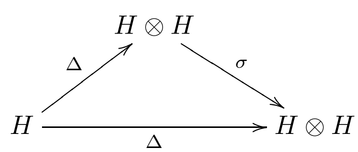


2.2. **The category $\mathbf{Hopf}_{K,coc}$ of cocommutative Hopf K-algebras.**
The category we study in this article is the category of Hopf K-algebras over a field K, denoted by $\mathbf{Hopf}_K$. The objects in $\mathbf{Hopf}_K$ are Hopf K-algebras, i.e. sextuples $(H, M, u, \Delta, \epsilon, S)$ where (H, M, u) is a K-algebra and (H, Δ, ϵ) is a K-coalgebra, such that these two structures are compatible, i.e. maps M and u are K-coalgebras morphisms, making $(H, M, u, \Delta, \epsilon)$ a K-bialgebra. The linear map S is called the *antipode*, and makes the following diagram commute:



Morphisms in $\mathbf{Hopf}_K$ are exactly morphisms of K-bialgebras (i.e. morphisms that are both morphisms of K-algebras and K-coalgebras), as morphisms of K-bialgebras always preserve antipodes.

To denote the comultiplication map of a Hopf algebra H, we will use the Sweedler notation: $\forall h \in H$, $\Delta(h) = h_1 \otimes h_2$ by omitting the summation sign. A Hopf algebra H is said to be *cocommutative* if its comultiplication map Δ makes the following diagram commute, where $\forall x, y \in H$, $\sigma(x \otimes y) := y \otimes x$



The category of cocommutative Hopf K-algebras will be denoted by $\mathbf{Hopf}_{K,coc}$. In the category $\mathbf{Hopf}_{K,coc}$ there are two full subcategories which will be of importance for our work: the category $\mathbf{GrpHopf}_K$ of group Hopf algebras, and the category $\mathbf{PrimHopf}_K$ of primitive Hopf algebras, whose definitions we are now going to recall.

(1) The *group Hopf algebra on a group G*, denoted by $K[G]$, is the free vector space on G over the field K, i.e. $K[G] = \{\sum_{g \in G} \alpha_g g$, where $(\alpha_g)_{g \in G}$ is a family of scalars with only a finite number being non zero$\}$ and $\{g \mid g \in G\}$ is a basis of $K[G]$. The group Hopf algebra $K[G]$ can be equipped with a structure of cocommutative Hopf algebra, by taking the multiplication induced by the group law, and comultiplication $\Delta : K[G] \longrightarrow K[G] \otimes K[G]$, counit $\epsilon : K[G] \to K$ and antipode $S : K[G] \to K[G]$ the linear maps defined on the base elements respectively by $\Delta(g) = g \otimes g$, $\epsilon(g) = 1$ and $S(g) = g^{-1}$, $\forall g \in G$.

This assignment defines a functor $K[-]\colon \mathbf{Grp} \to \mathbf{Hopf}_K$ from the category of groups to the category of Hopf algebras, which has a right adjoint $\mathcal{G}\colon \mathbf{Hopf}_K \to \mathbf{Grp}$, that associates to any Hopf algebra H its group of grouplike elements $\mathcal{G}(H) = \{x \in H \mid \Delta(x) = x \otimes x, \epsilon(x) = 1\}$. If we restrict ourselves from the category of Hopf algebras to the full subcategory $\mathbf{GrpHopf}_K$ of group Hopf algebras, then the functor $K[-]\colon \mathbf{Grp} \to \mathbf{GrpHopf}_K$ is by construction surjective on objects and moreover even an isomorphism of categories. Indeed, let us recall why the functor $\mathcal{G} \circ K[-]$ is the identity on the category of groups. If G is a group and $K[G]$ the group Hopf K-algebra on G, one clearly has that $G \subseteq \mathcal{G}\big(K[G]\big)$ (in fact, this inclusion is the unit of the adjunction described above). Conversely, let $x = \sum_{g\in G} \alpha_g g \in K[G]$ be a group-like element. We have $\Delta(x) = \sum_{g\in G} \alpha_g g \otimes g$ but also $\Delta(x) = x \otimes x = \sum_{g,h\in G} \alpha_g \alpha_h g \otimes h$. Since $\{g \otimes h \mid g, h \in G\}$ forms a basis of $K[G] \otimes K[G]$, we have $\forall g, h \in G$: $\alpha_g^2 = \alpha_g$, and $\alpha_g \alpha_h = 0$ if $g \neq h$. It follows that all α_g's should be zero except one that should be 1_K, thus x is in G.

(2) The *universal enveloping algebra of a Lie algebra L*, denoted by $U(L)$, is defined by the quotient $U(L) = T(L)/I$, where $T(L)$ is the tensor algebra on the vector space underlying L, and I is the two-sided ideal of $T(L)$ generated by the elements of the form $x \otimes x' - x' \otimes x - [x, x']$, $\forall x, x' \in L$. Remark that the elements of L generate $U(L)$ as an algebra. The universal enveloping algebra $U(L)$ can be equipped with a structure of cocommutative (and non commutative) Hopf algebra, by taking the concatenation as multiplication, and comultiplication $\Delta : U(L) \longrightarrow U(L) \otimes U(L)$, counit $\epsilon : U(L) \to K$ and antipode $S : U(L) \to U(L)$ the algebra maps defined on the generators by $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\epsilon(x) = 0$ and $S(x) = -x$, $\forall x \in L$.

Recall that for any Hopf algebra H an element $x \in H$ is called a *primitive element* if $\Delta(x) = x \otimes 1 + 1 \otimes x$ (and consequently, $\epsilon(x) = 0$). The above constructions lead to a pair of adjoint functors, where the functor $U : \mathbf{LieAlg}_K \to \mathbf{Hopf}_K$ is a left adjoint to $P : \mathbf{Hopf}_K \to \mathbf{LieAlg}_K$. We can now consider the category $\mathbf{PrimHopf}_K$, which is the full subcategory of $\mathbf{Hopf}_K$ whose objects are primitive Hopf algebras, that is Hopf algebras generated as algebra by primitive elements. In the case where K is of characteristic 0, the category $\mathbf{PrimHopf}_K$ is known to be isomorphic to the category $\mathbf{LieAlg}_K$ of Lie K-algebras [MM65, Theorem 5.18].

Remark 2.2. As can be seen from the formula for comultiplication, both group Hopf algebras and primitive Hopf algebras are cocommutative. Therefore the categories $\mathbf{GrpHopf}_K$ and $\mathbf{PrimHopf}_K$ are also full subcategories of $\mathbf{Hopf}_{K,coc}$. The functors $U, P, K[-], \mathcal{G}$ and their adjunctions are represented in the following diagram:

$$\mathbf{Grp} \underset{\mathcal{G}}{\overset{K[-]}{\underset{\perp}{\rightleftarrows}}} \mathbf{Hopf}_{K,coc} \underset{U}{\overset{P}{\underset{\top}{\rightleftarrows}}} \mathbf{LieAlg}_K$$

3. The category of cocommutative Hopf algebras over a field of characteristic zero is homological

The category $\mathbf{Hopf}_{K,coc}$ is certainly pointed, with the zero object K, that will be denoted by 0, from now on. $\mathbf{Hopf}_{K,coc}$ is complete and cocomplete, since it is locally presentable [Por11]. We will now establish its protomodularity and regularity.

3.1. **Protomodularity of the category $\mathbf{Hopf}_K$.** Let us consider the following commutative diagram of short exact sequences in the category $\mathbf{Hopf}_{K,coc}$:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & C_1 & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id_A} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle id_B} & & \\
0 & \longrightarrow & A & \longrightarrow & C_2 & \longrightarrow & B & \longrightarrow & 0
\end{array}
\tag{3.1}
$$

Thanks to the explicit descriptions of equalizers and coequalizers given in [AD95] one can easily prove that the kernel and the cokernel of θ are the zero object. This proves that θ is a monomorphism and an epimorphism of Hopf K-algebras. Since monomorphisms are injections and epimorphisms are surjections in the category $\mathbf{Hopf}_{K,coc}$ [Chi10, NT94], this shows that θ is an isomorphism of Hopf K-algebras and so the category $\mathbf{Hopf}_{K,coc}$ is protomodular.

The category $\mathbf{Hopf}_{K,coc}$ is actually even *strongly protomodular*, [Bou00] since it has finite limits and it can be viewed as the category of internal groups in the category of cocommutative K-coalgebras (see [Str07], for instance. The category of cocommutatitve K-coalgebras is studied in detail in [GP87]). This argument is general and can be applied to cocommutatitve Hopf algebras in any braided monoidal category.

Nevertheless, this result holds more generally for the category $\mathbf{Hopf}_K$ of arbitrary Hopf K-algebras, that is also protomodular. This follows from the following result (Lemma 3.2.19 in [AD95]) by taking into account the fact that any split extension induces a *cleft exact sequence* in the sense of by Andruskiewitsch and Devoto [AD95]:

Theorem 3.1. *Consider a diagram of the form* (3.1) *in the category of Hopf K-algebras, where the above exact sequence is cleft. Then the bottom exact sequence is also cleft and θ is an isomorphism.*

3.2. **Semi-direct products of cocommutative Hopf algebras.** Let B be a cocommutative Hopf algebra. A *B-module Hopf algebra* is an Hopf algebra A that is at the same time a left B-module with action $\rho : B \otimes A \to A, \rho(b \otimes a) = b \cdot a$ such that ρ is a morphism of bialgebras. The semi-direct product (also known as *smash product*) of B and A, denoted by $A \rtimes B$, is the Hopf algebra whose underlying vector space is the tensor product $A \otimes B$ and with the following structure. The unit is $u_{A \rtimes B} = u_A \otimes u_B$ and multiplication given by

$$(a \otimes b)(a' \otimes b') = a(b_1 \cdot a') \otimes b_2 b',$$

for all $a, a' \in A$ and $b, b' \in B$. The coalgebra structure is given by the tensor product coalgebra, i.e. $\Delta_{A \rtimes B} = (id_A \otimes \sigma \otimes id_B)(\Delta_A \otimes \Delta_B)$ and $\epsilon_{A \rtimes B} = \epsilon_A \otimes \epsilon_B$. The antipode is given by $S_{A \rtimes B}(a \otimes b) = S_B(b_1) \cdot S_A(a) \otimes S_B(b_2)$.

The following Lemma is a reformulation of Theorem 4.1 in [Mol77]:

Lemma 3.2. *Every split short exact sequence in $\mathbf{Hopf}_{K,coc}$*

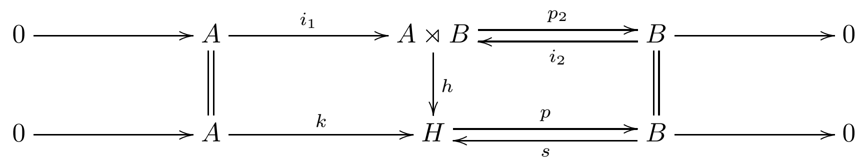
$$0 \longrightarrow A \xrightarrow{\;k\;} H \underset{s}{\overset{p}{\rightleftarrows}} B \longrightarrow 0$$

is canonically isomorphic to the semi-direct product exact sequence

$$0 \longrightarrow A \xrightarrow{\;i_1\;} A \rtimes B \underset{i_2}{\overset{p_2}{\rightleftarrows}} B \longrightarrow 0$$

where $i_1 = id_A \otimes u_B$, $i_2 = u_A \otimes id_B$ and $p_2 = \epsilon_A \otimes id_B$.

Proof. The arrow $h\colon A \rtimes B \to H$ in the diagram below is given by $h(a \otimes b) = k(a)s(b)$ for all $a \otimes b \in A \rtimes B$.



This is a morphism of split short exact sequences, and therefore h is an isomorphism by protomodularity of $\mathbf{Hopf}_{K,coc}$. □

We use this lemma to reformulate the well-known structure theorem for cocommutative Hopf algebras over an algebraically closed field of characteristic zero (see for instance [Swe69], page 279 in combination with Lemma 8.0.1(c)) in terms of split exact sequences.

Theorem 3.3 (Cartier-Gabriel-Moore-Milnor-Kostant)**.** *Every cocommutative Hopf K-algebra H, over an algebraically closed field K of characteristic 0, is isomorphic to the semi-direct product*

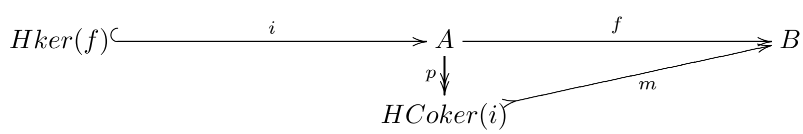
$$H \cong U(L_H) \rtimes K[G_H]$$

of the universal enveloping algebra of a Lie algebra $U(L_H)$ with the group Hopf algebra $K[G_H]$, where L_H and G_H are given respectively by the space of primitive elements and set of group-like elements of H. Consequently, for each H, there exists a canonical split exact sequence of cocommutative Hopf algebras of following form

$$0 \longrightarrow U(L_H) \xrightarrow{i_H} H \underset{s_H}{\overset{p_H}{\rightleftarrows}} K[G_H] \longrightarrow 0$$

3.3. Regularity of the category $\mathbf{Hopf}_{K,coc}$.

3.3.1. *The regular epimorphism/monomorphism factorization in* $\mathbf{Hopf}_{K,coc}$. Let $f : A \longrightarrow B$ be a morphism of cocommutative Hopf K-algebras. By the protomodularity of $\mathbf{Hopf}_{K,coc}$, it is well known that regular epimorphisms are the same as cokernels, i.e. normal epimorphisms. Thus, to construct the regular epimorphism/monomorphism factorization of the morphism f, we consider the kernel $i : Hker(f) \to A$ of f and the cokernel $p : A \to HCoker(i)$ of i, both computed in the category of $\mathbf{Hopf}_{K,coc}$:



The existence of this factorization m such that $m \circ p = f$ follows from the universal property of the cokernel p of i. It remains to prove that m is a monomorphism, which is equivalent in $\mathbf{Hopf}_{K,coc}$ to showing that m is an injection. In the category $\mathbf{Hopf}_{K,coc}$ the above factorization is obtained as in the category of vector spaces since $HCoker(i) = \dfrac{A}{A\big(Hker(f)\big)^{+}A}$ (by [AD95]), and $ker(f) = A\big(Hker(f)\big)^{+}A$, (by [Shu88, New75]).

Note that any epimorphism of cocommutative Hopf K-algebras is then a normal epimorphism, and the following classes of epimorphisms coincide in $\mathbf{Hopf}_{K,coc}$:

normal epis = regular epis = epis = surjective morphisms.

Remark 3.4. The two full subcategories $\mathbf{GrpHopf}_K$ and $\mathbf{PrimHopf}_K$, of $\mathbf{Hopf}_{K,coc}$, are closed under quotients since morphisms of Hopf K-algebras preserve group-like and primitive elements.

3.3.2. *Pullback stability of regular epimorphisms in the category* $\mathbf{Hopf}_{K,coc}$. To prove the pullback stability of regular epimorphisms in the category $\mathbf{Hopf}_{K,coc}$, the approach we follow is to apply the pullback stability of regular epimorphisms in the two full subcategories $\mathbf{GrpHopf}_K$ and $\mathbf{PrimHopf}_K$ of $\mathbf{Hopf}_{K,coc}$, which are both semi-abelian, and closed under pullbacks and quotients in $\mathbf{Hopf}_{K,coc}$. From the regularity of these two categories and the decomposition Theorem 3.3 we deduce the regularity of $\mathbf{Hopf}_{K,coc}$

Remark 3.5. In the following we shall assume that K is an algebraically closed field. It can be checked that this is not a restriction: indeed, given a field K and $\phi : K \to \overline{K}$ an embedding of K in an algebraic closure $\overline{K}$, one has the adjunction

$$\mathbf{Hopf}_{\overline{K},coc} \underset{R_\phi}{\overset{L_\phi}{\underset{\perp}{\rightleftarrows}}} \mathbf{Hopf}_{K,coc}$$

where R_ϕ is the "restriction of scalars functor" and $L_\phi = - \otimes_K \overline{K}$ its left adjoint, the "extension of scalars" functor. Being a left adjoint, L_ϕ preserves regular epimorphisms and moreover L_ϕ reflects regular epimorphisms and preserves finite limits. Accordingly, knowing that $\mathbf{Hopf}_{\overline{K},coc}$ is regular (respectively, exact), one can deduce from this that $\mathbf{Hopf}_{K,coc}$ is regular (resp. exact) as well.

The following result concerning split short exact sequences in $\mathbf{Hopf}_{K,coc}$ will be useful in the proof of the regularity of this category:

Lemma 3.6. *Given the following commutative diagram of split short exact sequences in* $\mathbf{Hopf}_{K,coc}$*:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \xrightarrow{i_{H_1}} & H_1 & \underset{p_{H_1}}{\overset{s_{H_1}}{\leftrightarrows}} & B_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h_A} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h_B} & & \\
0 & \longrightarrow & A_2 & \xrightarrow{i_{H_2}} & H_2 & \underset{p_{H_2}}{\overset{s_{H_2}}{\leftrightarrows}} & B_2 & \longrightarrow & 0
\end{array}$$

We have that h is surjective if and only if both h_A and h_B are surjective.

Proof. We apply Lemma 3.2 to the exact sequences in the statement of the Lemma, we obtain the following commutative diagram which is canonically isomorphic to the previous one:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \underset{i_1}{\overset{\xi_1}{\leftrightarrows}} & A_1 \rtimes B_1 & \underset{p_1}{\overset{s_1}{\leftrightarrows}} & B_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h_A} & & \downarrow{\scriptstyle h_A \otimes h_B} & & \downarrow{\scriptstyle h_B} & & \\
0 & \longrightarrow & A_2 & \underset{i_2}{\overset{\xi_2}{\leftrightarrows}} & A_2 \rtimes B_2 & \underset{p_2}{\overset{s_2}{\leftrightarrows}} & B_2 & \longrightarrow & 0
\end{array}$$

Hence, the morphism h is surjective if and only if $h_A \otimes h_B \colon A_1 \rtimes B_1 \to A_2 \rtimes B_2$ is surjective.

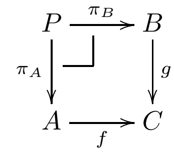
If h_A and h_B are surjective, then $h_A \otimes h_B$ is surjective by considering this morphism on its underlying vector space. For the converse implication, if $h_A \otimes h_B$ is surjective, let us note that for each semi-direct product $A_i \rtimes B_i$, the underlying coalgebra is exactly the categorical product of the coalgebras A_i and B_i; we denoted $\xi_i = id_{A_i} \otimes \epsilon_{B_i}$ for the coalgebra-projection of $A_i \rtimes B_i$ onto A_i (which is not a Hopf algebra morphism).

It is clear that $h_A \circ \xi_1 = \xi_2 \circ h$ (as coalgebra morphisms). Since ξ_2 is a split epimorphism and h is surjective, we conclude that h_A is surjective. It is clear that h_B is surjective whenever h is.

□

Theorem 3.7. *Consider the following pullback (P, π_A, π_B) in the category $\mathbf{Hopf}_{K,coc}$:*

(3.2)

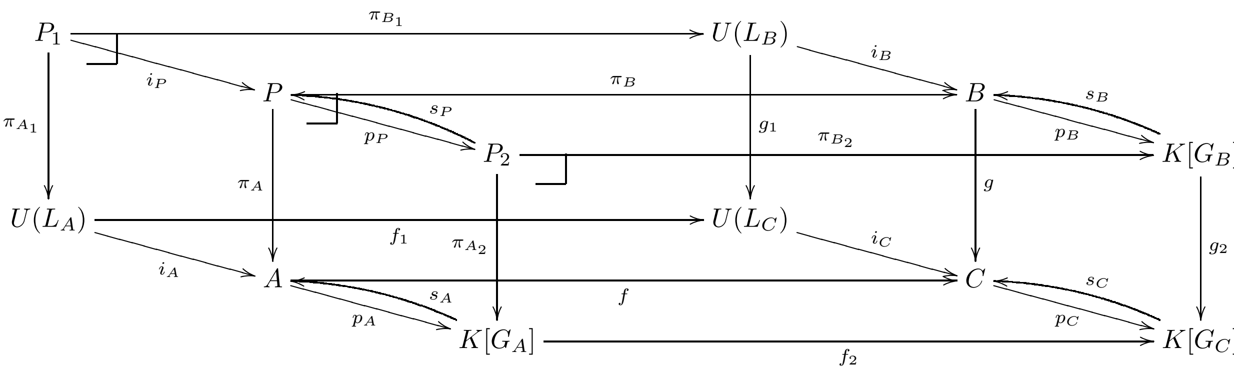


if f is a regular epimorphism then π_B is a regular epimorphism.

Proof. The fact that the subcategory $\mathbf{GrpHopf}_K$ (resp., $\mathbf{PrimHopf}_K$) is semi-abelian and closed in $\mathbf{Hopf}_{K,coc}$ under pullbacks and regular epimorphisms implies that regular epimorphisms are pullback stable whenever the Hopf algebras A,B and C in diagram of the form (3.2) belong to $\mathbf{GrpHopf}_K$ (or to $\mathbf{PrimHopf}_K$, respectively).

Let us now consider A, B and C cocommutative Hopf K-algebras over a field K of characteristic zero. By Theorem 3.3, we have: $A \cong U(L_A) \rtimes K[G_A]$, $B \cong U(L_B) \rtimes K[G_B]$ and $C \cong U(L_C) \rtimes K[G_C]$.

We will consider the following commutative diagram where $(P_1, \pi_{A_1}, \pi_{B_1})$ is the pullback of f_1 and g_1, $(P_2, \pi_{A_2}, \pi_{B_2})$ is the pullback of f_2 and g_2.



When f is surjective, the surjectivity of f_1 and f_2 follow both from Lemma 3.6 applied to the lower part of the diagram. The front and back faces of the diagram are in $\mathbf{GrpHopf}_K$ and in $\mathbf{PrimHopf}_K$, respectively, thus π_{B_1} and π_{B_2} are surjective. Applying Lemma 3.6 again (in converse direction), we obtain that π_B is also surjective.

□

4. A TORSION THEORY IN THE CATEGORY $\mathbf{Hopf}_{K,coc}$

In the non-abelian context of homological categories it is natural to define and study a general notion of torsion theory, that extends the one introduced by S.E. Dickson in the frame of abelian categories [Dic98]. This study was first initiated in [BG06], and further developed in [DEG12] [EG14], also in relationship with semi-abelian homology theory.

Let us recall the definition of a torsion theory in the homological context:

Definition 4.1. In a homological category $\mathbf{C}$, a *torsion theory* is given by a pair $(\mathbf{T},\mathbf{F})$ of full and replete (i.e. isomorphism closed) subcategories of $\mathbf{C}$ such that:

i. For any object X in $\mathbf{C}$, there exists a short exact sequence:

$$0 \longrightarrow T \xrightarrow{t_X} X \xrightarrow{\eta_X} F \longrightarrow 0$$

where 0 is the zero object in $\mathbf{C}$, $T \in \mathbf{T}$ and $F \in \mathbf{F}$.

ii. The only morphism $f : T \longrightarrow F$ from $T \in \mathbf{T}$ to $F \in \mathbf{F}$ is the zero morphism.

When $(\mathbf{T},\mathbf{F})$ is a torsion theory, $\mathbf{T}$ is called the *torsion* subcategory of $\mathbf{C}$, and $\mathbf{F}$ its *torsion-free* subcategory. Among the many examples in the homological context, let us just mention the following ones:

Example 4.2.

(1) Every torsion theory in an abelian category $\mathbf{C}$. For instance: the pair $(\mathbf{Ab}_t,\ \mathbf{Ab}_{tf})$ in the category of abelian groups $\mathbf{Ab}$. Where $\mathbf{Ab}_t$ and $\mathbf{Ab}_{tf}$ denote the full and replete subcategories of the category of abelian groups whose objects are torsion and torsion-free abelian groups, respectively.

(2) The pair $(\mathbf{NilCRng},\ \mathbf{RedCRng})$ in the category of commutative rings $\mathbf{CRng}$, where $\mathbf{NilCRng}$ and $\mathbf{RedCRng}$ denote the full subcategories of nilpotent commutative rings, and of reduced commutative rings (i.e. commutative rings without non trivial nilpotent elements), respectively.

(3) The pair $(\mathbf{Grp}(\mathbf{Ind}),\ \mathbf{Grp}(\mathbf{Haus}))$ in the category of topological groups $\mathbf{Grp}(\mathbf{Top})$, where $\mathbf{Grp}(\mathbf{Ind})$ and $\mathbf{Grp}(\mathbf{Haus})$ denote the full subcategories of indiscrete groups and of Hausdorff groups, respectively.

From now on, in the homological category of cocommutative Hopf K-algebras, $\mathbf{T}$ will always denote the (full) subcategory $\mathbf{PrimHopf}_K$ of primitive Hopf algebras, and $\mathbf{F}$ the (full) subcategory $\mathbf{GrpHopf}_K$ of group Hopf algebras.

Thanks to Theorem 3.3, we know that we can associate the following short exact sequence with any cocommutative Hopf K-algebra H:

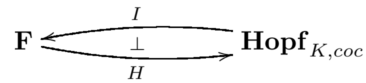
$$0 \longrightarrow U(L_H) \xrightarrow{i_H} H \xrightarrow{p_H} K[G_H] \longrightarrow 0$$

Any morphism from $T \in \mathbf{T}$ to $F \in \mathbf{F}$ is the zero morphism $\mathbf{Hopf}_{K,coc}$ since any primitive Hopf algebra $U(L)$ is generated by its primitive elements, which are preserved by a morphism of Hopf algebras and a group Hopf algebra $K[G]$ does not contain any primitive element. It follows that $(\mathbf{T},\mathbf{F})$ is a torsion theory which is actually hereditary, namely the torsion subcategory $\mathbf{T}$ is closed in $\mathbf{Hopf}_{K,coc}$ under subobjects:

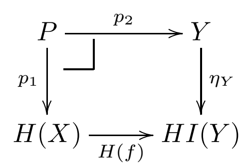
Proposition 4.3. *The pair $(\mathbf{T},\mathbf{F})$ is a hereditary torsion theory in $\mathbf{Hopf}_{K,coc}$.*

Proof. Given any monomorphism $m : A \rightarrowtail U(L)$ in $\mathbf{Hopf}_{K,coc}$ with codomain a primitive Hopf algebra in $\mathbf{T}$. The morphism m preserves group-like elements as a Hopf algebra morphism, and the group of group-like elements is trivial in a primitive Hopf algebra [Sch94]: since m is injective, by applying Theorem 3.3 to A we see that A can not contain any group-like element (different from 1_A) and therefore has to be primitive. □

As it follows from the results in [BG06] the reflector I in the adjunction



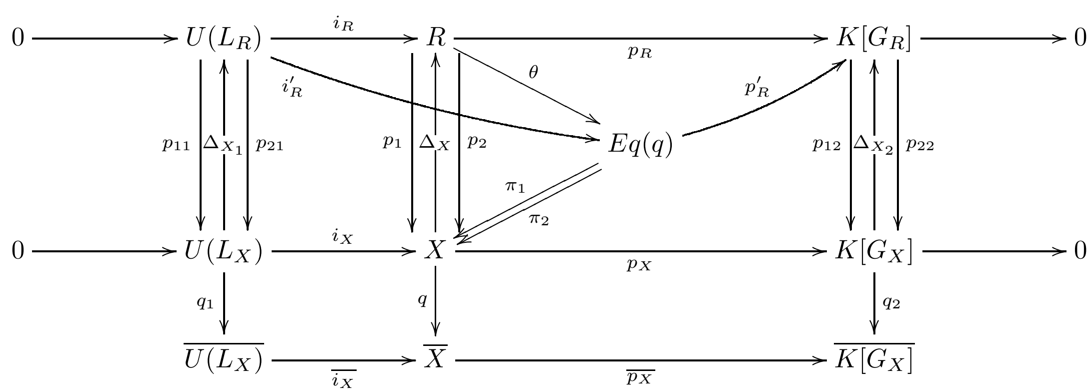
is semi-left-exact in the sense of Cassidy-Hebert-Kelly [CHK85], i.e. it preserves all pullbacks of the form



where η_Y is the Y-component of the unit of the adjunction and f lies in the subcategory $\mathbf{F}$. The adjunction is then *admissible* in the sense of categorical Galois theory [Jan91]: this opens the way to further investigations in the direction of semi-abelian homology [DEG12]. The fact that the torsion theory is hereditary and $\mathbf{Hopf}_{K,coc}$ a homological category implies that the corresponding *Galois coverings* are precisely those regular epimorphisms $f\colon A \to B$ in $\mathbf{Hopf}_{K,coc}$ with the property that the kernel $Hker(f)$ is in $\mathbf{F}$ (by applying Theorem 4.5 in [GR07]). This fact is crucial to describe generalized Hopf formulae for homology, as explained in [EG14].

5. The category of cocommutative Hopf algebras over a field of characteristic zero is semi-abelian

In order to prove that $\mathbf{Hopf}_{K,coc}$ is semi-abelian, it remains to show that equivalence relations are effective. For this, we shall show that any equivalence relation R as in diagram (2.1) in $\mathbf{Hopf}_{K,coc}$ is the kernel pair of its coequalizer $q\colon X \to \overline{X}$. We first apply Theorem 3.3 to the equivalence relation R, obtaining the following commutative diagram, where the morphisms q_1, q_2 and q are the coequalizers of p_{11} and p_{21}, p_{12} and p_{22}, p_1 and p_2, respectively, and $(Eq(q),\pi_1,\pi_2)$ is the kernel pair of q.



The subcategory $\mathbf{PrimHopf}_K$ of primitive Hopf algebras is semi-abelian, is closed under products [MM65] and subobjects (Proposition 4.3), thus under pullbacks in $\mathbf{Hopf}_{K,coc}$; since it is also closed under quotients, the left column is exact, i.e. $U(L_R) = Eq(q_1)$. On the other hand, let us then explain why the right column is also exact. First observe that $K[G_R]$ is a reflexive relation on $K[G_X]$: for this, one can use Lemma 5.2 in [BG06] and the fact that the reflector $I\colon \mathbf{Hopf}_{K,coc} \to \mathbf{GrpHopf}_K$ preseves binary products. The category $\mathbf{GrpHopf}_K$ of group Hopf algebras is exact Mal'tsev and closed under pullbacks and quotients in $\mathbf{Hopf}_{K,coc}$: it follows that $K[G_R]$ is the kernel pair of q_2 in $\mathbf{Hopf}_{K,coc}$.

The universal property of $(Eq(q),\pi_1,\pi_2)$ gives the unique arrow $\theta\colon R \to Eq(q)$ with $\pi_1 \circ \theta = p_1$ and $\pi_2 \circ \theta = p_2$. By applying the Split Short Five Lemma to the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U(L_R) & \xrightarrow{i_R} & R & \xrightarrow{p_R} & K[G_R] & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle id_{U(L_R)}} & & \Big\downarrow{\scriptstyle \theta} & & \Big\downarrow{\scriptstyle id_{K[G_R]}} & & \\
0 & \longrightarrow & U(L_R) & \xrightarrow[i'_R]{} & Eq(q) & \xrightarrow[p'_R]{} & K[G_R] & \longrightarrow & 0
\end{array}$$

it follows that the morphism θ is an isomorphism, and the equivalence relation R is effective. One accordingly has the following:

Theorem 5.1. *For any field K of characteristic 0, the category $\mathbf{Hopf}_{K,coc}$ of cocommutative Hopf K-algebras is semi-abelian.*

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