On the Ranking of Bilateral Bargaining Opponents*

Ross Cressman† and Maria Gallego‡

June 24, 2007

Abstract: We fix the status quo ($Q$) and one of the bilateral bargaining agents to examine how shifting the opponent’s ideal point away from $Q$ in a unidimensional space affects the Nash (NS) and the Kalai-Smorodinsky (KS) bargaining solutions. As anticipated, the bargainer whose ideal point is farthest from $Q$ prefers an opponent whose ideal is closest to his own. On the other hand, the ranking of opponents by the player closest to $Q$ depends on the opponent’s intensity of preferences (or absolute risk aversion, ARA). An intuitive ranking emerges when the opponent preferences exhibits increasing ARA. However, under decreasing ARA, the player closest to $Q$ prefers a more extreme opponent since the farther the opponent’s ideal is from $Q$, the less intense her preferences and the easier she is to satisfy.

Keywords: Game Theory, Nash bargaining problems, bargaining solutions, rankings, monotone comparative statics.

JEL: C7, C71, C78

*This research was partly carried out while Ross Cressman was a Visiting Professor at the University of Vienna and Maria Gallego was visiting the University of Toulouse and while both authors where Fellows at the Collegium Budapest. The authors thank the institutions for their hospitality and research support. Also acknowledged is financial support from the Society of Management Accountants of Ontario and the Natural Sciences and Engineering Research Council of Canada. Many thanks to Ehud Kalai, Marc Kilgour, Hervé Moulin and William Thomsson for their useful comments. The usual caveat applies.

†Department of Mathematics, Wilfrid Laurier University, Waterloo, Ontario N2L 3C5 Canada; email: rcressma@wlu.ca

‡author of correspondence: Department of Economics, Wilfrid Laurier University, Waterloo, Ontario N2L 3C5 Canada; email: mgallego@wlu.ca
1 Introduction

In social choice environments, bilateral bargaining situations occur for example when voters delegate bargaining to between party negotiations in three-party legislatures, to governments in international negotiations or to central and sub-national authorities in intergovernmental negotiations; or when unions represent workers in wage negotiations. Though not concerned with delegation and its incentive problems, we study situations where negotiators use their preferences to bargain over outcomes in a unidimensional space. Choosing a delegate requires ranking the agreements reached between different bargaining pairs.

A bilateral bargaining problem is defined by the set of feasible utility payoffs ($S$) for each agent including the disagreement outcome ($d$) that prevails if negotiations fail over a set of alternatives ($X$). There are several solutions\(^1\) to the bargaining problem ($S, d$). Although we concentrate on the Nash (NS) and Kalai-Smorodinsky (KS) solutions in this paper, our results also apply to the Egalitarian (EG) and Perles-Maschler (PM) solutions (see Remarks 3 and 4 below). In addition, we assume the disagreement outcome is outside the bargainers control.\(^2\)

We show that the presence of heterogeneous bargainers renders ranking agreements and consequently the choice of bargaining partner more difficult.

In order to rank the agreements reached between different bargaining pairs,

---

\(^1\)For excellent discussions on bargaining solutions see Peters [1992a] and Thomson [1994].

\(^2\)In social choice environments the disagreement outcome is usually determined by previous agreements reached by perhaps a different pair of agents. The disagreement outcome represents the bargainers’ fallback position when negotiations fail and is determined by the status quo policy. We use these terms throughout the paper.
we must understand how three crucial elements affecting the opponent’s preferences determine her bargaining position: her most preferred agreement or ideal point, the degree of concavity of her utility function and her disagreement outcome when bargaining fails. These three characteristics interact with each other to determine a bargainer’s toughness in negotiations.

We assume players have different ideal points in a unidimensional space. A bargainer may face opponents with ideal points on either side of her ideal who may also differ in their toughness in negotiations. This, in conjunction with the idea that players prefer agreements closer to their ideals, naturally leads to assuming that bargainers have single-peaked preferences.\(^3\)

In Austen-Smith and Banks (1988), voters delegate policy making to a three-party legislature. Parties have single-peaked quadratic preferences with different ideal policies over a unidimensional space. In minority situations, each party, chosen in the order of vote shares, makes a proposal to the legislature. If no party gets its proposal approved, all parties receive a zero payoff. In equilibrium, the bilateral bargaining agreement between parties follows the ranking of the party’s ideal policies. We show that these two rankings may not coincide when preferences are not quadratic.

In Gallego and Scoones (2007), voters elect one of three parties to represent them in intergovernmental negotiations. Since the elected State formateur engages in Nash (1950) bargaining over a unidimensional policy with its Federal counterpart, voters must rank anticipated agreements. Gallego and Scoones find that agreements depend on the identity and the toughness

\(^3\)Single-peaked utility functions are commonly used in social choice and political economy models (see Austen-Smith and Banks [1999,2005] or Persson and Tabellini [2000]). For example Alesina and Rosenthal [1995,1996] and Baron and Ferejohn [1989] assume political parties have single-peaked preferences.
of the formateurs. When parties have quadratic utility functions, the ranking of agreements and party’s ideal policies coincide. However, if the centre and right parties are more risk averse than the left party, the rankings of agreements and party’s ideal policies do not coincide. We show this unintuitive case emerges when the agents’ type and fallback position matter, thus extending and adding insights to their findings in a broader framework.

We study the effect that varying the opponent’s type has on the bargaining solutions. We assume opponent’s type is defined by her ideal point but maintain the remaining characteristics of the opponent’s utility function constant. Opponents’ utilities are then just perfect translations of each other. Even under this very strong assumption the ranking of agreements and bargainers’ ideal points may not coincide. Opponents’ preferences may not always be perfect translations of each other since they may differ in more ways than their ideals. Allowing opponents to have more general preferences can however only complicate matters. We opt then for perfect translations as it clearly conveys the main message of our paper. Moreover, under the assumption of perfect translations of opponents, their preferences satisfy the single-crossing (SC) condition given in Milgrom and Shannon (1994).\footnote{Ashworth and Bueno de Mesquita (2006), Alesina and Rosenthal (1995), Austen-Smith and Banks (1988) and Gans and Smart (1996)—among others in political economy and social choice—use translated single-peaked utility functions to illustrate the SC condition.}

We have a family of problems indexed by pairs of negotiators. We assume the existence of a fallback position completely outside the agents’ control. We cast our model in a complete information riskless framework to isolate the effect of types and the disagreement outcome on the different solutions.

Let $L$ and $R$ represent the two bargainers. To rank agreements between
different bargaining pairs, we fix one agent and shift the opponent’s ideal point by $\alpha \geq 0$ units, $\alpha$ identifies the opponent’s type. We fix $R$ ($L$) and shift $L$ ($R$) to the right. To avoid repetition of cases that are reflections of those we consider, we assume the status quo $Q$ is to the left of $R$’s ideal point. Single-peakedness and agents’ types bring out the role of the status quo (relative to the ideal points) in these solutions.\(^5\)

We find that the NS and KS solutions may not be monotone in opponents’ ideals. That is, these bargaining solutions may rank opponents in opposite order of their ideals. The ranking depends on whether the opponent is the one closest or farthest from $Q$. In the following summary, $Q$ is to the left of $L$’s ideal. When we fix $R$ and shift $L$, $R$ prefers an opponent whose ideal is closer to her own. When we fix $L$ and shift $R$, things change. Whether $L$ prefers $R$ or the shift to $R_\alpha$ depends on $R$’s toughness in negotiations (a property connected to the concavity of $R$’s utility function $u_R$ and its absolute risk aversion, ARA, Peters 1992b). When $u_R$ exhibits IARA (increasing ARA), $L$ prefers $R$. When $u_R$ has DARA (decreasing ARA), $L$ prefers $R_\alpha$. The “unintuitive” DARA result contrasts with Kihlstrom et al.’s (1981) finding that, for a fixed set of alternatives,\(^6\) agents prefer more risk averse, easier to satisfy, opponents (Kobberling and Peters 2003).

In our model, fallback positions affect bargaining outcomes. The solutions move closer to $R$’s ideal point when $L$’s ideal shifts to the right. This is in the spirit of Thomson (1987) where the NS and KS solutions hurt the player whose fallback position worsens for a fixed $S$.\(^7\) In contrast, if $u_R$ has IARA,

\(^5\)Under single-peakedness the status quo may constrain the set of feasible alternatives.

\(^6\)In our model the set of feasible alternatives may change. Kannai (1977) first observed the results of Kihlstrom et al. (1981) and Roth (1979).

\(^7\)Others also examine the effect of disagreement outcomes on bargaining solutions (see
as $R_\alpha$’s disagreement point worsens, her bargaining position improves.

In the applied bargaining literature, preferences are characterized mathematically through explicit utility functions. Since quadratic utilities exhibit IARA, the ranking of solutions and opponent’s ideal points coincide.\(^8\) Under DARA (e.g., for logarithmic utilities), $L$ ranks opponents opposite to their ideals. This counter-intuitive ranking of opponents in the DARA case is due to such an opponent with her ideal farther from $Q$ being more willing to compromise and easier to satisfy than if she has IARA preferences.

The DARA result contrasts with those in the monotone comparative statics (MCS) literature. Milgrom and Shannon (1994) show that when (1) the choice set and the parameters of the model\(^9\) can be rank-ordered and (2) players’ preferences satisfy the single-crossing (SC) condition, the solution to their optimization problem is monotonic in these parameters (i.e. the solution satisfies MCS). Athey (2002) extends this result by showing that if preferences exhibit DARA they satisfy the SC condition. Even though our single-peaked translated utility functions always satisfy SC, we show that bargaining solutions do not always follow the ordering of opponents’ ideals, i.e., may not satisfy MCS with respect to opponents’ ideals.

In a unidimensional space, rank-ordering bargainers according to their ideal points is not enough. In our set up, changes in the opponent’s ideal affects other components of the bargaining problem: the concavity of the opponents’ utility function at a given point and his disagreement outcome.

---

\(^8\)This supports the assumption made in multiparty ($> 2$) models where policy is modeled as a convex combination of the ideal policies of parties involved in negotiations.

\(^9\)In our model, the choice set corresponds to the set of feasible agreements and the single parameter set is the opponent’s ideal point.
These three components jointly determine a player’s toughness in negotiations, which in turn determines whether the ranking of agreements and of opponents’ ideals coincide.

The analysis leading to the above counterintuitive results rely on translated utility functions that exhibit DARA. More importantly, we show that even under IARA preferences, a minor departure from IARA near the solution is enough to “upset” the ranking of agreements and the preferences over opponents’ types (see Example 2 below). Furthermore, if we allow opponents to have more general (non-identical, non-uniform ARA) preferences, the ranking of agreements and opponents will depend on whether preferences exhibit IARA or DARA near the solution and, when all opponents have DARA preferences, on comparisons of the relative degree of DARA of their utility functions. Since many perturbations of utility functions that initially exhibit IARA near the solution will reverse the ranking, we conclude that the preference of bargainers, whose ideal is closest to the status quo $Q$, for opponents with ideals farther away is a quite general phenomenon.

## 2 The Model

Two bargainers $L$ and $R$ have different preferences over a unidimensional set of alternatives $X$. Their unimodal utility functions, $u_L$ and $u_R$, are defined on the compact interval $X = [a, b]$.\(^{10}\) For arbitrary utility functions, let $\hat{L} \equiv \arg \max u_L(x)$ and $\hat{R} \equiv \arg \max u_R(x)$ be their ideal points located in the interior of $X$ with $\hat{L} < \hat{R}$ (see quadratic utilities example in Figure 1).

\(^{10}\)In fact, we assume that the domain of these utility functions contains a larger interval than $X$. The reason for the broader domain will become clear later on.
If $L$ and $R$ simultaneously demand the payoff pair $(v_L, v_R)$ (i.e., $L$ demands $v_L$ and $R$ demands $v_R$ where $v_L$ and $v_R$ are real numbers) for which $v_L \leq u_L(x)$ and $v_R \leq u_R(x)$ for some $x \in X$, they both receive their demands. Otherwise (i.e., if, for all $x \in X$, either $v_L > u_L(x)$ or $v_R > u_R(x)$), they receive the default payoffs $(u_L(x_0), u_R(x_0))$ for some fixed status quo $x_0 \in X$ outside of their control.\footnote{The status quo may represent agreements reached in previous negotiations.} We avoid repetition by assuming $x_0 < \hat{R}$.

In our Nash (1950) bargaining problem, the feasible set $S$ is

$$\{(v_L, v_R) \mid u_L(x_0) \leq v_L \leq u_L(x) \text{ and } u_R(x_0) \leq v_R \leq u_R(x) \text{ for some } x \in X\}.$$  

Given concave utility functions, it can be shown $S$ is a compact convex subset of $\mathbb{R}^2$ (as in Figure 2 below\footnote{Figure 2(a) shows the feasible set $S$ for the quadratic utility functions of Example 1. In Figure 2(b) the feasible set $S$ is bound on the right by the parametric curve $\{(u_L(x), u_R(x)) \mid x \in [a, b], u_L(x) \geq u_L(x_0), u_R(x) \geq u_R(x_0)\}$. The solutions are unaffected by which feasible set is taken. We take the latter in our figures.})). The disagreement point is $d = (u_L(x_0), u_R(x_0))$. Every other point in $S$ weakly dominates $d$ (i.e., neither agent is worse off at this other point and at least one is better off). The pair $(S, d)$ constitute the bargaining problem (where $S$ and $d$ depend on $x_0$).

The Pareto optimum set is then given by

$$PO(S) \equiv \{(v_L, v_R) \in S \mid \text{either } v'_L < v_L \text{ or } v'_R < v_R \text{ for all other } (v'_L, v'_R) \in S\}$$

and the subset of alternatives $A \equiv \{x \in X \mid (u(x), v(x)) \in PO(S)\}$ is the bargaining set. The NS and KS solutions belong to $PO(S)$ and correspond to (possibly different) points $x^* \in A$.

When the status quo lies between the ideal points of the two bargainers, $x_0 \in [\hat{L}, \hat{R}]$, the bargaining solution is $x^* = x_0$ regardless of the solution concept used, since the bargaining set is $A = \{x_0\}$. 


For \( x \in [x_0, \hat{L}] \), both \( L \) and \( R \) prefer agreements to the right of \( x \). Thus, agreements acceptable to both cannot lie to the left of \( \hat{L} \) (i.e., \( A \subset [\hat{L}, b] \)). Let \( \overline{x_0} > \hat{L} \) be the agreement that keeps \( L \) indifferent to the status quo, \( u_L(\overline{x_0}) = u_L(x_0) \).\(^\text{13}\) Whether \( \overline{x_0} \) constrains the bargaining set depends on the location of \( R \)'s ideal relative to \( \overline{x_0} \). When \( \overline{x_0} \in (\hat{L}, \hat{R}] \), \( L \) rejects any proposal to the right of \( \overline{x_0} \), thus \( A = [\hat{L}, \overline{x_0}] \). When \( \hat{R} < \overline{x_0} \), for \( x \in [\hat{R}, \overline{x_0}] \), both prefer agreements to the left of \( \overline{x_0} \), \( A = [\hat{L}, \hat{R}] \) in this case.

We now illustrate the NS and KS solutions using quadratic utilities.

**Example 1** Suppose \( u_L \) and \( u_R \) are given by
\[
\begin{align*}
    u_L(x) &= -(x - 1)^2 + 1 \\
    u_R(x) &= -(x - 2)^2 + 1 \\
\end{align*}
\]
\[\text{(1)}\]

Clearly, the agents want to agree on a payoff pair \((u_L(x), u_R(x))\) for some \( x \) between the vertices of these parabolas (i.e. for some \( x \in A = [1, 2] \)) since every other point \((v_L, v_R)\) in the feasible set is dominated by a point of this form. When \( x_0 \in [1, 2] \), the agents agree on \( x_0 \), the only Pareto optimum point. When \( L \) wants to increase his payoff by demanding something higher than \( u_L(x_0) \) (i.e., \( x < x_0 \)), \( R \) opts for \( x_0 \) to avoid a decrease in her payoff.

When \( x_0 < 1 \) (or \( x_0 > 2 \)), both increase their payoff if they agree on \((u_L(x), u_R(x))\) for some \( x \in A = [1, 2] \). Figure 2 plots the boundary of the feasible set \( S \) when \( x_0 = 0 \) (i.e., \( \overline{x_0} = 2 \)). Every point in the interior of \( S \) has another point on the boundary in the first quadrant that dominates it. This boundary represents the Pareto optimum set.

\[\text{Figure 2 about here}\]

\[^{13}\]If there is no such \( \overline{x_0} \) (i.e. \( u_L(x) > u_L(x_0) \) for all \( x > \hat{L} \) that are in \( X \)), we take \( \overline{x_0} \) as the right-hand endpoint of \( X \) (i.e. \( \overline{x_0} = b \)).
The Nash solution (NS) is the unique point on the Pareto set corresponding to the \( x \) that maximizes the Nash product (see Section 3)

\[
\max_{x \in A} [u_L(x) - u_L(x_0)] [u_R(x) - u_R(x_0)].
\]

A straightforward calculation with \( x_0 = 0 \) yields \( x^* \approx 1.2192 \).

The Kalai-Smorodinsky (KS) solution is the unique point corresponding to \( x \) on the Pareto set for which (see Section 4)

\[
\frac{u_L(x) - u_L(x_0)}{u_L(L) - u_L(x_0)} = \frac{u_R(x) - u_R(x_0)}{u_R(R) - u_R(x_0)},
\]

i.e., \( x^* \) satisfies \( 4u_L(x) = u_R(x) + 3 \) when \( x_0 = 0 \), implying that \( x^* = 4/3 \).

In Figure 2, the NS and KS solutions are the intersection of the boundary of the feasible set \( S \) with the relevant Nash level curve (NLC) and the ray from the disagreement point to the maximum utilities (KS ray).

As is well known, an agent’s risk attitude, associated with the concavity of his utility function for given ideal points, can affect the solutions (Kannai 1977, Kihlstrom et al. 1981, Roth 1979, de Koster et al. 1983). Applying their analysis to Example 1, if \( u_L \) is fixed and \( u_R \) maintains her ideal point but increases in concavity (e.g. \( u_R(x_0) = -c(x - 2)^2 + 1 \) for some \( c > 1 \)), then NS moves closer to \( L \)’s ideal point. We are not interested how changes in a player’s risk attitude for given ideal points affect the solutions; rather, we consider what happens when the ideal points change through horizontal translations of the utility functions for a given status quo. We state the problem in technical terms.

Problem: Suppose \( u(x) \) (where \( u \) is either \( u_L \) or \( u_R \)) is a concave utility
function defined on \( x \in X = [a, b] \) with its maximum in the interior.\(^{14}\) Let \( x_0 \) be the fixed status quo for some \( x_0 \in [a, \hat{L}] \). If \( u_{-\alpha}(x) = u(x - \alpha) \) for some \( \alpha > 0 \) (i.e. a horizontal translation to the right by \( \alpha \) units), describe how the solution \( x(\alpha) \) depends on \( \alpha \). Does the bargaining solution \( x(\alpha) \) of a horizontal shift of \( \alpha \) units increase as \( \alpha \) increases?

We rephrase in terms of ranking opponents. Suppose \( L \) can choose between two \( R \) opponents whose utility functions are perfect translations of each other with different ideal points. In this case, the conditions of Milgrom and Shannon (1994) (given in the Introduction) are satisfied. The best for \( L \) is that \( R \)'s ideal coincides with his own to agree on their common ideal. However, if this is not possible, it seems intuitive that the agent whose ideal is closest to \( L \)'s should provide \( L \) with the better payoff for any bargaining solution. We expect the ranking of agreements and opponents’ ideals to coincide (i.e., that the solution to our bargaining problem satisfies MCS).

Since bargainers agree only on \( x_0 \) when \( x_0 \in [\hat{L}, \hat{R}] \), from now on assume \( x_0 < \hat{L} \). Sections 3 and 4 rank the NS and KS solutions respectively.

### 3 The Nash Solution

In his seminal paper, Nash (1950) shows the existence of a unique solution (NS) to his bargaining problem satisfying Pareto optimality, symmetry (invariance under all permutations of agents), contraction independence (independence of irrelevant alternatives), and scale invariance (invariance to positive affine transformations). Nash (1953) shows how the threat point or

\(^{14}\)Since the utility functions are defined over a broader domain than \( X \), the horizontal translations of these utility functions that we consider are still defined for all \( x \in X \).
disagreement outcome \( d \) affects the solution. Let us now define the Nash solution to our bargaining problem \((S, d)\) in formal terms.

Let \( x^* \in A \) be the agreement associated with the pair of equilibrium payoffs \( NS(A) \) of the NS. That is, \( x \) maximizes the Nash product

\[
NP(x) \equiv [u_L(x) - u_L(x_0)] [u_R(x) - u_R(x_0)].
\]

Though our problem differs from Nash’s (1950) (symmetry and scale invariance no longer hold), there is a single NS solution (i.e., \( NS(A) \) is well defined) since the feasible set \( S \) is compact and convex for any \( u_L \) and \( u_R \). Assuming these utility functions are sufficiently smooth (i.e., continuous and differentiable), the NS can be found by setting the derivative of the Nash product to zero. Thus, \( x^* \in A \) is the unique solution to

\[
\frac{u'_R(x)}{u'_L(x)} = -\frac{u_R(x) - u_R(x_0)}{u_L(x) - u_L(x_0)}
\]

and corresponds to the point where the level curve \( NP(x^*) = C \) for some constant \( C > 0 \) meets the feasible set \( S \) at a point of tangency (Figure 2).

### 3.1 Ranking R’s opponents

We fix \( R \) and vary \( L \)’s ideal point, so that \( u_{L,\alpha}(x) = u_L(x - \alpha) \) for some \( \alpha > 0 \). The \( x(\alpha) = x \) associated with the Nash solution in this case satisfies

\[
\frac{u'_R(x)}{u'_L(x - \alpha)} = -\frac{u_R(x) - u_R(x_0)}{u_L(x - \alpha) - u_L(x_0 - \alpha)}.
\]

Given the *rightward* shift of the feasible alternatives \( A_{L,\alpha} \), we anticipate \( R \) prefers opponents’ with ideals closer to her own. The following confirms that \( R \) orders the \( x(\alpha) \) as she orders \( L \)’s ideals (i.e. \( x(\alpha) > x(\hat{\alpha}) \) iff \( \alpha > \hat{\alpha} \geq 0 \)).
Theorem 1 Suppose \( u_L \) is a unimodal utility function with \( \arg \max u_L = \hat{L} \) and the status quo is at \( x_0 < \hat{L} \). Let \( u_{L,\alpha} \) be the horizontal translation of \( u_L \) to the right by \( \alpha \) units (i.e. \( u_{L,\alpha}(x) = u_L(x - \alpha) \)). Suppose \( u_R \) is an increasing utility function for all \( x \in [x_0, \hat{L} + \alpha] \). Then the agreement \( x(\alpha) \) associated with the NS between \( L_\alpha \) and \( R \) satisfies \( \frac{dx(\alpha)}{d\alpha} > 0 \). Thus, if \( u_R \) is unimodal and \( \hat{R} > \hat{L} + \alpha \), \( R \) prefers a less extreme opponent, i.e., an opponent whose ideal point is closer to her own.

The analytic proof of this theorem (and all others) can be found in the Appendix. To gain a more conceptual understanding of this result, we study how translating \( u_L \) affects the components of the bargaining problem; namely,

(i) the set of feasible alternatives \( A_{L,\alpha} \),

(ii) the disagreement outcomes \( d_{L,\alpha} \) and

(iii) the set of feasible payoffs \( S_{L,\alpha} \)

and how these affect the Pareto set \( PO_{L,\alpha} \) and the Nash product curves.

An increase in \( \alpha \) causes a rightward shift in \( A_{L,\alpha} \). Had nothing else changed, \( R \) can only benefit, i.e., \( \frac{dx(\alpha)}{d\alpha} \geq 0 \) (Thomson and Myerson 1980). Moreover, \( L_\alpha \)'s disagreement outcome worsens as \( \alpha \) increases. Had nothing else changed, \( L_\alpha \)'s bargaining position worsens (Thomson 1987). Finally, had nothing else changed, at high levels of \( x \in A_{L,\alpha} \), the upward shift of \( L_\alpha \)'s utility relative to \( L \)'s makes it easier for \( R \) to satisfy \( L_\alpha \) rather than \( L \) (Thomson and Myerson 1980). Though each of these effects on its own does not guarantee that \( \frac{dx(\alpha)}{d\alpha} > 0 \), the following qualitative description of their separate effects on the feasible set and the Nash product curves does. We illustrate the effects for the quadratic utility functions of Example 1 but they are similar for all unimodal utilities.
We first study how increasing $\alpha$ affects the feasible set ignoring its effect on the Nash product curves. From Figure 3, it is clear that the agreement reached between $L$ and $R$ is inside the Pareto frontier of $L_\alpha$ and $R$, $x^* \in PO(S_{L,\alpha})$. As $\alpha$ increases, the Pareto set shifts in such a way that $R$ can reach agreements with $L_\alpha$ that give $R$ higher utility levels than when facing $L$. Thus, $R$ cannot be made worse off when bargaining with $L_\alpha$ instead of $L$. This implies the existence of a range where $PO(S_{L,\alpha})$ strictly dominates $PO(S)$. An increase in $\alpha$ can only benefit $R$, i.e., $\frac{dx(\alpha)}{d\alpha} \geq 0$.

**Figure 3 about here**

Now, we analyze the effect of increasing $\alpha$ on the Nash product curves ignoring its effect on the feasible set. Given $x_0$ and the upward shift of $L_\alpha$’s utility in $A_{L,\alpha}$, the trade-off in the Nash product curves $NP(A_{L,\alpha})$ moves in $R$’s favor (the thin and thick equilibrium Nash level curves in Figure 3(a)). So that changes that affect the implementation of the NS also favor $R$.

The simultaneous changes in the feasible set and in the Nash product curves reinforce one another to $R$’s benefit. It is easier for $R$ to satisfy $L_\alpha$ instead of $L$ at high $x \in A_{L,\alpha}$. $R$ gets a higher payoff when facing $L_\alpha$ rather than $L$ (thick versus thin horizontal lines intersecting the $u_R$ axis in Figure 3(a)) and prefers an opponent whose ideal is closer to her own. In this case, the NS satisfies MCS.
3.2 Ranking L’s opponents

Fix $L$ and vary $R$, so that $u_{R,\alpha}(x) = u_R(x - \alpha)$ for some $\alpha > 0$. The Nash solution $x(\alpha) = x$ satisfies

$$\frac{u'_R(x - \alpha)}{u'_L(x)} = -\frac{u_R(x - \alpha) - u_R(x_0 - \alpha)}{u_L(x) - u_L(x_0)}.$$ 

The set of feasible alternatives $A_{R,\alpha}$ expands away from $\hat{L}$ up to $\hat{R} = x_0$ as $\alpha$ increases. Since $x^* \in A$ and $A_{R,\alpha} \supset A$, then $x^* \in A_{R,\alpha}$. With no change in $L$’s disagreement outcome, ideal point or utility function, had nothing else changed, $L$ has no means on its own to improve its bargaining position, this suggests $\frac{dx(\alpha)}{d\alpha} \geq 0$ (Thomson and Myerson 1980). Moreover, as $R_\alpha$’s disagreement outcome worsens relative to $R$’s, had nothing else changed, $R_\alpha$ has more to lose from failed negotiations than $R$, so that $R_\alpha$ is in a worse bargaining position than $R$, this suggests $\frac{dx(\alpha)}{d\alpha} \leq 0$ (Thomson 1987). Finally, had nothing else changed, the downward shift of $R_\alpha$’s utility relative to $R$’s makes it easier for $L$ to satisfy $R_\alpha$ rather than $R$, suggesting $\frac{dx(\alpha)}{d\alpha} \leq 0$ (Thompson and Myerson 1980). Their simultaneous effect on the bargaining outcome is however unclear, $\frac{dx(\alpha)}{d\alpha}$ may be positive or negative.

Our main contribution below is to show that the concavity of $u_R$ (i.e., $R$’s absolute risk aversion) affects the ranking of opponents. Theorem 2 shows that $R_\alpha$’s bargaining position vis à vis $L$, i.e., whether $\frac{dx(\alpha)}{d\alpha} \geq 0$, depends on the simultaneous effect these factors have on $R_\alpha$’s utility. Thus, the concavity of $R_\alpha$’s utility, $R_\alpha$’s strength or intensity of preference,\(^{15}\) is a

\[^{15}\]Peters [1992b] defines \textit{strength of a preference relation} as follows. For a player facing four choices $\{a, b, c, d\} \in A$, let the binary relation $\succ^*$ be a complete transitive binary relation on $A \times A$. If $(a, b) \succ^* (c, d)$, then the player prefers the change from $b$ to $a$ to the change from $d$ to $c$, i.e., for utility function $u$, $u(a) - u(b) > u(c) - u(d)$. He proves that
major determinant of $\frac{dx(\alpha)}{d\alpha}$. To show this we use the Arrow-Pratt coefficient of absolute risk aversion (ARA) as it measures changes in concavity that are invariant to positive linear transformations even in riskless environments such as ours (Mas-Colell et al. 1995). $R_\alpha$’s coefficient of ARA is

$$ARA_{R,\alpha} \equiv -\frac{u''_{R,\alpha}}{u'_{R,\alpha}}.$$  

Since we allow $R_\alpha$’s ideal point to shift as far as the upper bound of $X = [a, b]$, we examine situations where $R$’s ARA is always increasing or always decreasing over the relevant range. By definition, $u_R$ exhibits increasing (decreasing) ARA (respectively IARA and DARA) when $\frac{d}{dx} \left( -\frac{u''_R}{u'_R} \right) > (<) 0$.

**Figure 4 about here**

We are now ready to state one of the main results of our paper.

**Theorem 2** Suppose $u_L$ is a unimodal utility function with $\arg\max u_L = \hat{L}$ and the status quo is $x_0 < \hat{L}$. Suppose $u_R$ is an increasing utility function for all $x \in [x_0, \hat{L}]$. Let $u_{R,\alpha}$ be the horizontal translation of $u_R$ to the right by $\alpha$ units (i.e. $u_{R,\alpha}(x) = u_R(x - \alpha)$). If $u_R$ has increasing absolute risk aversion, the $x(\alpha)$ associated with the NS between $L$ and $R_\alpha$ satisfies $\frac{dx(\alpha)}{d\alpha} > 0$. If $u_R$ is unimodal, $L$ prefers an opponent whose ideal is closer to his own. If $u_R$ has decreasing absolute risk aversion, the $x(\alpha)$ associated with the NS between $L$ and $R_\alpha$ satisfies $\frac{dx(\alpha)}{d\alpha} < 0$, $L$ prefers a more extreme opponent.

**DARA** (Figure 5(b)). The intuition for the DARA case is as follows. With $R_\alpha$’s utility being a perfect translation of $R$’s, $R$’s utility is a concave for two players the utility function of the player with the weaker strength of preference relation is a concave transformation of the other player’s utility.
transformation of $R_\alpha$’s (Mas-Colell et al. 1995). Thus, $R_\alpha$’s utility increases faster than $R$’s for $x \in A_{R_\alpha}$. Since $R_\alpha$ has less intense preferences (Peters 1992b), $R_\alpha$ is less tough in negotiations than $R$. $L$ can more easily satisfy $R_\alpha$ than $R$ at low levels of $x$ since, to avoid the breakdown of negotiations, $R_\alpha$ accepts a bigger compromise than $R$. The trade-off of the feasible set and the Nash product curves improves in $L$’s favor increasing $L$’s payoff. $L$ prefers an opponent whose ideal point is farther from his own. Consequently, though in this case our problem fits the framework of Milgrom and Shannon (1994, stated in the Introduction), the agreement does not satisfy MCS.

**Figure 5 about here**

**IARA** (Figure 4(a)). $R_\alpha$ is tougher in negotiations (has more intense preferences) than $R$. $L$ gives up more when bargaining with $R_\alpha$ than with $R$, so that $\frac{dx(\alpha)}{d\alpha} > 0$ and $L$ prefers $R$ to $R_\alpha$ so the NS satisfies MCS.

As stated in the Introduction, only the DARA result contrasts with that of Kihlstrom et al. (1981) where for a given set of alternatives a player prefers a more risk averse opponent. In our framework, we work in a domain where strength of preferences matters and interpersonal comparisons are possible. Thus, Nash’s (1950) scale invariance axiom no longer holds (Thomson 1994).

The above theory can be applied to common utility functions. For instance, suppose $u_L$ and $u_R$ are quadratic as in Example 1. Since $\frac{d}{dx} \left( \frac{-u''}{u'} \right) = \frac{(u'')^2 - u''u'}{(u')^2}$, and $u''' = 0$, quadratic utility functions exhibit IARA and so both $L$ and $R$ prefer an opponent whose ideal is closer to their own. On the other hand, the utility function $u_R(x) \equiv \ln x$ used in Figure 5 (where the set of alternatives $X = [a, 3]$ with $a$ positive and close to 0 and $\alpha < 0.5$) satisfies
\[
\frac{d}{dx} \left( -\frac{u_R''}{u_R'} \right) = -\frac{1}{x^2} < 0. \text{ } u_{R,\alpha} \text{ exhibits DARA and so } \frac{dx(\alpha)}{d\alpha} < 0 \text{ by Theorem 2.}
\]

**Remark 1.** Our use of ARA should not be interpreted as asserting players are risk averse\textsuperscript{16} in our model. We use the Arrow-Pratt coefficient of ARA because it measures changes in the concavity of \(R_\alpha\)’s utility relative to \(R\)’s, i.e., are associated with \(R_\alpha\)’s intensity of preferences relative to \(R\)’s (Peters 1992b) and not to changes in \(R_\alpha\)’s risk aversion since there is no risk in our model. The ARA coefficient reflects the responsiveness of \(R_\alpha\)’s utility to the combined effect of changes in the components of the bargaining problem and in the Nash product curves. In our context, intensity of preferences decreases as the concavity of the utility function increases (as ARA increases).

The proofs of Theorems 1 and 2 in the Appendix rely heavily on the assumption that the coefficient of ARA does not change sign on the interval \([x_0, x(\alpha)]\) (i.e., the utility function is either always IARA or always DARA). We now ask what happens if ARA changes sign somewhere in the set of alternatives. For instance, from the Appendix, we see that \(x(\alpha) = x\) decreases in \(\alpha\) when \(R\)’s utility function is translated if and only if

\[
u'_L(x) [u'_R(x) - u'_R(x_0)] + [u_L(x) - u_L(x_0)] u''_R(x) > 0. \tag{2}
\]

Thus, if utilities are risk neutral (or close to being risk neutral) near \(x(\alpha)\), then \(u''_R(x) = 0\) and (2) is satisfied. In particular, a small change in a utility function that always exhibits IARA to one that is risk neutral near the NS will change the sign of \(\frac{d}{d\alpha}(x(\alpha))\). This is illustrated in the following example where, for clarity, the interval of risk neutrality is fairly large.\textsuperscript{17}


\textsuperscript{17}If \(u_R\) is risk neutral for all alternatives in \(X\), then \(x(\alpha)\) is constant (i.e. \(\frac{d}{d\alpha}(x(\alpha)) = 0\)).
Example 2  For mathematical convenience, we change $u_L(x)$ in Example 1 to $u_L(x) = -x^2$ and take $u_{R,\alpha}(x) = u_L(x - \alpha)$. Let the status quo $x_0 = -7$. If $\alpha = 3$, the NS is $x(3) = 1$. We make both players risk neutral near this NS by taking secants to the downward parabola for $u_L(x)$ in the intervals $x \in [-3, -1]$ and $x \in [0, 2]$. Thus, redefine $u_L(x)$ as

$$u_L(x) = \begin{cases} 4x + 3 & \text{if } -3 \leq x \leq -1 \\ -2x & \text{if } 0 \leq x \leq 2 \\ -x^2 & \text{otherwise} \end{cases}$$

and we keep $u_{R,\alpha}(x) = u_L(x - \alpha)$.

For $\alpha$ close to $3$, the NS is close to $1$. Near $x = 1$, we maximize $f_\alpha(x) \equiv [u_L(x) - u_L(-7)][u_R(x) - u_R(-7)]$. For $\alpha = 3$, $f_3(x) = [-2x + 49][4(x - 3) + 3 + 100] = -8x^2 + 14x + 4459$ which has a maximum when $-16x + 14 = 0$. The new solution is $x = 7/8$. As $\alpha$ varies slightly from $3$, we maximize

$$f_\alpha(x) = [-2x + 49][4(x - \alpha) + 3 + (-7 - \alpha)^2] = (-2x + 49)[4x + 52 + 10\alpha + \alpha^2].$$

Now $\frac{df_\alpha(x)}{dx} = -16x + 92 - 20\alpha - 2\alpha^2 = 0$ when $x(\alpha) = \frac{46 - 10\alpha - \alpha^2}{8}$. Clearly, as $\alpha$ increases (i.e. $u_R(x)$ is translated farther to the right), $x(\alpha)$ decreases (i.e. moves to the preferred solution of player L).

Remark 2. The above discussion using $u_R(x) = \ln x$ as in Figure 5 illustrates Theorem 2 must be applied with care in the DARA case since the utility function that is always DARA must be increasing and so cannot be unimodal. From the proof in the Appendix, clearly we only need DARA on the interval $[x_0, x(\alpha)]$ to conclude $\frac{dx(\alpha)}{d\alpha} < 0$. Thus, we can modify $u_R(x)$ for $x > x(\alpha)$ so that it becomes unimodal outside this interval without affecting the statement of Theorem 2. By a similar argument, if $L$ and $R$ have ideal
points sufficiently close, \( L \) always prefers \( R \) to \( R_\alpha \) since \( u_R \) cannot be DARA near its maximum value.

Care must also be taken when applying the above theory to functions that are not always differentiable. For instance, Euclidean preferences \( (u(x) = -|x - \hat{x}|) \) common in political economy models, exhibit constant ARA everywhere except at \( \hat{x} \). When \( u_R(x) = -|x - \hat{R}| \), the NS between \( L \) and \( R_\alpha \) may equal \( \hat{R}_\alpha \) (i.e. \( \frac{dx_\alpha}{d\alpha} = \frac{d\hat{R}_\alpha}{d\alpha} > 0 \)). However, for \( \alpha \) sufficiently large, \( x(\alpha) < \hat{R}_\alpha \) and then \( x(\alpha) \) is constant \( (\frac{dx_\alpha}{d\alpha} = 0) \) as in the risk neutral situation.\(^{18}\)

### 4 The Kalai-Smorodinsky Solution

Since the results for our Problem with respect to the NS solutions differ, it is important to consider other solutions as well. We now analyze our Problem \((S, d)\) for the Kalai and Smorodinsky (KS, 1975) solution. Kalai and Smorodinsky replace contraction independence in Nash’s solution with individual monotonicity, requiring that a player must benefit from any expansion of her feasible alternatives/payoffs. KS selects the Pareto point in \( S \) at which the utility gains for each agent from the disagreement point \( d \) are proportional to their maximum possible utility gains, i.e., proportional to the difference between the maximum utilities achievable within \( S \) and the disagreement outcome.

Let \( KS(A) \) be the pair of equilibrium payoffs of the KS solution and let \( x^* \) be the agreement associated with \( KS(A) \). Let \( u_L(\bar{L}) = \max\{u_L(x) \mid \)

\(^{18}\)In Theorem 1 when \( L_\alpha \) has Euclidean preferences, for small values of \( \alpha \), \( x(\alpha) > \hat{L}_\alpha \) and then \( x(\alpha) \) is constant. Once \( \alpha \) is sufficiently large, \( x(\alpha) = \hat{L}_\alpha \) and increases in \( \alpha \).
\((u_L(x), u_R(x)) \in S\) and \(u_R(\bar{R}) = \max\{u_R(x) \mid (u_L(x), u_R(x)) \in S\}\) be respectively the maximum utilities \(L\) and \(R\) can achieve within \(S\). Note that \(\bar{L} \geq \hat{L}\) with equality when \(\hat{L} \in A\). Similarly, \(\bar{R} \leq \hat{R}\). Then, \(x^* \in A\) satisfies

\[
\frac{u_R(x^*) - u_R(x_0)}{u_L(x^*) - u_L(x_0)} = \frac{u_R(\bar{R}) - u_R(x_0)}{u_L(\bar{L}) - u_L(x_0)}.
\]

(3)

The RHS of (3), \(\frac{u_R(x) - u_R(x_0)}{u_L(x) - u_L(x_0)} \equiv s(x_0)\) is the agreement on how utility gains must be shared. KS is the \(x \in A\) equating the ratio of maximum utility gains achievable within \(S\) (LHS) to \(s(x_0)\). \(K S(A)\) is the point where \(S\) intersects the ray from the disagreement point \(d\) to the maximum utilities \(m = (u_L(\bar{L}), u_R(\bar{R}))\) (Figure 2). \(s(x_0)\) determines how gains must be shared (the KS ray) and the feasible set determines where along the ray the solution is located. KS is the agreed proportion of the maximum possible utility gains.

In Sections 4.1 and 4.2, Theorems 3 and 4 show that the NS results (Theorems 1 and 2) extend to the KS solution as well.

### 4.1 Ranking R’s opponents

Fix \(R\) and vary \(L\), \(u_{L,\alpha}(x) = u_L(x - \alpha)\) for some \(\alpha > 0\). Define \(x_{0(\alpha)}\) as the solution of \(u_{L,\alpha}(x) = u_{L,\alpha}(x_0)\) for \(x > \hat{L}_\alpha \equiv \hat{L} + \alpha\).\(^{19}\) The KS solution depends on whether \(\bar{R} \geq \bar{x}_{0(\alpha)}\) as this determines whether the bargaining set is \(A_{L,\alpha} = [\bar{L}_\alpha, \bar{R}]\) or \(A_{L,\alpha} = [\bar{L}_\alpha, x_{0(\alpha)}]\). When \(\bar{R} < \bar{x}_{0(\alpha)}\), \(\{\bar{L}_\alpha, \bar{R}\} \subset A_{L,\alpha}\), and the KS solution \(x(\alpha) = x\) satisfies

\[
\frac{u_R(x) - u_R(x_0)}{u_L(x - \alpha) - u_L(x_0)} = \frac{u_R(\bar{R}) - u_R(x_0)}{u_L(\bar{L}) - u_L(x_0 - \alpha)}.
\]

\(^{19}\)Again, if no such \(x_{0(\alpha)}\) exists, we set \(x_{0(\alpha)} = b\).
When \( \hat{R} > \bar{x}_0(\alpha) \), we know \( \hat{R} \notin A_{L,\alpha} \), so that the solution \( x(\alpha) = x \) satisfies

\[
\frac{u_R(x) - u_R(x_0)}{u_L(x - \alpha) - u_L(x_0)} = \frac{u_R(\bar{x}_0(\alpha)) - u_R(x_0)}{u_L(\hat{L}) - u_L(x_0 - \alpha)}
\]
since the maximum utility \( R \) can achieve in \( S_{L,\alpha} \) is \( u_R(\bar{x}_0(\alpha)) \).

Regardless of \( \hat{R} \)’s location, given the rightward shift in \( A_{L,\alpha} \), we anticipate \( R \)’s ranking of opponents monotonically increases in \( \alpha \), i.e., satisfies MCS.

**Theorem 3** Under the assumptions of Theorem 1, the agreement \( x(\alpha) \) corresponding to the KS bargaining solution between \( L_\alpha \) and \( R \) satisfies \( \frac{dx(\alpha)}{d\alpha} > 0 \).

While the statements of Theorem 1 and 3 are the same exchanging NS with KS, the solution concepts and intuitions differ. The intuition is similar for \( \hat{R} > \bar{x}_0(\alpha) \), so we only discuss that pertaining to \( \hat{R} < \bar{x}_0(\alpha) \). As in the NS, \( \alpha \) affects the feasible set and the share of the maximum possible utility gains (the KS ray, the implementation of the KS solution).

We examine how increasing \( \alpha \) affects the feasible set disregarding changes in the agreed upon share. For \( \alpha > 0 \), there is a range where \( PO(S_{L,\alpha}) \) strictly dominates \( PO(S) \) (Figure 3(b)). The trade-off at the Pareto frontier favors \( R \) when facing \( L_\alpha \) rather than \( L \). As in the NS, these changes favor \( R \) but are not sufficient to guarantee \( \frac{dx(\alpha)}{d\alpha} > 0 \), but the are other changes that do.

We now study the effect of increasing \( \alpha \) on \( s_{L,\alpha}(x_0) \), \( R \) and \( L_\alpha \)’s agreed upon share of the maximum utility gains, ignoring changes in the feasible set. By assumption, the maximum utilities satisfy \( (u_{L,\alpha}(\hat{L}_\alpha), u_R(\hat{R})) = (u_L(\hat{L}), u_R(\hat{R})) \). And although \( L_\alpha \)’s fallback position worsens relative to \( L \’s \), \( L_\alpha \)’s maximum utility gain increases, \( u_{L,\alpha}(\hat{L}_\alpha) - u_{L,\alpha}(x_0) > u_L(\hat{L}) - u_L(x_0) \).

To avoid the breakdown of negotiations \( L_\alpha \) agrees to \( R \)’s demand of a bigger
share, \( s_{L,\alpha}(x_0) < s(x_0) \) (KS\(_{L,\alpha}\) ray flattens in Figure 3(b)). Ceteris paribus, the redistribution of the share in \( R\)'s favor increases her payoff in \( S\).

The combined changes to the feasible set and to \( R\)'s share leads to an additional effect that also favors \( R\). Since \( PO(S_{L,\alpha}) \) strictly dominates \( PO(S) \) over the relevant range, \( L \) and \( R\_\alpha \) agree to take a higher proportion of \( L\_\alpha\)'s greater maximum utility gains (relative to \( L\)'s). KS\(_{L,\alpha}\) ray intersects \( PO(S_{L,\alpha}) \) at a point where \( x(\alpha) > x^* \) (Figure 3(b)). Thus, \( R\)'s payoff increases in \( \alpha \). So that, in this case, the KS solution satisfies MCS.

### 4.2 Ranking L’s opponents

Fix \( L \) and vary \( R \), so that \( u_{R,\alpha}(x) = u_R(x - \alpha) \) for some \( \alpha > 0 \). We take into account that beyond a certain \( \alpha \) the set of feasible alternatives changes. For \( \alpha \) small enough such that \( \hat{R}_\alpha < \overline{x_0} \), we have that \( u_L(\hat{R}_\alpha) = u_L(\hat{R} + \alpha) > u_L(\overline{x_0}) \) and the bargaining set is \( A_{R,\alpha} = [\hat{L}, \hat{R}_\alpha] \). The KS solution, \( x(\alpha) = x \), satisfies

\[
\frac{u_R(x - \alpha) - u_R(x_0 - \alpha)}{u_L(x) - u_L(x_0)} = \frac{u_R(\hat{R}) - u_R(x_0 - \alpha)}{u_L(\hat{L}) - u_L(x_0)}. \tag{4}
\]

For large enough \( \alpha \), \( \hat{R}_\alpha > \overline{x_0} \), we have \( u_L(\hat{R}_\alpha) = u_L(\hat{R} + \alpha) < u_L(\overline{x_0}) = u_L(x_0) \), so that \( A_{R,\alpha} = [\hat{L}, \overline{x_0}] \). The KS solution \( x(\alpha) = x \) satisfies

\[
\frac{u_R(x - \alpha) - u_R(x_0 - \alpha)}{u_L(x) - u_L(x_0)} = \frac{u_R(\overline{x_0}) - u_R(x_0 - \alpha)}{u_L(\hat{L}) - u_L(x_0)}. \tag{5}
\]

The relative location of \( \hat{R}_\alpha \) and \( \overline{x_0} \) determines which equation, (4) or (5), is relevant for our analysis. Theorem 4 shows that the qualitative behavior of \( \frac{dx(\alpha)}{d\alpha} \) is independent of their location. As in the NS, \( \alpha \) affects the components of the bargaining problem and the share of the maximum possible utility gains (the KS ray, the implementation of KS) through \( R\_\alpha\)'s preference intensity.
Theorem 4  Suppose the assumptions of Theorem 2 hold. If $u_R$ has IARA, the KS solution $x(\alpha)$ between $L$ and $R_\alpha$ satisfies $\frac{dx(\alpha)}{d\alpha} > 0$. $L$ prefers a less extreme opponent. If $u_R$ has DARA, the KS solution $x(\alpha)$ between $L$ and $R_\alpha$ satisfies $\frac{dx(\alpha)}{d\alpha} < 0$. $L$ prefers a more extreme opponent.

Since the intuition of Theorem 4 is similar when $\hat{R}_\alpha > \bar{x}_0$, assume $\hat{R}_\alpha < \bar{x}_0$. Note that $A_{R,\alpha}$ expands as $\alpha$ increases until $\hat{R}_\alpha = \bar{x}_0$. We now study how increasing $\alpha$ affects the share $s_{R,\alpha}(x_0)$ ignoring changes in the feasible set. $R_\alpha$’s maximum gain in utility increases as her fallback position worsens. Thus, $L$ demands a bigger share as “compensation” for bargaining with a $R_\alpha$ rather than $R$, $s_{R,\alpha}(x_0) < s(x_0)$ (a steeper $KS_{R,\alpha}$ ray in Figures 4b and 5c). Within $S$, $L$’s bigger share ensures $L$ a higher payoff, suggesting $\frac{dx(\alpha)}{d\alpha} < 0$.

Consider the effect of increasing $\alpha$ on the feasible set ignoring its effect on share. Since $PO(S)$ strictly dominates $PO(S_{R,\alpha})$ over some range, if players agree on $s(x_0)$, they must take a smaller proportion of the maximum utility gains. This however does not imply $L$’s payoff must fall, so that $\frac{dx(\alpha)}{d\alpha} \not\geq 0$.

The combined changes to the agreed share and the feasible set is unclear, i.e., $\frac{dx(\alpha)}{d\alpha} \not\geq 0$. One of our major contributions is to show that $L$’s bargaining position depends on $R_\alpha$’s toughness in negotiations, i.e., on $u_R$’s ARA.

**DARA** (Figure 5(c)). Since some payoffs in $S$ are no longer available to $L$ and $R_\alpha$, a redistribution of share in $L$’s favor is **not** enough. Players must take a smaller proportion of $R_\alpha$’s maximum gains in utility (a lower point along $KS_{R,\alpha}$ ray). Since $R_\alpha$ is less tough in negotiations than $R$, $\frac{dx(\alpha)}{d\alpha} < 0$, failing to satisfy MCS.

**IARA** (Figure 4(b)). $R_\alpha$ is tougher than $R$, $\frac{dx(\alpha)}{d\alpha} > 0$, satisfying MCS.

**Remark 3.** (The Egalitarian Solution) Kalai (1977) shows the ex-
istence of a unique egalitarian (EG) solution to Nash’s bargaining problem satisfying weak Pareto optimality,\textsuperscript{20} symmetry and strong monotonicity.\textsuperscript{21} It is the unique \((v_L, v_R)\) in the weak Pareto optimal set that equates the gain in utilities (from the disagreement outcomes) for the two bargainers. EG may or may not correspond to an \(x\) on the bargaining set \(A\). For instance, if \(u_R\) is a translation of \(u_L\) (such as in Example 1), then concavity implies \(u_L(x) - u_L(x_0) < u_R(x) - u_R(x_0)\) for all \(x \in A\) and so EG is not in \(A\).\textsuperscript{22} When EG is in the interior of the Pareto set, it can be shown that \(R\) prefers a less extreme opponent whereas \(L\) always prefers a more extreme opponent.\textsuperscript{23}

\textbf{Remark 4. (Perles-Maschler Solution)} Perles-Maschler (1981) show the existence of a unique solution (PM) to Nash’s bargaining problem satisfying Pareto optimality, symmetry, scale invariance, continuity and supper-additivity.\textsuperscript{24} Perles-Maschler propose that bargainers make stepwise concessions from the maximum utilities, \(u_L(\tilde{L})\) and \(u_R(\tilde{R})\), achievable within \(S\). Thus, \(L\)’s initial proposal is \(d_L = (u_L(\tilde{L}), u_R(\tilde{L}))\) rather than \(d\) for \(\tilde{L} = \tilde{L} \in A\). Given that \(L\)’s fallback position may constrain the feasible set, \(R\) proposes \(d_R = (u_L(\tilde{R}), u_R(\tilde{R}))\) rather than \(d\) where \(\tilde{R} = \min \{x_0, \hat{R}\}\). With demands \(d_L\) and \(d_R\) being incompatible, concessions move bargainers towards the in-

\textsuperscript{20}The weak Pareto optimal set is \(\{(u, v) \in S \mid \text{either } u' \leq u \text{ or } v' \leq v \text{ for all } (u', v') \in S\}\).

\textsuperscript{21}Strong monotonicity requires that if \(S' \supseteq S\), then \(EG(S') \geq EG(S)\).

\textsuperscript{22}The relevance of an EG that is not in \(A\) is somewhat controversial (Thomson 1994) since it seems unreasonable bargaining would result in an outcome where one agent could improve his utility with no loss in utility for his opponent. In such cases, the boundary point in \(A\) closest to EG is often taken as the bargaining solution.

\textsuperscript{23}The formal proof for the EG solution can be found in the 2006 working paper version of this paper available at http://www.wlu.ca/sbe/gallego/. The concluding statement of Remark 4 is also found there.

\textsuperscript{24}Given two feasible sets \(S_1\) and \(S_2\) and their solutions \(PM(S_1) \in PO(S_1)\) and \(PM(S_2) \in PO(S_2)\), supper-additivity implies \(PM(S_1 + S_2) \geq PM(S_1) + PM(S_2)\).
terior of $PO(S)$. Under PM bargainers agree to maintain the same constant concession velocities, i.e., $-du_L du_R = k > 0$.

With the Pareto optimal set parameterized as $(u_L(x), u_R(x))$ for $x \in [\tilde{L}, \tilde{R}]$, the PM solution corresponds to the $x^* \in A$ that satisfies

$$\int_{u^S}^{d_R} \sqrt{-du_L(x)} du_R(x) = \int_{u^S}^{d_L} \sqrt{-du_L(x)} du_R(x)$$

where the integrals are taken along the arcs of the Pareto set. For instance, in Example 1, the PM solution is $x^* = \frac{3}{2}$ since $\sqrt{-du_L du_R} = \sqrt{-4(x-1)(x-2)} dx$ and

$$\int_{x^*}^{\tilde{R}} \sqrt{-4(x-1)(x-2)} dx = \int_{1}^{\frac{3}{2}} \sqrt{-4(x-1)(x-2)} dx.$$

The statements of Theorems 1 and 2 remain the same when PM is used in place of the NS as the bargaining solution.

5 Conclusions

We examine four solutions (NS, KS, EG and PM) to a bilateral bargaining problem when an agent faces an opponent who may be one of an infinite type and where the status quo is outside the agents’ control. To isolate the effect of types and the status quo in a model where the entire utility function matters, we make opponents identical in every respect except their ideal points. When the status quo lies between the players’ ideal points, the degenerate bargaining problem leaves the status quo in place. Otherwise, the ranking of opponents depends on who is doing the ranking.

Assuming (as in the text) $x_0 < \tilde{L}$, our model predicts the following. $R$ always prefers a less extreme opponent. A similar intuitive ranking emerges
for $L$ when $u_R$ exhibits IARA for the NS, KS and PM solutions. This contrasts with the EG solution where $L$ always prefers an opponent whose ideal point is farthest from his own. It also contrasts with the NS, KS and PM solutions when $u_R$ exhibits DARA. In addition, we show in Example 2 that any minor departure of $u_R$ from IARA near the NS solution is enough to upset the ranking. Analogous conclusions are possible for the other bargaining solutions as well. In general, $L$ prefers the less tough opponent.

We show under two assumptions – opponents’ preferences being perfect translations of each other and preferences exhibiting uniform ARA – that the ranking of opponents and their ideals may not coincide. Maintaining perfect translations but relaxing uniform IARA around the solution as in Example 2 reverses the ranking of opponents. When we relax perfect translations but maintain uniform ARA, the ranking depends on whether each opponent will have IARA or DARA near the solution and when all opponents have DARA on the ranking of the relative degree of DARA of their preferences. Simultaneously relaxing these two assumptions can only complicate matters. In general, ranking opponents in unidimensional bilateral bargaining situations is a complex problem.

With translated preferences in our unidimensional bilateral bargaining problem, we only change one parameter, the opponents’ ideal points shifts to the right, but maintain all the other characteristics of the opponents’ utility function constant. Thus, ideal points can be used to rank-order opponents. Moreover, the perfectly translated utility functions satisfy the SC condition. Using Milgrom and Shannon (MS, 1994), we would expect the agreements to satisfy MCS with respect to the opponents’ ideals. However, in our problem
the translation of opponents’ utilities interacts with two other components of the bargaining problem, namely, the concavity of the opponent’s utility function at any given point and his disagreement outcome. The effect of changing the opponent’s ideal point on the components of the bargaining problem determine a player’s toughness in negotiation. Under IARA preferences, the bargaining solution satisfies MCS. In contrast to Athey (2002), we find that when the opponents whose ideals are farther from the status quo have DARA preferences, the bargaining solution fails MCS.

Our DARA result also contrasts with those obtained by Gans and Smart (1996) for majority voting. They show that when conditions (1)-(2) of MS (stated in the Introduction) hold, the majority preference relation is quasitransitive, the median voter theorem applies and thus satisfies MCS. Moreover, Saporiti and Tohmé (2006, p.364) state that the SC condition

... is not only technically convenient, but it also makes sense in many political settings. In few words, the single-crossing property used in the context of voting, which is similar to that used in the principal-agent literature, says that, given any two policies, one of them more to the right than the other, the more rightist an individual is (with respect to another individual) the more he will “tend to prefer” the right-wing policy over the left-wing one.

While this is true for individuals, the rank-order of the agreements under bilateral bargaining depends on the effect that changes in the opponent’s ideal point has on her strength of preference and disagreement outcome. Thus, even though our model fits the general framework of MS, agreements may not satisfy MCS.

Finally, our results complement those of Austen-Smith and Banks (1988) and Gallego and Scoones (2006). When parties have quadratic utilities
(IARA), the agreement satisfies MCS (i.e., the agreement follows the ranking of the party’s ideal policies). However, for general utilities, the bargaining solution may fail MCS as this depends on how changes in the parameter affect the components of the bargaining problem.

6 Appendix

Proof of Theorem 1. Let $A \subset X$ be the bargaining set\(^{25}\) described in Section 2. The NS is the $x$ that maximizes the Nash product

$$\max_{x \in A} NP(x) = [u_L(x) - u_L(x_0)] [u_R(x) - u_R(x_0)].$$

The NS between $L_\alpha$ and $R$ is $x(\alpha) \in A_{L,\alpha}$ and the unique solution of

$$F(x, \alpha) = 0$$

where $F(x, \alpha) \equiv u_{L,\alpha}(x)[u_R(x) - u_R(x_0)] + [u_{L,\alpha}(x) - u_{L,\alpha}(x_0)]u'_R(x)$. By implicit differentiation,

$$\frac{dx(\alpha)}{d\alpha} = -\frac{\partial F(x, \alpha)/\partial \alpha}{\partial F(x, \alpha)/\partial x}$$

$$= \frac{-u''_L(x - \alpha)[u_R(x) - u_R(x_0)] + [-u'_L(x - \alpha) + u'_L(x_0 - \alpha)]u'_R(x)}{u''_L(x - \alpha)[u_R(x) - u_R(x_0)] + 2u'_L(x - \alpha)u'_R(x) + [u_L(x - \alpha) - u_L(x_0 - \alpha)]u''_R(x)}.$$

Now, $\frac{dx(\alpha)}{d\alpha} > 0$ since the denominator is negative for $x \in A_{L,\alpha}$ and

$$u''_L(x - \alpha)[u_R(x) - u_R(x_0)] + [u'_L(x - \alpha) - u'_L(x_0 - \alpha)]u'_R(x) < 0.$$

\(^{25}\)That is, $A$ is a compact interval with left endpoint $\hat{L}$. Notice that

$$\frac{d}{dx}([u_L(x) - u_L(x_0)](u_R(x) - u_R(x_0))] = u'_L(x)[u_R(x) - u_R(x_0)] + [u_L(x) - u_L(x_0)]u'_R(x)$$

is positive at $x = \hat{L}$ and negative at the right endpoint (since either $u_L(x) - u_L(x_0) = 0$ or $u'_R(x) = 0$ there). Since $[u_L(\hat{L}) - u_L(x_0)]u_R(\hat{L}) - u_R(x_0)] > 0$ and $\frac{d^2}{dx^2}([u_L(x) - u_L(x_0)](u_R(x) - u_R(x_0))] < 0$, the NS corresponds to a unique $x^*$ in the interior of $A$. 

29
Proof of Theorem 2. For $L$ and $R_\alpha$, the NS $x(\alpha)$ is the solution of

$$G(x, \alpha) = 0$$

where $G(x, \alpha) \equiv u'_L(x)[u_{R,\alpha}(x) - u_{R,\alpha}(x_0)] + [u_L(x) - u_L(x_0)]u'_{R,\alpha}(x)$. Now

$$\frac{dx(\alpha)}{d\alpha} = -\frac{\partial G(x, \alpha)}{\partial \alpha} \frac{\partial \alpha}{\partial x}$$

$$= -\frac{u'_R(x - \alpha) + u'_R(x_0 - \alpha) - u_L(x) - u_L(x_0)u''_R(x - \alpha)}{u''_R(x - \alpha)u_R(x - \alpha) + u'_L(x) - u'_L(x_0) - 2u'_L(x)u'_R(x - \alpha)}.$$ 

The denominator is again negative but now the numerator,

$$u'_L(x)[u'_R(x - \alpha) - u'_R(x_0 - \alpha)] + [u_L(x) - u_L(x_0)]u''_R(x - \alpha) \geq 0.$$ 

By substituting $G(x, \alpha) = 0$, this numerator is (with $x(\alpha) = x$)

$$u'_L(x)[u'_R(x - \alpha) - u'_R(x_0 - \alpha)] - u'_L(x)u''_R(x - \alpha)\frac{u_R(x - \alpha) - u_R(x_0 - \alpha)}{u'_R(x_0 - \alpha)}$$

$$= u'_L(x)[u_R(x - \alpha) - u_R(x_0 - \alpha)]\left[\frac{u'_R(x - \alpha) - u'_R(x_0 - \alpha)}{u_R(x - \alpha) - u_R(x_0 - \alpha)} - \frac{u''_R(x - \alpha)}{u'_R(x_0 - \alpha)}\right] < 0$$

if and only if

$$\frac{u''_R(x - \alpha)}{u'_R(x_0 - \alpha)} < \frac{u'_R(x - \alpha) - u'_R(x_0 - \alpha)}{u_R(x - \alpha) - u_R(x_0 - \alpha)}.$$ 

Since $u'_R \neq 0$ for all $x \in A_{R,\alpha}$, by Cauchy’s Mean Value Theorem, $\frac{u''_R(x - \alpha) - u''_R(x_0 - \alpha)}{u'_R(x - \alpha) - u'_R(x_0 - \alpha)} = \frac{u''_R(x - \alpha) - u''_R(x_0 - \alpha)}{u'_R(x - \alpha) - u'_R(x_0 - \alpha)}$ for some $x_0 - \alpha < \xi - \alpha < x(\alpha) - \alpha$. If $u_R$ is IARA (i.e., $\frac{d}{dx} - \frac{u''_R}{u'_R} > 0$),

$$\frac{u''_R(x - \alpha) - u''_R(x_0 - \alpha)}{u'_R(x - \alpha) - u'_R(x_0 - \alpha)} > \frac{u''_R(x - \alpha)}{u'_R(x_0 - \alpha)}$$

and so the numerator is negative (i.e., $\frac{dx(\alpha)}{d\alpha} > 0$). If $u_R$ is DARA, then $\frac{dx(\alpha)}{d\alpha} < 0$.\footnote{Note that $u_R$ only needs to be IARA (respectively DARA) for all $x \in [x_0, x(\alpha)]$ (and not for all $x \in X$) to apply this method of proof. In fact, if $\hat{R}_\alpha \in [x_0, x(\alpha)]$, then $u_R$ cannot be DARA for all $x \in X$ since $\frac{d}{dx} \left( -\frac{u''_R}{u'_R} \right) \equiv \left( -\frac{u''_R}{u'_R} \right)^2 > 0$ near $\hat{R}_\alpha$ (see Remark 2).}
Proof of Theorem 3. We consider whether \( \hat{R} \in A_{L,\alpha} \) (case a) or \( \hat{R} \notin A_{L,\alpha} \) (case b). Typically, (b) holds for small \( \alpha \), (a) for large \( \alpha \).

(a) Here \( u_{L,\alpha}(\hat{R}) > u_{L,\alpha}(x_0) \). The KS \( x(\alpha) = x \) between \( L_\alpha \) and \( R \) satisfies

\[
\frac{u_R(x) - u_R(x_0)}{u_{L,\alpha}(x) - u_{L,\alpha}(x_0)} = \frac{u_R(\hat{R}) - u_R(x_0)}{u_{L}(\hat{L}) - u_{L,\alpha}(x_0)}.
\]

Since \( \{\hat{L}_\alpha, \hat{R}\} \in A_{L,\alpha} \), the KS solution \( x(\alpha) \) also satisfies

\[
F(x, \alpha) = 0
\]

where

\[
F(x, \alpha) \equiv \left[u_R(\hat{R}) - u_R(x_0)\right][u_L(x - \alpha) - u_L(x_0 - \alpha)] - \left[u_R(x) - u_R(x_0)\right][u_L(\hat{L}) - u_L(x_0 - \alpha)].
\]

By implicit differentiation,

\[
\frac{dx(\alpha)}{d\alpha} = -\frac{\partial F(x, \alpha)}{\partial x} = -\frac{\left[u_R(\hat{R}) - u_R(x_0)\right][u_L(\hat{L}) - u_L(x_0 - \alpha)] - \left[u_R(x) - u_R(x_0)\right][u_L(x - \alpha) - u_L(x_0 - \alpha)]}{\left[u_R(\hat{R}) - u_R(x_0)\right][u_L(x - \alpha) - u_L(x_0 - \alpha)] - \left[u_R(x) - u_R(x_0)\right][u_L(\hat{L}) - u_L(x_0 - \alpha)].}
\]

Since the denominator is negative, \( \frac{dx(\alpha)}{d\alpha} > 0 \) if and only if

\[
\left[u_R(\hat{R}) - u_R(x_0)\right][u_L(x - \alpha) - u_L(x_0 - \alpha)] - \left[u_R(x) - u_R(x_0)\right][u_L(\hat{L}) - u_L(x_0 - \alpha)] > 0.
\]

Substituting \( F(x, \alpha) = 0 \) and \( u_L'(\hat{L}) = 0 \), this is true if and only if

\[
\frac{\left[u_L(\hat{L}) - u_L(x_0 - \alpha)\right]}{u_L(x - \alpha) - u_L(x_0 - \alpha)} \left[u_R(\hat{R}) - u_R(x_0)\right][u_L'(x - \alpha) - u_L'(x_0 - \alpha)] < \left[u_R(x) - u_R(x_0)\right][u_L'(\hat{L}) - u_L'(x_0 - \alpha)]
\]

if and only if

\[
\frac{u_L'(x - \alpha) - u_L'(x_0 - \alpha)}{u_L(x - \alpha) - u_L(x_0 - \alpha)} < \frac{u_L'(\hat{L}) - u_L'(x_0 - \alpha)}{u_L(\hat{L}) - u_L(x_0 - \alpha)}.
\]
Let \( h(x) \equiv \frac{u'_L(x) - u'_L(x_0)}{u_L(x) - u_L(x_0)} \). Then
\[
h'(x) \equiv \frac{u''_L(x) [u_L(x) - u_L(x_0)] - [u'_L(x) - u'_L(x_0)] u'_L(x)}{[u_L(x) - u_L(x_0)]^2} < 0.
\]
Since \( x(\alpha) > \hat{L} \), \( h(x(\alpha)) < h(\hat{L}) \). The numerator is negative and so \( \frac{dx(\alpha)}{d\alpha} > 0 \).

(b) For \( \alpha \) small, \( u_{L,\alpha}(\hat{R}) = u_L(\hat{R} - \alpha) < u_L(x_0 - \alpha) = u_{L,\alpha}(x_0) \). Let \( x_0(\alpha) \) be such that \( u_{L,\alpha}(x_0) = u_{L,\alpha}(\bar{x}_0(\alpha)) \) with \( \hat{L} < \bar{x}_0(\alpha) \). The KS satisfies
\[
\frac{u_R(x) - u_R(x_0)}{u_L(x - \alpha) - u_L(x_0 - \alpha)} = \frac{u_R(\bar{x}_0(\alpha)) - u_R(x_0)}{u_L(\hat{L}) - u_L(x_0 - \alpha)}.
\]
Thus, the \( x(\alpha) \) between \( L_\alpha \) and \( R \) is the solution of
\[
F(x, \alpha) = 0
\]
where \( F(x, \alpha) \equiv [u_R(\bar{x}_0(\alpha)) - u_R(x_0)] [u_L(x - \alpha) - u_L(x_0 - \alpha)] - [u_R(x) - u_R(x_0)] [u_L(\hat{L}) - u_L(x_0 - \alpha)] \). By implicit differentiation,
\[
\frac{dx(\alpha)}{d\alpha} = -\frac{\partial F(x, \alpha)}{\partial \alpha}\frac{\partial F(x, \alpha)}{\partial x} = -\left[ \frac{u_R(\bar{x}_0(\alpha)) - u_R(x_0)}{u_L(x - \alpha) - u_L(x_0 - \alpha)} [u'_L(x - \alpha) - u'_L(x_0 - \alpha)] - \frac{u'_R(\bar{x}_0(\alpha)) \frac{dx_0(\alpha)}{d\alpha}}{u_L(x - \alpha) - u_L(x_0 - \alpha)} [u_L(\hat{L}) - u_L(x_0 - \alpha)] \right].
\]
Since the denominator is negative, \( \frac{dx(\alpha)}{d\alpha} > 0 \) if and only if
\[
\left[ u_R(\bar{x}_0(\alpha)) - u_R(x_0) \right] [-u'_L(x - \alpha) + u'_L(x_0 - \alpha)] - [u_R(x) - u_R(x_0)] u'_L(x_0 - \alpha)
+ u'_R(\bar{x}_0(\alpha)) \frac{dx_0(\alpha)}{d\alpha} [u_L(x - \alpha) - u_L(x_0 - \alpha)] > 0.
\] (A.1)
Substituting $F(x, \alpha) = 0$ and $u'_L(\hat{L}) = 0$, the first two terms of (A.1) are

$$\left[u_R(x_0(\alpha)) - u_R(x_0)\right] \left[-u'_L(x - \alpha) + u'_L(x_0 - \alpha)\right] - \left[u_R(x) - u_R(x_0)\right] u'_L(x_0 - \alpha)$$

$$= -\left[u_L(\hat{L}) - u_L(x_0 - \alpha)\right] \left[u_R(x) - u_R(x_0)\right] \frac{u'_L(x - \alpha) - u'_L(x_0 - \alpha)}{u_L(x - \alpha) - u_L(x_0 - \alpha)}$$

$$+ \left[u_R(x) - u_R(x_0)\right] \left[u'_L(\hat{L}) - u'_L(x_0 - \alpha)\right] \left[u'_L(x_0 - \alpha) - u'_L(x_0)\right] \left[u'_L(x_0 - \alpha) - u'_L(x)\right] \frac{u'_L(x_0 - \alpha)}{u_L(x_0 - \alpha)}$$

Let $h(z) \equiv \frac{u'_L(z) - u'_L(z_0 - \alpha)}{u_L(z) - u_L(x_0 - \alpha)}$. Then, for $z > \hat{L} + \alpha$, we have

$$h'(z) = \frac{u''_L(z - \alpha) \left[u_L(z - \alpha) - u_L(x_0 - \alpha)\right] - \left[u'_L(z - \alpha) - u'_L(x_0 - \alpha)\right] u'_L(z - \alpha)}{u_L(z - \alpha) - u_L(x_0 - \alpha)^2} < 0.$$ 

Since $x(\alpha) > \hat{L} + \alpha$, the two terms are positive. The third term of (A.1)

$$u'_R(x_0(\alpha)) \frac{dx_0(\alpha)}{d\alpha} \left[u_L(x - \alpha) - u_L(x_0 - \alpha)\right]$$

is positive if $\frac{dx_0(\alpha)}{d\alpha} > 0$. This follows since $u_L$ is unimodal and concave. Alternatively, $x_0(\alpha)$ satisfies $H(x_0(\alpha), \alpha) = 0$ where $H(x, \alpha) \equiv u_L(x_0 - \alpha) - u_L(x - \alpha)$. Thus,

$$\frac{dx_0(\alpha)}{d\alpha} = -\frac{\partial H(x_0(\alpha), \alpha)/\partial \alpha}{\partial H(x_0(\alpha), \alpha)/\partial x_0(\alpha)} = \frac{u'_L(x_0 - \alpha) - u'_L(x_0(\alpha) - \alpha)}{-u'_L(x_0(\alpha) - \alpha)} > 0.$$  

**Proof of Theorem 4.** As in Theorem 2, there are two cases for the KS solution, except that case (a) is now for small $\alpha$ and case (b) for large $\alpha$.

(a) For $\alpha$ small, $u_L(\hat{R} + \alpha) > u_L(x_0)$. The $x(\alpha)$ between $L$ and $R_\alpha$ satisfies

$$G(x, \alpha) \equiv 0$$

33
where \( G(x, \alpha) \equiv \left[ u_R(\hat{R}) - u_R(x_0 - \alpha) \right] \left[ u_L(x) - u_L(x_0) \right] - \left[ u_R(x - \alpha) - u_R(x_0 - \alpha) \right] \left[ u_L(\hat{L}) - u_L(x_0) \right] \). By implicit differentiation,

\[
\frac{dx(\alpha)}{d\alpha} = -\frac{\partial G(x, \alpha)/\partial \alpha}{\partial G(x, \alpha)/\partial x} = -\frac{u'_R(x_0 - \alpha) \left[ u_L(x) - u_L(x_0) \right] + \left[ u'_R(x - \alpha) - u'_R(x_0 - \alpha) \right] \left[ u_L(\hat{L}) - u_L(x_0) \right]}{u_R(\hat{R}) - u_R(x_0 - \alpha) \left[ u_L(\hat{L}) - u_L(x_0) \right] - \left[ u_R(x - \alpha) - u_R(x_0 - \alpha) \right] \left[ u'_R(x_0 - \alpha) \right] \left[ u_L(x) - u_L(x_0) \right]}.
\]

The denominator is negative. On substituting \( G(x, \alpha) = 0 \) and \( u'_R(\hat{R}) = 0 \), the numerator of (A.2) becomes

\[
-u'_R(x_0 - \alpha) \frac{\left[ u_R(x - \alpha) - u_R(x_0 - \alpha) \right] \left[ u_L(\hat{L}) - u_L(x_0) \right]}{u_R(\hat{R}) - u_R(x_0 - \alpha)} - \left[ u'_R(x - \alpha) - u'_R(x_0 - \alpha) \right] \left[ u_L(\hat{L}) - u_L(x_0) \right] = \left[ u'_R(\hat{R}) - u'_R(x_0 - \alpha) \right] \left[ u_L(x) - u_L(x_0) \right] - \frac{u'_R(x - \alpha) - u'_R(x_0 - \alpha) \left[ u'_R(x_0 - \alpha) \right] \left[ u_L(x) - u_L(x_0) \right]}{u_R(x - \alpha) - u_R(x_0 - \alpha)}.
\]

Let \( k(z) = \frac{u'_R(z) - u'_R(x_0 - \alpha)}{u_R(z) - u_R(x_0 - \alpha)} \). Then

\[
k'(z) = \frac{u''_R(z - \alpha) \left[ u_R(z - \alpha) - u_R(x_0 - \alpha) \right] - \left[ u'_R(z - \alpha) - u'_R(x_0 - \alpha) \right] u'_R(z - \alpha)}{[u_R(z - \alpha) - u_R(x_0)]^2} = \frac{\left[ u''_R(z - \alpha) - u''_R(x_0 - \alpha) \right] \left[ u'_R(z - \alpha) - u'_R(x_0) \right]}{u_R(z - \alpha) - u_R(x_0)}
\]

for some \( x_0 - \alpha < \xi - \alpha < z - \alpha \). As in the proof of Theorem 2 above, \( \frac{dx(\alpha)}{d\alpha} > 0 \) if \( u_R \) is IARA and \( \frac{dx(\alpha)}{d\alpha} < 0 \) if \( u_R \) is DARA.
(b) For large $\alpha$, $u_L(\hat{R} + \alpha) < u_L(x_0)$. There is a unique $\hat{L} < \overline{x}_0 < \hat{R}$ so that $u_L(x_0) = u_L(\overline{x}_0)$. The $x(\alpha) = x$ between $L$ and $R_\alpha$ satisfies

$$G(x, \alpha) \equiv 0$$

where $G(x, \alpha) \equiv [u_R(\overline{x}_0 - \alpha) - u_R(x_0 - \alpha)][u_L(x) - u_L(x_0)]$

$$- [u_R(x - \alpha) - u_R(x_0 - \alpha)] [u_L(\hat{L}) - u_L(x_0)].$$

By implicit differentiation,

$$\frac{dx(\alpha)}{d\alpha} = -\frac{\partial G(x, \alpha)/\partial \alpha}{\partial G(x, \alpha)/\partial x}$$

is given by

$$\frac{-u'_R(\overline{x}_0 - \alpha) + u'_R(x_0 - \alpha)}{u_R(\overline{x}_0 - \alpha) - u_R(x_0 - \alpha)} \frac{u'_R(x - \alpha) - u'_R(x_0 - \alpha)}{u'_R(\overline{x}_0 - \alpha) - u'_R(x_0 - \alpha)} [u_L(\hat{L}) - u_L(x_0)].$$

The denominator is negative. On substituting $G(x, \alpha) = 0$, the numerator is

$$\frac{u'_R(\overline{x}_0 - \alpha) - u'_R(x_0 - \alpha)}{u_R(\overline{x}_0 - \alpha) - u_R(x_0 - \alpha)} [u_R(x - \alpha) - u_R(x_0 - \alpha)] [u'_R(\overline{x}_0 - \alpha) - u'_R(x_0 - \alpha)] [u'_R(x - \alpha) - u'_R(x_0 - \alpha)]$$

$$- [u_R(x - \alpha) - u_R(x_0 - \alpha)] [u'_R(\overline{x}_0 - \alpha) - u'_R(x_0 - \alpha)] [u_L(\hat{L}) - u_L(x_0)].$$

$$\times \frac{u'_R(\overline{x}_0 - \alpha) - u'_R(x_0 - \alpha)}{u_R(\overline{x}_0 - \alpha) - u_R(x_0 - \alpha)} [u'_R(x - \alpha) - u'_R(x_0 - \alpha)] [u'_R(x - \alpha) - u'_R(x_0 - \alpha)].$$

From part (a) where $k(z) \equiv \frac{u'_R(z - \alpha) - u'_R(x_0 - \alpha)}{u_R(z - \alpha) - u_R(x_0 - \alpha)}$, $k'(z) = \frac{u''_R(z - \alpha)}{u'_R(z - \alpha)} - \frac{u''_R(\xi - \alpha)}{u'_R(\xi - \alpha)} \frac{u''_R(z - \alpha)}{u_R(z - \alpha) - u_R(x_0)}$

for some $x_0 - \alpha < \xi - \alpha < z - \alpha$. As in the proof of Theorem 2 above, $\frac{dx(\alpha)}{d\alpha} > 0$

if $u_R$ is IARA and $\frac{dx(\alpha)}{d\alpha} < 0$ if $u_R$ is DARA. ■

References


Figure 1: The utility functions of Example 1. The thick left parabola is $u_L(x)$ and the thin right parabola is $u_R(x)$.

Figure 2: The feasible set $S$ for Example 1 with status quo $x_0 = 0$ is the region bounded by the vertical axis and the thick curve. The disagreement point is $d = (0, -3)$. (a) shows $S$ according to the definition whereas (b) replaces the right-hand boundary with the parametric curve $\{(u_L(x), u_R(x))| x \in [a, b], u_L(x) \geq u_L(x_0), u_R(x) \geq u_R(x_0)\}$. The Pareto set is the portion of $S$ that lies in the first quadrant. Also shown are the NS and KS for Example 1. The NS is the point of intersection of the feasible set $S$ with the Nash level curve (NLC). L’s and R’s payoffs at the NS, $x^* = 1.2192$, are given by the relevant horizontal and vertical line. KS is the point of intersection of the feasible set $S$ with the ray from the disagreement point $d$ to the maximum utilities $m$ (the KS ray). L’s and R’s payoffs at the KS solution, $x^* = 4/3$, are given by the relevant horizontal and vertical line.
Figure 3: The boundary of the feasible set for the quadratic utility functions of Example 1 with $x_0 = 0$ when $\alpha = 0$ ($S_0$, the thin curve corresponding to Figure 2(b)) and for $L$'s utility function shifted by $\alpha = 0.2$ ($S_{L,\alpha}$, the thick curve). The disagreement points are $d_0 = (0, -3)$ when $\alpha = 0$ and $d_{L,\alpha} = (-0.44, -3)$ when $\alpha = 0.2$, so that $x_0(\alpha) > \hat{R}$. In (a), the NS solutions correspond to the intersections of $S_0$ with $NLC_0$ for $\alpha = 0$ and of $S_{L,\alpha}$ with $NLC_{L,\alpha}$ for $\alpha = 0.2$. In (b), the KS solutions correspond to the intersections of $S_0$ with $KS_0$ray for $\alpha = 0$ and of $S_{L,\alpha}$ with $KS_{L,\alpha}$ray for $\alpha = 0.2$. $R$'s payoffs are shown by horizontal lines. Thus, $x(\alpha)$ increases in $\alpha$ since $u_R(x)$ is an increasing function near $x(\alpha)$. 
Figure 4: (a) shows the boundary of the feasible set for the quadratic utility functions of Example 1 with $x_0 = 0$ when $\alpha = 0$ ($S_0$, the thin curve corresponding to Figure 2(b)) and for $R$’s utility function shifted by $\alpha = 0.2$ ($S_{R,\alpha}$, the thick curve). The disagreement points are $d_0 = (0, -3)$ when $\alpha = 0$ and $d_{R,\alpha} = (0, -3.84)$ when $\alpha = 0.2$, so that $x_0 = 2 < \hat{R}_\alpha$. How the NS and KS change with $\alpha$ is unclear in (a) so the relevant region is blown up in (b) and (c). In (b), the NS solutions are the intersections of $S_0$ with $NLC_0$ when ($\alpha = 0$) and $S_{R,\alpha}$ with $NLC_{R,\alpha}$ when $\alpha = 0.2$. In (c), the KS solutions are the intersections of $S_0$ with $KS_0$, ray when $\alpha = 0$ and $S_{R,\alpha}$ with $KS_{R,\alpha}$ ray when $\alpha = 0.2$. L’s payoffs are shown by vertical lines. $x(\alpha)$ increases in $\alpha$ since $u_L(x)$ is a decreasing function near $x(\alpha)$. 

(a) The feasible Set  
(b) NS solutions  
(c) KS solutions
Figure 5: (a) shows the feasible set $S_0$ (the thin curve) for the utility function $u_L(x) = -(x - 1.5)^2 + 1$ and $u_R(x) = \ln x$ with status quo $x_0 = 0.5$ and disagreement point $d_0 = (0, -\ln 2)$ when $\alpha = 0$. The other feasible set $S_{R,\alpha}$ (the thick curve) is for $u_R$ shifted by $\alpha = 0.2$. Now $x_0(\alpha) < \bar{R}_\alpha$ (in fact, $\bar{R}_\alpha = \infty$ since $u_R$ is increasing). How the NS and KS change with $\alpha$ is unclear in (a) so the relevant region is blown up in (b) and (c). In diagrams (b) and (c) the vertical line shifts right as $\alpha$ increases, i.e., $L$’s payoff increases as $\alpha$ increases. Thus, $x(\alpha)$ decreases in $\alpha$. 

\[ u_L(x) = -(x - 1.5)^2 + 1 \]
\[ u_R(x) = \ln x \]