On Replete Graphs

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ABSTRACT

A graph G is k-clique replete if G has clique number k and every elementary homomorphism of G has clique number greater than k. Results on the order of k-clique replete graphs are presented, and bounds for the minimum degree and the maximum degree of such graphs are discussed.

1. INTRODUCTION

All graphs considered in this paper are finite and simple. For undefined concepts we refer the reader to [2].

An elementary homomorphism of a graph is the identification of two nonadjacent vertices of the graph. A homomorphism of a graph is the composition of a sequence of elementary homomorphisms. Thus a homomorphism of a graph G onto a graph H is a function \( \phi \) from \( V(G) \) onto \( V(H) \) such that

(i) if \( g_1 \) and \( g_2 \) are two adjacent vertices in G, then \( \phi(g_1) \neq \phi(g_2) \), and
(ii) two vertices \( h_1 \) and \( h_2 \) of H are adjacent if and only if there are two adjacent vertices \( g_1 \) and \( g_2 \) in G such that \( \phi(g_1) = h_1 \) and \( \phi(g_2) = h_2 \).

A graph G is vertex critical with respect to a graph parameter \( \gamma \) (\( \gamma \)-vertex critical for short) if \( \gamma(G - v) < \gamma(G) \) for every vertex \( v \) of G. A graph G is \( \gamma \)-edge critical if \( \gamma(G - e) < \gamma(G) \) for every edge \( e \) of G. A graph that is both \( \gamma \)-vertex critical and \( \gamma \)-edge critical is simply called \( \gamma \)-critical. On the other hand, if \( \gamma(G + ab) > \gamma(G) \) for every pair of nonadjacent vertices \( a \) and \( b \) of G (where \( ab \) is a new edge joining \( a \) and \( b \)), then G is called \( \gamma \)-saturated.

We define a graph G to be \( \gamma \)-replete if \( \gamma(e(G)) > \gamma(G) \) for every elementary homomorphism \( e \) of G.

In this paper we study graphs that are replete with respect to the clique number \( \omega \). The study of \( \omega \)-replete graphs is greatly facilitated by relationships that exist between \( \alpha \)-critical graphs (where \( \alpha \) is the vertex covering number), \( \omega \)-saturated graphs, and \( \omega \)-replete graphs. If G is \( \omega \)-replete (-satu-

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rated) with \(\omega(G) = k\), we say that \(G\) is k-clique replete (-saturated), and if \(G\) is \(\alpha\)-critical with \(\alpha(G) = k\) we say that \(G\) is k-covering critical.

For each natural number \(k\), the complete graph \(K_k\) is trivially k-clique replete. We prove that, for each \(k \geq 2\), the graph \(C_5 + K_{k-2}\) is the smallest noncomplete k-clique replete graph. (By “the smallest” we mean one of smallest order, and the only one of that order.) Furthermore, \(C_{2k+1}\) is the smallest k-clique replete graph without universal vertices. (A vertex \(v\) is universal in \(G\) if it is adjacent to every vertex of \(G - v\).) We also prove that, for each \(k \geq 2\) there exist k-clique replete graphs of arbitrarily large order that do not contain universal vertices.

It seems difficult to find good upper and lower bounds for the size of a k-clique replete graph of given order. As a result in this direction, we prove that every k-clique replete graph of order at least \([3k/2]\) contains a vertex of degree at least \([3k/2] - 1\). However, for every \(k \geq 2\) there exists a k-clique replete graph of arbitrarily large order that contains a vertex of degree 2.

Results concerning \(\omega\)-replete graphs were employed in [3] to determine a lower bound for the order of certain uniquely colorable graphs. Results on graphs that are replete with respect to other parameters might prove useful in solving other problems of a similar nature. The value of an elementary homomorphism for this purpose is that it decreases the order of a graph, while leaving certain properties of the graph unchanged.

2. PRELIMINARIES

We shall say that a clique \(A\) of a graph \(G\) is jointly owned by two nonadjacent vertices \(x\) and \(y\) of \(G\) if \(A\) contains neither \(x\) nor \(y\) but every vertex of \(A\) is adjacent to either \(x\) or \(y\) or both. Thus a graph \(G\) is k-clique replete if and only if \(\omega(G) = k\) and every pair of nonadjacent vertices of \(G\) jointly owns a k-clique.

It is well known that the graph parameters \(\alpha\), \(\beta\), \(\nu\), \(\omega\) (where \(\nu\) is the order and \(\beta\) the vertex independence number) are related as follows:

**Lemma 2.1.**

(i) \(\alpha(G) + \beta(G) = \nu(G)\).

(ii) \(\omega(G) = \beta(\overline{G})\).

The following relationships exist between \(\omega\)-saturated, \(\alpha\)-critical, and \(\omega\)-replete graphs.

**Lemma 2.2.**

(i) A graph \(G\) is k-clique saturated if and only if \(\overline{G}\) is \((\nu(G) - k)\)-covering edge critical.

(ii) If \(G\) is a k-clique replete subgraph without universal vertices, then \(\overline{G}\) is \((\nu(G) - k)\)-covering vertex critical.

**Proof.**

(i) The result follows directly from Lemma 2.1.

(ii) It follows from Lemma 2.1 that \(\alpha(\overline{G}) = \nu(G) - k\), and hence \(\overline{G}\) contains a \((\nu(G) - k)\)-covering vertex critical graph \(H\) as induced subgraph. Suppose \(V(H) \neq V(G)\). Then, since \(\alpha(H) = \alpha(\overline{G})\), the graph...
\[ G - V(H) \] consists only of isolated vertices, and hence \( G - V(H) \) is a complete graph. Let \( x \in V(G) - V(H) \). Since \( G \) contains no universal vertices, there is a vertex \( y \) in \( H \) that is not adjacent to \( x \) in \( G \). Let \( \varepsilon \) be the elementary homomorphism that identifies \( x \) and \( y \). Then

\[
\omega(\varepsilon(G)) \leq 1 + \omega(\overline{H} - y) + \omega(G - V(H) - x) \\
\leq 1 + \beta(H) + (\nu(G) - \nu(H) - 1) \quad \text{since} \quad \omega(\overline{H}) = \beta(H) \\
= \nu(G) - \alpha(H) \\
= \nu(\overline{G}) - \alpha(G) \\
= \beta(\overline{G}) \\
= k.
\]

This contradicts our assumption that \( G \) is \( k \)-clique replete, which proves that \( G = H \). \qed

Note that, for each \( k \geq 2 \), the graph \( kK_2 \) is \( \omega \)-saturated but not \( \omega \)-replete, whereas, for any \( n \geq 2 \) and \( k \geq 1 \) the lexicographic product \( \overline{C_{2n+1}} [K_k] \) is \( \omega \)-replete but not \( \omega \)-saturated (and its complement is \( \alpha \)-vertex critical but not \( \alpha \)-edge critical).

We shall need some results concerning \( \omega \)-saturated and \( \alpha \)-critical graphs.

**Theorem 2.3** (Hajnal [5] Theorem 1). If \( G \) is a \( k \)-clique saturated graph, then either \( G \) contains a universal vertex or \( \delta(G) \geq 2(k - 1) \). \qed

In view of Lemma 2.2(i) and that an \( \alpha \)-critical graph contains no isolated vertices, Theorem 2.3 can be restated for \( \alpha \)-critical graphs. (See also Theorem 3 of [7].)

**Theorem 2.4.** If \( G \) is a \( k \)-covering critical graph, then \( \Delta(G) \leq 2k - \nu(G) + 1 \). \qed

**Corollary 2.5.** If \( G \) is a connected \(( k + 1)\)-covering critical graph of order \( 2k + 1 \), then \( G = C_{2k+1} \). \qed

**Theorem 2.6** (Erdős and Gallai [4] Result 4.8). If \( G \) is a \(( k + 1)\)-covering vertex critical graph, then \( \nu(G) \leq 2k \), with equality only if \( G = kK_2 \). \qed

We shall also need some results concerning cliques of a graph.

**Lemma 2.7** (Hajnal [5]). If \( G \) is a graph with \( \omega(G) = k \) and \( A_1, \ldots, A_r \), are \( k \)-cliques of \( G \), then \( |\bigcap_{i=1}^{r} V(A_i)| \geq 2k - |\bigcup_{i=1}^{r} V(A_i)| \). \qed

The set of universal vertices of a graph \( G \) will be denoted by \( U_G \).
Lemma 2.8. If \( G \) is a \( k \)-clique replete graph, then \( U_G \) is the intersection of all the \( k \)-cliques of \( G \).

**Proof.** Let \( Z \) be the intersection of all the \( k \)-cliques of \( G \). Let \( x \in U_G \) and suppose \( A \) is a \( k \)-clique of \( G \) that does not contain \( x \). Then \( \{x\} \cup A \) is a \((k + 1)\)-clique, which contradicts \( \omega(G) = k \). Hence \( U_G \subseteq V(Z) \). Now suppose \( Z \) has a vertex \( z \) that is not universal in \( G \). Then \( G - V(Z) \) has a vertex \( v \) that is not adjacent to \( z \). But since \( G \) is \( k \)-clique replete, it follows that \( G \) contains a \( k \)-clique \( A \) that is jointly owned by \( z \) and \( v \), and then \( z \notin A \), contradicting our assumption that \( z \) lies in every \( k \)-clique of \( G \). \( \blacksquare \)

Our next result is easy to prove.

Lemma 2.9. A graph \( G \) is \( k \)-clique replete if and only if \( G + K_r \) is \((k + r)\)-clique replete for every positive integer \( r \). \( \blacksquare \)

Corollary 2.10. A graph \( G \) is \( k \)-clique replete if and only if \( G - S \) is \((k - |S|)\)-clique replete for each subset \( S \) of \( U_G \). \( \blacksquare \)

A graph \( G \) is called **vertex determining** if no two distinct vertices of \( G \) have the same neighborhood. (See [8].)

Lemma 2.11. If \( G \) is an \( \omega \)-replete graph, then \( G \) is vertex determining.

**Proof.** If two vertices are adjacent, they cannot have the same neighborhood. Suppose \( v \) and \( w \) are two nonadjacent vertices of \( G \) and \( \omega(G) = k \). Since \( G \) is \( k \)-clique replete there is a \( k \)-clique \( A \) of \( G \) that is jointly owned by \( v \) and \( w \). Then \( A \) contains a vertex \( a \) that is not adjacent to \( v \), otherwise \( A \cup \{v\} \) would induce a \((k + 1)\)-clique in \( G \). But then \( a \) is adjacent to \( w \), so \( v \) and \( w \) have different neighborhoods. \( \blacksquare \)

Our final result in this section concerns the chromatic number \( \chi \).

Lemma 2.12. If \( G \) is a noncomplete \( k \)-clique replete graph then \( \chi(G) > k \).

**Proof.** If \( \chi(G) = r \) then there exists a homomorphism \( \phi \) of \( G \) such that \( \phi(G) = K_r \). (See [6] Theorem 1.) Since \( \phi \) consists of the composition of a sequence of elementary homomorphisms, \( \omega(\phi(G)) > k \). Hence \( r > k \). \( \blacksquare \)

3. THE ORDER OF \( k \)-CLIQUE REPLETE GRAPHS

A \( k \)-clique saturated graph of order \( n \) contains at least \( 2k - n \) universal vertices. (See [5] Theorem 2.) We have a corresponding result for \( k \)-clique replete graphs.
Theorem 3.1. If $G$ is a $k$-clique replete graph of order $n > k$, then $G$ contains at least $2k + 1 - n$ universal vertices.

Proof. By Corollary 2.10 the graph $G - U_G$ is $(k - |U_G|)$-clique replete and hence by Lemma 2.2(ii), its complement, $\overline{G - U_G}$ is $(n - k)$-covering vertex critical. It now follows from Theorem 2.6 that either $\nu(G - U_G) < 2(n - k)$ or $\overline{G - U_G} = (n - k)K_2$. If the latter is true then $G - U_G = (n - k)K_2$. However, this is not the case, because $G - U_G$ is an $\omega$-replete graph whereas $(n - k)K_2$ is clearly not vertex determining and hence not $\omega$-replete by Lemma 2.11. We therefore have $\nu(G - U_G) < 2(n - k)$, which implies $\nu(U_G) > 2k - n$. \hfill \Box

Corollary 3.2. Every $k$-clique replete graph of order less than $2k + 1$ contains a universal vertex. \hfill \Box

Theorem 3.3. For every integer $k \geq 2$ the graph $C_5 + K_{k-2}$ is the smallest noncomplete $k$-clique replete graph.

Proof. The proof is by induction over $k$. It is easy to check that $C_5$ is the smallest $2$-clique replete graph. Now let $k > 2$ and suppose $G$ is a $k$-clique replete graph of smallest possible order. Then $\nu(G) \leq \nu(C_5 + K_{k-2}) = k + 3 < 2k + 1$ (because $C_5 + K_{k-2}$ is $k$-clique replete by Lemma 2.9). Hence $G$ contains a universal vertex $x$ by Corollary 3.2. But then it follows from Lemma 2.9 that $G - x$ is a $(k - 1)$-clique replete graph of smallest possible order. By our induction hypothesis we therefore have $G - x = C_5 + K_{k-3}$, and the result follows. \hfill \Box

Theorem 3.4. For every integer $k \geq 2$ the graph $C_{2k+1}$ is the smallest $k$-clique replete graph without universal vertices.

Proof. It is easy to see that $C_{2k+1}$ is a $k$-clique replete graph without universal vertices. By Corollary 3.2 it is such a graph of smallest possible order. We shall now prove that it is the only one of that order.

Suppose $G$ is a $k$-clique replete graph of order $2k + 1$ without universal vertices. Then $\overline{G}$ is $(k + 1)$-covering vertex critical by Lemma 2.2(ii). Hence $\overline{G}$ contains a $(k + 1)$-covering critical subgraph $H$ with $V(H) = V(\overline{G})$. Since $\nu(H) = 2k + 1$, it follows from Theorem 2.4 that $\Delta(H) \leq 2$.

We now prove that $H$ is a connected graph. Let $H_1, \ldots, H_r$ be the connected components of $H$. Then $H = \bigcup_{i=1}^r H_i$ and $H_i$ is $\alpha$-critical and $\Delta(H_i) \leq 2$ for each $i = 1, \ldots, r$. Hence each $H_i$ is an odd cycle or a $K_2$. Suppose $H_j = K_2$ for some $j$. Then the two vertices $x$ and $y$ of $H_j$ are nonadjacent in $G$. Let $\epsilon$ be the elementary homomorphism of $G$ that identifies $x$ and $y$. Since $G \subseteq \overline{H} = H_1 + \cdots + H_r$, we have $\omega(\epsilon(G)) = \sum_j \omega(H_j) + 1 = \omega(H) = \beta(H) = \nu(H) - \alpha(H) = 2k + 1 - (k + 1) = k$. This contradicts the assumption that $G$ is $k$-clique replete, proving that for each
For each $k \geq 3$ we shall now construct a $k$-clique replete graph that has the special properties required by Lemma 3.5 with $r = 2$.

**Example 3.6.** Let $k \geq 3$. Let $A$ be the graph of order $2k - 2$ with vertex set $\{a_1, \ldots, a_{2k-2}\}$ having $a_i$ adjacent to $a_j$ if and only if $|i - j| \neq k - 1$. (Thus $A$ is isomorphic to the complete graph $K_{2k-2}$ with a one-factor removed.) Let the graph $B$ with vertex set $\{b_1, \ldots, b_{2k-2}\}$ be similarly defined. For each $i = 1, \ldots, 2k - 2$ join $a_i$ to each of the vertices $b_i, b_{i+1}, \ldots, b_{i+k-2}$. Call the resulting graph $R_k$. Then $R_k$ is a $k$-clique replete graph, and $A$ and $B$ are (disjoint) maximal $K_k$-free induced subgraphs of $R_k$. Moreover, $\delta(R_k - A) > 1$ and $\delta(R_k - B) > 1$ since $k \geq 3$. 

**Theorem 3.7.** For each $k \geq 2$ there exist $k$-clique replete graphs of arbitrarily large order that contain no universal vertices.
Proof. For \( k \geq 3 \) the construction is as follows: Take \( n \) pairwise disjoint copies \( D_1, \ldots, D_n \) of the graph \( R_k \) described above, with \( n \geq 3 \) (and as large as you wish). For each \( i = 1, \ldots, n \) let \( A_i \) and \( B_i \) be two disjoint maximal \( K_k \)-free induced subgraphs of \( D_i \) with \( \delta(D_i - A_i) > 1 \) and \( \delta(D_i - B_i) > 1 \). Now take two sets of additional vertices \( \{a_1, \ldots, a_n\} \) and \( \{b_1, \ldots, b_n\} \) and, for each \( i = 1, \ldots, n \) join \( a_i \) to every vertex of \( B_i \) and also to every vertex of \( A_j \) for each \( j \neq i \). Likewise, join \( b_i \) to every vertex of \( A_i \) and also to every vertex of \( B_j \) for each \( j \neq i \). Call the resulting graph \( R_{k,n} \). By Lemma 3.5 the graph \( F_i = (D_i \cup \{a_i, b_i\}) \) is \( k \)-clique replete for each \( i = 1, \ldots, n \). In order to prove that \( R_{k,n} \) is \( k \)-clique replete, we now only need to show that every pair of nonadjacent vertices \( u_1 \) and \( u_2 \) with \( u_1 \in F_1 \) and \( u_2 \in F_2 \), jointly own a \( k \)-clique. We distinguish six cases. In each case we exhibit a vertex \( x \) and a \((k - 1)\)-clique \( X \) such that \( \{x \cup X\} \) is a \( k \)-clique jointly owned by \( u_1 \) and \( u_2 \).

Case 1. \( u_1 \in A_1 \) and \( u_2 \in A_2 \): Let \( x = a_1 \) (which is adjacent to \( u_2 \)) and let \( X \) be a \((k - 1)\)-clique of \( B_1 \), all of whose vertices are adjacent to \( u_1 \). (The existence of such a \((k - 1)\)-clique is guaranteed by the fact that \( B_1 \) is a maximal \( K_k \)-free induced subgraph of \( F_1 \).)

Case 2. \( u_1 \in A_1 \) and \( u_2 \in B_2 \): Let \( x = b_3 \) (which is adjacent to \( u_2 \)) and let \( X \) be as in (i).

Case 3. \( u_1 \in A_1 \) and \( u_2 = a_2 \): Let \( x = b_1 \) and let \( X \) be any \((k - 1)\)-clique in \( B_2 \).

Case 4. \( u_1 \in A_1 \) and \( u_2 = b_2 \): Let \( x = a_3 \) and let \( X \) be any \((k - 1)\)-clique in \( B_3 \).

Case 5. \( u_1 = a_1 \) and \( u_2 = a_2 \): Let \( x \) be any vertex in \( A_2 \) and let \( X \) be a \((k - 1)\)-clique in \( B_2 \), all of whose vertices are adjacent to \( x \).

Case 6. \( u_1 = a_1 \) and \( u_2 = b_2 \): Let \( x \) be a vertex in \( A_3 \) and let \( X \) be a \((k - 1)\)-clique in \( B_3 \), all of whose vertices are adjacent to \( x \).

For \( k = 2 \) we need a slightly different construction since we were unable to find a suitable 2-clique replete graph that satisfies the requirements of Lemma 3.5. In this case let each of the graphs \( A_1, \ldots, A_n, B_1, \ldots, B_n \) be a copy of \( K_1 \), and for each \( i = 1, \ldots, n \) join the vertex of \( A_i \) to the vertex of \( B_i \). Now proceed exactly as in the case \( k \geq 3 \), but also add three new vertices \( v, w, \) and \( z \). Join \( v \) to each of the vertices \( a_1, \ldots, a_n \), and join \( w \) to each of the vertices \( b_1, \ldots, b_n \). Join \( z \) to \( v \) and \( w \) only. Call the resulting graph \( R_{2,n} \). It is easy to prove that \( R_{2,n} \) is \( k \)-clique replete.

Note that, for \( k \geq 2 \) and \( n \geq 3 \), the graph \( R_{k,n} \) constructed above contains smaller \( k \)-clique replete graphs as subgraphs. We now define a graph
to be a minimal \(k\)-clique replete graph if it is \(k\)-clique replete but contains no \(k\)-clique replete graph as proper subgraph. We conjecture that for each \(k \geq 1\) there are only finitely many minimal \(k\)-clique replete graphs.

The only minimal 1-clique replete graph is \(K_1\), and the only minimal 2-clique replete graphs are \(K_2\) and \(C_3\). If \(k = k_1 + k_2\) and \(G_i\) is a minimal \(k_i\)-clique replete graph for \(i = 1, 2\), then the join \(G_1 + G_2\) is a minimal \(k\)-clique replete graph. Also, if \(k = nm\) and \(G\) is a minimal \(m\)-clique replete graph, then the lexicographic product \(G[K_n]\) is a minimal \(k\)-clique replete graph. A minimal \(k\)-clique replete graph that cannot be obtained from \(\omega\)-replete graphs with smaller clique numbers by forming such a join or a lexicographic product will be called a new minimal \(k\)-clique replete graph.

For each \(k \geq 3\) the only known new minimal \(k\)-clique replete graphs are \(C_{2k+1}\) and the graph \(R_k\) constructed in Example 3.6.

4. MAXIMUM AND MINIMUM DEGREES OF \(k\)-CLIQUE REPELETE GRAPHS

Theorem 4.1. If \(G\) is a \(k\)-clique replete graph of order at least \(\lfloor 3k/2 \rfloor\) then \(\Delta(G) \geq \lfloor 3k/2 \rfloor - 1\).

Proof. Suppose that for some integer \(k\) there exist \(k\)-clique replete graphs of order at least \(\lfloor 3k/2 \rfloor\) with maximum degree less than \(\lfloor 3k/2 \rfloor - 1\). Let \(G\) be such a graph of smallest order. Then \(G\) contains no universal vertices and hence we have, by Lemma 2.8:

\[
\text{If } G_1, \ldots, G_m \text{ are all the } k\text{-cliques of } G, \text{ then } \bigcap_{i=1}^m G_i = \phi. \quad \text{(A)}
\]

By our assumption that \(\Delta(G) < \lfloor 3k/2 \rfloor - 1\) we have

\[
\text{If } H_1, \ldots, H_i \text{ are } k\text{-cliques of } G \text{ such that } \bigcap_{i=1}^\infty H_i \neq \phi, \text{ then } \bigcup_{i=1}^\infty H_i \leq \lfloor 3k/2 \rfloor - 1. \quad \text{(B)}
\]

Hence, by Lemma 2.7:

\[
\text{If } H_1, \ldots, H_i \text{ are as in (B), then } \bigcap_{i=1}^\infty H_i \geq \lfloor k/2 \rfloor + 1. \quad \text{(C)}
\]

Also, by Lemma 2.7:

\[
\text{If } F_1, \ldots, F_i \text{ are } k\text{-cliques of } G \text{ such that } \bigcup_{i=1}^\infty F_i \leq 2k - 1 \text{ then } \bigcap_{i=1}^\infty F_i \neq \phi. \quad \text{(D)}
\]

We distinguish two cases.

Case 1. Every pair of \(k\)-cliques of \(G\) have a nonempty intersection. Let \(G_1\) and \(G_2\) be any two \(k\)-cliques of \(G\). Since \(G_1 \cap G_2 \neq \phi\), and in view of (B), it follows that \(k + 1 \leq |G_1 \cup G_2| \leq \lfloor 3k/2 \rfloor - 1 \leq 2k - 1\).
Arguing inductively, suppose that \( s \geq 2 \) and that \( G_1, \ldots, G_s \) are \( k \)-cliques such that \( k + s - 1 \leq |\cup_{i=1}^s G_i| \leq 2k - 1 \). Then it follows from (D) that \( \cap_{i=1}^s G_i \neq \emptyset \) and so we may choose \( x_i \in \cap_{i=1}^s G_i \). By (A) there exists a \( k \)-clique \( G_{s+1} \) of \( G \) such that \( x \notin G_{s+1} \). Now \( G_{s+1} \cup \{ x_i \} \) would induce a \( (k + 1) \)-clique in \( G \). Therefore \( |\cup_{i=1}^{s+1} G_i| \geq |\cup_{i=1}^s G_i| + 1 \geq k + (s + 1) - 1 \). Moreover, \( |\cup_{i=1}^s G_i| \leq [3k/2] - 1 \) by (B) and \( |G_{s+1} \cap G_i| \geq [k/2] + 1 \) by (C), and so

\[
|\cup_{i=1}^{s+1} G_i| \leq ([3k/2] - 1) + (k - [k/2] - 1) \leq 2k - 1.
\]

Therefore \( G_1, \ldots, G_{s+1} \) are \( k \)-cliques such that

\[
k + (s + 1) - 1 \leq |\cup_{i=1}^{s+1} G_i| \leq 2k - 1.
\]

Therefore, by induction, for each positive integer \( s \geq 2 \) there exist \( k \)-cliques \( G_1, \ldots, G_s \) such that

\[
k + s - 1 \leq |\cup_{i=1}^s G_i| \leq 2k - 1.
\]

This is clearly absurd when \( s = k + 1 \).

**Case 2.** There is a pair of disjoint \( k \)-cliques in \( G \). Let \( A \) and \( B \) be \( k \)-cliques of \( G \) such that \( A \cap B = \emptyset \). Let \( \mathcal{H} = \{ H | H \) is a \( k \)-clique of \( G \) and \( H \cap A \neq \emptyset \} \) and \( \mathcal{F} = \{ F | F \) is a \( k \)-clique in \( G \) and \( F \cap B \neq \emptyset \} \). Now let \( X = \langle \cup_{H \in \mathcal{H}} H \rangle \) and \( Y = \langle \cup_{F \in \mathcal{F}} F \rangle \).

First we prove that \( X \) is a \( k \)-clique replete graph. This is clear if \( X = K_k \) so suppose that \( X \neq K_k \). Let \( v \) and \( w \) be any two nonadjacent vertices of \( X \). Let \( M \) be a \( k \)-clique of \( G \) that is jointly owned by \( v \) and \( w \). Then \( v \) or \( w \), say \( v \), is adjacent to at least \( [k/2] \) vertices of \( M \). But \( v \) is contained in a \( k \)-clique \( H \) of \( X \), and is therefore adjacent to \( k - 1 \) vertices of \( H \). Hence, by our assumption on \( \Delta(G) \), we have \( M \cap H \neq \emptyset \), so that \( |M \cap H| \geq [k/2] + 1 \) by (C). But also \( |A \cap H| \geq [k/2] + 1 \) by (C), and hence \( A \cap M \neq \emptyset \). By the definition of \( X \) we therefore have \( M \subseteq X \). This proves that \( X \) is \( k \)-clique replete.

By the same argument, \( Y \) is also \( k \)-clique replete. By our choice of \( G \), we therefore have \( |X| \leq [3k/2] - 1 \) and \( |Y| \leq [3k/2] - 1 \). If \( X \neq K_k \) it now follows from Theorem 3.1 that \( |U_X| \geq [k/2] + 2 \). This is also true if \( X = K_k \). It is similarly true that \( |U_Y| \geq [k/2] + 2 \). But \( A \) is a \( k \)-clique of \( X \), so that \( U_X \subseteq A \) by Lemma 2.8. Likewise, \( U_Y \subseteq B \). Now if \( x \in U_X \), then \( x \) is adjacent to \( k - 1 \) vertices of \( A \), and since \( d(x) < [3k/2] - 1 \) and \( A \cap B = \emptyset \), at most \( [k/2] - 1 \) vertices of \( B \) are adjacent to \( x \). Hence \( U_x \) contains a vertex \( y \) that is nonadjacent to \( x \). There is therefore a \( k \)-clique \( D \) in \( G \) that is jointly owned by \( x \) and \( y \). At least one of \( x \) and \( y \), say \( x \), is then adjacent to at least \( [k/2] \) vertices of \( D \). But then \( A \cap D \neq \emptyset \), otherwise \( d(x) \geq ...
For all even \( k \) the graph \( C_2 [K_{k/2}] \) is an example of a \( k \)-clique replete graph whose maximum degree attains the bound given in Theorem 4.1. However, for \( k \)-clique replete graphs of large order (compared to \( k \)), this bound is probably poor.

**Theorem 4.2.** For each \( k \geq 2 \) there exists a \( k \)-clique replete graph \( G \) of arbitrarily large order such that \( \delta (G) = 2 \).

**Proof.** Take the graph \( R_{k,n} \) (with \( n \) as large as you wish) constructed in Theorem 3.7. For \( k \geq 3 \) add three new vertices \( u, w, \) and \( z \) to \( R_{k,n} \). Join \( u \) to every vertex of each \( A_i, i = 1, \ldots, n \). Also join \( w \) to every vertex of each \( B_i, i = 1, \ldots, n \). Now join \( z \) to \( u \) and \( w \) only. The resulting graph is \( k \)-clique replete and \( d(z) = 2 \). For \( k = 2 \) the graph \( R_{2,n} \) already contains a vertex of degree 2.

The graph constructed above is, of course, not a minimal \( k \)-clique replete graph. For minimal \( k \)-clique replete graphs we have

**Theorem 4.3.** If \( G \) is a minimal \( k \)-clique replete graph then \( \delta (G) \geq k \).

**Proof.** Let \( x \) be any vertex of \( G \). Then there exists a pair of nonadjacent vertices \( u \) and \( w \) in \( G \) such that \( x \) is contained in every \( k \)-clique that is jointly owned by \( u \) and \( w \) (otherwise \( G - x \) would be \( k \)-clique replete). Hence \( x \) is adjacent to \( k - 1 \) vertices of a \( k \)-clique, and also to \( u \) or \( w \).

Andrásvai, Erdős, and Sós proved that if \( G \) is a graph of order \( n \) with \( \omega (G) = k \) and \( \chi (G) > k \), then \( \delta (G) \leq (3k - 4)n/(3k - 1) \). (See [1] Theorem 1.1.) By Lemma 2.12 this is then also an upper bound for the minimum degree of a noncomplete \( k \)-clique replete graph. When \( (3k - 1)(3k - 4)n \) the (unique) extreme graphs belonging to the theorem just mentioned are not vertex determining when \( n > 5 \), and hence not \( k \)-clique replete by Lemma 2.11. We therefore have

**Theorem 4.4.** If \( G \) is a noncomplete \( k \)-clique replete graph of order \( n > 5 \) then \( \delta (G) < (3k - 4)n/(3k - 1) \).

For \( k \geq 2 \) the graph \( \overline{C}_{2k+1} \) is a \( k \)-clique replete graph of order \( n = 2k + 1 \) and \( \delta (\overline{C}_{2k+1}) = 2k - 2 = [(2k - 4)n/(3k - 1)] \). However, when \( n \) is large compared to \( k \), this upper bound is probably poor.
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