Natural Compression and Expansion Characteristics of Asynchronous Sigma-Delta ADC

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Abstract—The paper analyzes the impact of non-linearity of asynchronous Sigma-Delta analog-to-digital converter (ASD-ADC) characteristics on the quantization process. The ASD-ADC is considered as time-interleaved multi-resolution converter that employs two different characteristics for signal processing alternately. As shown in the paper, the classical uniform quantization of the asynchronous Sigma-Delta modulator (ASDM) output signal by time-to-digital converter is ineffective due to the different shapes of ASDM characteristics that provide disparity of converter resolution transformed into the signal value domain. A need to introduce extra compensation techniques to cope with the problem of balancing the ASD-ADC resolution and shaping the signal-to-noise ratio is emphasized.

Keywords: asynchronous Sigma-Delta modulation; analog-to-digital conversion; companding; non-uniform quantization

I. INTRODUCTION

To overcome problems with decreasing accuracy of amplitude quantization in low-voltage circuits and systems, the conventional voltage signal processing is proposed to be substituted by the time-mode signal processing (TMSP) developed in recent years [1]. In the context of analog-to-digital conversion, the postulate of the TMSP led to introduce a new class of devices called asynchronous analog-to-digital converters (A-ADCs). In the A-ADCs, the mapping of an analog signal into time domain rather than into amplitude domain is used [2]-[10]. In general, the concept of time encoding of a signal amplitude is not new since it was used for example in the well-known dual-slope ADCs, the frequency-to-code converters, Sigma-Delta converters, spike-based sensors and even ultra-wideband (UWB) signal transmission [11]. The other important example of time encoding techniques are data converters based on the pulse width modulation (PWM) used in power electronics [12]-[13].

In some developments of time-based synchronous energy-efficient ADCs, the time is an intermediate signal variable (e.g. [14]). A significant difference introduced in the A-ADCs is that the time is used as the output variable and the binary codewords that appear on the converter output irregularly represent a sequence of time intervals instead of a series of signal amplitude samples. Consequently, the quantization process is moved in the A-ADCs from the signal amplitude domain to the time domain.

One of main postulates of asynchronous analog-to-digital conversion technology is an elimination of a global clock from the converter architecture [2],[15]. The conventional periodic sampling is replaced with asynchronous mapping of analog signal value into the time domain by level-crossing sampling [16],[2],[7]-[9], asynchronous analog-to-digital modulator [3]-[5] as a type of time encoding machine (TEM) [17],[18], or using a hybrid approach [10].

Many time encoding circuits (including the asynchronous Sigma-Delta modulator [17],[3]) use non-linear mapping of signal values into time parameters. In such cases, the uniform quantization of time intervals (regarded as discrete samples of analog information) in the time domain implies a non-uniform quantization in the signal amplitude domain. It has several repercussions on the signal-to-noise ratio and dynamic range of the signal.

The present paper contains an analysis of non-uniform quantization process in the asynchronous Sigma-Delta analog-to-digital converters. The quantization is considered in the context of their consistency with companding technique used for decades to reduce the signal dynamic range and to obtain the flat characteristics of the signal-to-noise ratio [19].

II. ASYNCHRONOUS SIGMA-DELTA ANALOG-TO-DIGITAL CONVERTER (ASD-ADC)

The asynchronous Sigma-Delta modulation is one of promising methods of self-timed mean value analog-to-digital conversion investigated in recent years [3]-[5],[10]. The asynchronous Sigma-Delta modulator (ASDM) translates a continuous-time band-limited signal into a binary signal through a joint frequency and the duty cycle modulation.

A. ASD-ADC Operation

The concept of the asynchronous Sigma-Delta analog-to-digital conversion (ASD-ADC) investigated in the present paper is illustrated in Fig. 1 [3],[4].

![Figure 1. Block diagram of asynchronous Sigma-Delta analog-to-digital converter (ASD-ADC).](image-url)
A two-level conversion scheme is utilized in the ASD-ADC. First, the analog signal $x(t)$ amplitude is mapped to a sequence of pulses $z(t)$ in the asynchronous Sigma-Delta modulator (ASDM). The width of a particular pulse depends on the input signal value. The signal $z(t)$ on the ASDM output is discrete in the amplitude and continuous in the time. Thus, the ASDM converts the analog signal value to the ‘quasi-digital’ (frequency/time) domain since these parameters combine both analog and digital signal properties [20]. The width of pulses has both a lower, and an upper bound [17],[3].

In order to provide the digital output, the pulse widths are next quantized in time-to-digital converter (TDC). The quantization is based on counting periods of a reference clock during the time intervals defined by slopes of the square waveform on the ASDM output [3],[29]. The advanced architecture of the TDC with rate-based data flow control obtained by double output buffering for the ASD-ADC was reported in [29]. On the other hand, clockless TDCs may be also employed in the ASD-ADC [30]-[32]. Finally, the digital code on the ASD-ADC output represents the actual pulse width. Due to integration in the modulator, the ASD-ADC belongs to a class of the mean value converters [3].

B. Asynchronous Sigma-Delta Modulation (ASDM)

The asynchronous Sigma-Delta modulator (ASDM) realizes a joint frequency and the duty cycle modulation. The ASDM consists of the lowpass filter (integrator), and the Schmitt trigger operating in a negative feedback loop (Fig. 1). For zero input signal $x(t)$, the square wave $z(t)$ on the ASDM output oscillates with the self-oscillation period equal to $2T$ and $\frac{1}{2}$ duty cycle. For non-zero input signal $x(t)$, the square wave $z(t)$ has a duty cycle different from $\frac{1}{2}$ which means the $z(t)$ consists of alternate long and short pulses [17],[3]. The self-oscillation period $2T$ termed sometimes in the literature as a limit cycle [4],[5],[21],[22] is a key ASDM design parameter.

The waveform on the integrator output $y(t)$ and on the ASDM output $z(t)$ for a given signal $x(t)$ is presented in Fig. 2. Both integration time intervals depend on the input voltage according to the non-linear relationships as follows.

If the sign of the input voltage $x(t)$ and the reference voltage $(b)$ on the Schmitt trigger output is the same, the integration time $t_i$ equals [3]:

$$t_i = \frac{T}{1 + |\eta|}$$  \hspace{1cm} (1)

where $T$ is the integration time for $x(t)=0$ (the half of the self-oscillation period), and $\eta = x/b$ while $x$ is the mean value of the input signal $x(t)$ during the integration time interval $t_i$. The maximum value of $|\eta|$ is called the modulation depth $\eta_{\text{max}}$ where $\eta_{\text{max}}<1$.

On the other hand, if the signs of the input voltage $x(t)$ and the reference voltage $(b)$ are different, the integration time $t_i$ equals [3]:

$$t_i = \frac{T}{1 - |\eta|}$$  \hspace{1cm} (2)

The asynchronous Sigma-Delta modulation ASDM may be decomposed into two modulation components producing the pulses of $t_i$ and $t_d$ duration respectively. We denote by ASDM(+)$ the modulation component related to generation of $t_i$ pulses, and accordingly by ASDM(-) the component producing $t_d$ pulses. We refer sometimes to ASDM(+) pulses as short pulses since $t_i \leq T$, and respectively to ASDM(-) pulses as long pulses because $t_d \geq T$. If the input signal bandwidth is low compared to the ASDM limit frequency defined by the self-oscillation period, then the ASDM(+) and ASDM(-) pulses alternate in the signal $z(t)$ on the ASDM output.

C. ASD-ADC Decomposition: ASDM(+) versus ASDM(-)

In the time-to-digital converter TDC which is the second functional block of the ASD-ADC (Fig. 1), a digital value is assigned to each ASDM pulse width. We denote by $k$, the digital numbers corresponding to ASDM(+) pulses, and by $k_i$ the numbers representing ASDM(-) pulses.

Similarly, throughout the present paper, we refer by ASD-ADC(+) and ASD-ADC(-) to encoding systems quantizing ASDM(+) and ASDM(-) pulses. Thus, the ASD-ADC may be considered as time-interleaved converter since it employs two different characteristics for signal processing defined respectively by ASD-ADC(+) and ASD-ADC(-).

D. Related Works

The idea of the asynchronous Sigma-Delta modulation was formulated in the 60s [21]. The analog-to-digital conversion using the ASDM as duty-cycle modulator and signal recovery based on classical lowpass filtering was proposed by Roza [21]. Considerable amount of research on ASDM theory and design was done in last years [23]-[28]. Since the ASDM belongs to a class of time-encoding machines (TEMs) [17], the input signal may be perfectly recovered based on knowledge of...
ASDM output using the method developed by Lazar and Tóth [17].

E. Paper Contribution

In the previous works, the ASD-ADC implementations have used coding of ASDM\(^{(-)}\) and ASDM\(^{(+)}\) pulses into the codeword of the same number of bits. However, due to disparity of ASDM\(^{(-)}\) and ASDM\(^{(+)}\) pulse widths, \(t_s\) and \(t_d\) the digital numbers representing \(t_s\) and \(t_d\) use only a fraction of the quantization space. As a result, real resolution of coding of the ASDM\(^{(-)}\) pulses with uniform quantization is much lower than the precision of digital representation of the ASDM\(^{(+)}\) pulses. Thus, the signal-to-noise ratio of the ASD-ADCs with unified resolution is low and determined only by the ASD-ADC\(^{(-)}\) conversion component.

In the present paper, we propose to introduce multi-resolution coding of the ASDM time parameters meaning that the ASD-ADC\(^{(-)}\) employs coding with higher number of bits than the ASD-ADC\(^{(+)}\) although the size of the least significant bit (LSB) represented by the period of the reference clock used for time quantization is the same both in ASD-ADC\(^{(-)}\) and ASD-ADC\(^{(+)}\).

Furthermore, we show that the classical uniform quantization of the ASDM output signal even with multi-resolution coding is ineffective due to non-linearity of ASDM characteristics. Whereas the shape of the ASD-ADC\(^{(-)}\) quantization harmonize with the compressor characteristics, the shape of the quantization in ASD-ADC\(^{(+)}\) is opposite and provides expansion capability which is inconsistent with the well-known principles of companded quantization.

III. CODING SCHEME IN ASDM\(^{(-)}\) AND ASDM\(^{(+)}\)

A. ASDM\(^{(+)}\) Pulse Width

As stated, in the ASDM\(^{(+)}\) modulation, the analog signal absolute value \(|\eta|\) is transformed into an ASDM\(^{(+)}\) pulse (or simply a short pulse) width \(t_s\) according to (1). As follows from (1), the minimum width of short pulses \(t_{s\min}\) depends on the modulation depth \(\eta_{\max}\) where \(\eta_{\max} = \max|\eta|\):

\[
t_{s\min} = \frac{T}{1 + \eta_{\max}}
\]

and \(T\) is the half of the self-oscillation period. In particular, if \(\eta_{\max} \to 1\), then \(t_{s\min} \to T/2\).

On the other hand, the short pulse width reaches its maximum \(t_{s\max}\) for \(\eta = 0\), thus:

\[
t_{s\max} = T
\]

Summing up, the short pulse width \(t_s\) depends non-linearly on the analog signal absolute value \(|\eta|\) and ranges from \(t_{s\min}\) stated by (3) to \(t_{s\max} = T\) where \(t_{s\min}\) approaches \(T/2\) if the modulation depth \(\eta_{\max}\) reaches one.

B. Coding Representation of ASDM\(^{(-)}\) Pulse Width

We propose to code the time interval \(d_s\) instead of the short pulse width \(t_s\) where by \(d_s\) we denote the ASDM\(^{(-)}\) pulse (short pulse) width representation as follows:

\[
d_s = t_s - t_s = T - t_s
\]

After setting (1) to (5):

\[
d_s = \frac{T|\eta|}{1 + |\eta|}
\]

Furthermore:

\[
d_{s\min} = T - t_s = 0
\]

\[
d_{s\max} = T - t_s = \frac{T}{1 + \eta_{\max}} = \frac{T\eta_{\max}}{1 + \eta_{\max}}
\]

so finally: \(0 \leq d_s \leq T|\eta|/(1 + \eta_{\max})\).

The relationship between the short pulse width \(t_s\) and its representation \(d_s\) being coded in the ASDM\(^{(+)}\) is illustrated in Fig. 3.

![Figure 3. Relationship between ASDM\(^{(+)}\) short pulse width \(t_s\) and its representation \(d_s\) being coded.](image)

The plots of the \(t_s/T\) and \(d_s/T\) versus the analog signal absolute value \(|\eta|\) according to (1) and (6) is illustrated in Fig. 4. For example, \(d_s(|\eta| = 0.8) = 0.447T\).

![Figure 4. Plots of \(t_s/T\) and \(d_s/T\) vs. analog signal absolute value \(|\eta|\).](image)
The mapping of the analog signal absolute value $|\eta|$ into $d_t$ has several advantages compared to the coding $d_t$ into $t$:
- $d_t$ does not contain offset because $d_{t_{\text{min}}} = 0$ ,
- $d_t$ is an increasing function of the signal absolute value $|\eta|$ ,
- derivative of the $d_t$ with respect to $|\eta|$ decreases with growing $|\eta|$ which provides a higher sensitivity of mapping small signal absolute values and respectively a lower sensitivity of mapping large absolute values $|\eta|$.

The latter attribute indicates that the ASDM$^{(+)}$ offers a natural signal compression capability providing a reduction of the signal dynamic range. The compression as a part of companding has been used as a well-known signal processing technique in wide range of applications. In particular, the logarithmic compressors have been utilized for decades in order to obtain flat characteristics of the signal-to-noise ratio especially for coding audio signals [34]. On the other hand, the companding has been proposed as a method to control the dynamic range in low-voltage circuits [35].

C. ASDM$^{(+)}$ Pulse Width

In the ASDM$^{(+)}$ modulation, the analog signal absolute value $|\eta|$ is transformed into an ASDM$^{(+)}$ pulse (or simply a long pulse) width $t_t$ according to the formula (2).

On the basis of (2), the maximum width $t_{t_{\text{max}}}$ of long pulses depends on the $\eta_{\text{max}}$:

$$t_{t_{\text{max}}} = \frac{T}{1 - \eta_{\text{max}}} \quad (9)$$

In particular, if $\eta_{\text{max}} \to 1$, then $t_{t_{\text{max}}} \to \infty$.

The minimum width $t_{t_{\text{min}}}$ of long pulses does not depend on $\eta_{\text{max}}$ and equals $T$:

$$t_{t_{\text{min}}} = T \quad (10)$$

Summing up, the long pulse width $t_t$ depends non-linearly on the analog signal absolute value $|\eta|$ and ranges from $t_{t_{\text{min}}} = T$ to $t_{t_{\text{max}}}$ stated by (9) whereas $t_{t_{\text{max}}}$ diverges to infinity if the modulation depth $\eta_{\text{max}}$ is close to one.

D. Coding Representation of ASDM$^{(+)}$ Pulse Width

The $t_{t_{\text{min}}}$ represents the offset for measurements of $t_t$. To eliminate this offset, we propose to code the time interval $d_t$ instead of $t_t$ where $d_t$ is defined simply as $d_t = t_t - t_{t_{\text{min}}}$. By the convention, we term $d_t$ as the ASDM$^{(+)}$ pulse (long pulse) width representation. Taking into account (10), it is evident that:

$$d_t = t_t - t_{t_{\text{min}}} = \frac{T|\eta|}{1 - |\eta|} \quad (11)$$

The minimum value $d_{t_{\text{min}}}$ of long pulse representation equals zero ($d_{t_{\text{min}}} = 0$) and the maximum values $d_{t_{\text{max}}}$ is a function of $\eta_{\text{max}}$:

$$d_{t_{\text{max}}} = t_{t_{\text{max}}} - T = \frac{T}{1 - \eta_{\text{max}}} - T = \frac{T\eta_{\text{max}}}{1 - \eta_{\text{max}}} \quad (12)$$

The relationship between the long pulse width $t_t$ and its representation $d_t$ being coded in the ASDM$^{(+)}$ is illustrated in Fig. 5.

![Figure 5. Relationship between ASDM$^{(+)}$ (long) pulse width $t_t$ and its representation $d_t$ being coded.](image-url)

The plots of the $t_t/T$ and $d_t/T$ versus the analog signal absolute value $|\eta|$ according to (2) and (11) is illustrated in Fig. 6. In particular, as follows from Fig. 6, $d_t(|\eta| = 0.8) = 4T$. By comparison, $d_t(|\eta| = 0.8) = 0.44T$ as computed in Sect. III.B. Thus, if $|\eta| = 0.8$, the $d_t$ is almost one order of magnitude greater than the $d_t$.

![Figure 6. Plots of $t_t/T$ and $d_t/T$ vs. analog signal absolute value $|\eta|$](image-url)

Similarly as for $d_t$, the characteristics of mapping the signal absolute value $|\eta|$ into $d_t$ does not contain offset ($d_{t_{\text{min}}} = 0$) and is a non-linear increasing function of $|\eta|$. However in contrast with $d_t$, the derivative of the $d_t$ with respect to $|\eta|$ increases with growing $|\eta|$ (Fig. 6). It provides a lower sensitivity of mapping small analog signal values and respectively a higher sensitivity of mapping large analog signal values $|\eta|$ which corresponds to the characteristics of the
This attribute is disadvantageous and opposite to principles of classical companding where the source is encoded using the compressor characteristics.

E. Proposed Multi-Resolution Coding Scheme

We assume that both ASDM\(^3\) and ASDM\(^4\) analog signal representations, \(d_t\) and \(d_s\), are quantized uniformly into a digital states \(k_t\) and \(k_s\) by counting the numbers of reference clock periods \(T_0\) respectively within \(d_t\) and \(d_s\) time intervals using modulo counters.

Since the \(d_t\) is a few times as long as the \(d_s\) for typical values of the modulation depth \(\eta_{\text{max}}\) (e.g., \(d_{\text{max}}^t = 4T\) and \(d_{\text{max}}^s = 0.44T\) for \(\eta_{\text{max}} = 0.8\)), we propose to code the \(d_s\) into the codeword of higher number of bits than the coding resolution of \(d_t\). More precisely, we propose to introduce a multi-resolution coding in the ASD-ADC where the signal is coded with \(n+p\) bits of precision in the ASD-ADC\(^3\) and respectively with \(n+p\) bits of precision in the ASD-ADC\(^4\).

We assume that the period \(T_0\) of the reference clock used for quantization is selected according to the maximum width of the \(d_s\) as follows:

\[
T_0 = \frac{d_{\text{max}}^s}{2^n} \tag{13}
\]

In order to simplify the analysis, we suppose that the start of the quantized time interval \(d_s\) is synchronized with the initial edge of the reference clock. The parameter \(p\) is selected such that:

\[
d_{\text{max}}^s \leq 2^p d_{\text{max}}^s \tag{14}
\]

The optimal use of the capacity of the space of quantization levels corresponds the situation when:

\[
d_{\text{max}}^s = 2^p d_{\text{max}}^s \tag{15}
\]

To meet the requirement (15) in practice, the value \(p\) can be selected as 1, 2, or 3 because larger values of \(p\) correspond to too high disparity between \(d_{\text{max}}^s\) and \(d_{\text{max}}^s\) (compare Figs. 4 and 6).

The \(p\) value is selected to the modulation depth \(\eta_{\text{max}}\). For example, if \(\eta_{\text{max}} = 0.6\), then \(d_{\text{max}}^s = 3T/8\) and \(d_{\text{max}}^s = 1.5T\) according to (8) and (12) respectively. Because \(d_{\text{max}}^s/d_{\text{max}}^s = 4\), the \(p=2\) should be selected and this value according to (15) corresponds to the optimal utilization of the quantization space at the same time.

F. Selection of Optimal Modulation Depth

By setting (3) and (6) to (15), we can find the optimal values \(\eta_{\text{max opt}}\) of modulation depth according to the following formula:

\[
\eta_{\text{max opt}} = \frac{2^p - 1}{2^p + 1} \tag{16}
\]

The optimal modulation depth \(\eta_{\text{max opt}}\) guarantees the optimal use of the space of quantization levels which means that the digital value \(k_{\text{max}}\) equal to \(2^n p - 1\) is obtained for the signal absolute value \(|\eta| = \eta_{\text{max opt}}\). In Table I, the numerical values of the optimal modulation depth \(\eta_{\text{max opt}} = \{1/3, 3/5, 7/9\}\) for \(p = \{1, 2, 3\}\) respectively are listed on the basis of (16). In particular, \(\eta_{\text{max opt}} (p = 2) = 3/5\) as stated previously.

<table>
<thead>
<tr>
<th>(P)</th>
<th>(\eta_{\text{max opt}})</th>
<th>(t_{\text{min}})</th>
<th>(t_{\text{max}})</th>
<th>(d_{\text{max}})</th>
<th>(d_{\text{max}})</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>1/3</td>
<td>1/4 T</td>
<td>1.5 T</td>
<td>3/4 T</td>
<td>0.5 T</td>
</tr>
<tr>
<td>2</td>
<td>3/5</td>
<td>5/8 T</td>
<td>2.5 T</td>
<td>3/8 T</td>
<td>1.5 T</td>
</tr>
<tr>
<td>3</td>
<td>7/9</td>
<td>9/16 T</td>
<td>4.5 T</td>
<td>7/18 T</td>
<td>3.5 T</td>
</tr>
<tr>
<td>(\infty)</td>
<td>1</td>
<td>1/2 T</td>
<td>(\infty)</td>
<td>1/2 T</td>
<td>(\infty)</td>
</tr>
</tbody>
</table>

IV. ASD-ADC\(^3\) QUANTIZATION

Due to non-linearity of mapping the analog signal value in ASDM\(^3\) and ASDM\(^4\) as defined by non-linear relationships of \(d_t\) and \(d_s\) against the signal absolute value \(|\eta|\) (see (6) and (11)), the uniform quantization in the time domain yields non-uniform quantization in the signal value domain. In the forthcoming sections, we will derive the shape of the ASD-ADC quantization characteristics in the signal value domain and discuss their impact on real converter resolution.

Since we have assumed that the initial edge of the quantized pulse is synchronized with the initial edge of the reference clock, the proposed discretization corresponds to rounding down quantization scheme [33].

A. Quantization Characteristics in ASD-ADC\(^3\)

The digital value \(k_s\) assigned to the representation of the short pulse width \(d_s\) in a result of the uniform quantization in the time domain in ASD-ADC\(^3\) is stated by the proportion:

\[
k_s = \left\lfloor \frac{d_s}{d_{\text{max}}^s} \right\rfloor \tag{17}
\]

where \(\left\lfloor x \right\rfloor\) denotes the floor function of \(x\) (i.e., the largest integer not greater than \(x\)).

By setting (1) and (8) to (17), the \(k_s\) is expressed as a function of the analog signal absolute value \(|\eta|\):

\[
k_s(\eta) = 2^n \left( \left\lfloor \frac{1 + \eta_{\text{max opt}} |\eta|}{1 + |\eta|} \right\rfloor \right) \tag{18}
\]

The relationship of \(k_s\) versus \(|\eta|\) according to (18) defines the quantization characteristics in the ASD-ADC\(^3\) and is illustrated in Fig. 7 for \(n = 5\) and \(p = \{1, 2, 3\}\). The relevant plots correspond to the set of optimal modulation depths \(\eta_{\text{max opt}} = \{1/3, 3/5, 7/9\}\) for each \(p\) according to Table I.

The characteristics in Fig. 7 are staircase functions that define the sequence of non-uniform quantization steps in the analog signal value \(|\eta|\) domain in the ASD-ADC\(^3\). The analog signal absolute values \(|\eta|\) range from zero to \(\eta_{\text{max opt}}\) for each
As shown in Fig. 7, the small absolute signal values \(|p|\) are quantized in the ASD-ADC\(^{\text{A}}\) with higher precision than large signal values \(|p|\) because the quantization steps grows with \(|p|\). The resolution increase is proportional to the decrease of the slope of the relevant graphs respectively for small and large \(|p|\). Thus, the ASD-ADC\(^{\text{A}}\) realizes a classical companded quantization which is obtained by a non-linear signal compression in the ASDM\(^{\text{A}}\) modulator and subsequent uniform quantization in the time domain.

**B. Conversion Resolution in ASD-ADC\(^{\text{A}}\)**

Now we will examine how the resolution in the signal value domain varies over the signal range in the ASD-ADC\(^{\text{A}}\).

The absolute value of the quantized analog signal \(|\eta_s|\) corresponding to the binary digital word \(k_s\) is defined on the basis of (18) as:

\[
|\eta_s(k_s)| = \frac{k_s \eta_{\text{max}}}{2^n(1 + \eta_{\text{max}}) - k_s \eta_{\text{max}}}
\]  
(19)

The index \(s\) in the symbol \(\eta_s\) indicates only that the relevant signal absolute value is derived from ASD-ADC\(^{\text{A}}\) digital representation \(k_s\).

The quantization step size \(\Delta \eta_s(k_s)\) in the ASD-ADC\(^{\text{A}}\) is found as a difference:

\[
\Delta \eta_s(k_s) = |\eta_s(k_s + 1) - |\eta_s(k_s)|
\]  
(20)

The relative quantization step size \(\Delta \hat{\eta}_s(k_s)\) is defined as:

\[
\Delta \hat{\eta}_s(k_s) = \frac{\Delta \eta_s(k_s)}{\eta_{\text{max}}}
\]  
(21)

After setting (20) and (19) to (21), the formula (21) can be expressed as:

\[
\Delta \hat{\eta}_s(k_s) = \frac{(k_s + 1)(2^p + 1) - k_s(2^p + 1)}{2^{n+p+1} - (k_s + 1)(2^p - 1) - k_s(2^p - 1)}
\]  
(22)

The relative size \(\Delta \hat{\eta}_s(0)\) of the first quantization step for small analog signal values in ASD-ADC\(^{\text{A}}\) can be found as:

\[
\Delta \hat{\eta}_s(0) = \frac{2^p - 1}{2^{n+p+1} - 2^p + 1} \approx \frac{2^p}{2^{n+p+1}}
\]  
(23)

By comparison, the relative quantization step size for a classical uniform quantization with \(n\)-bit resolution is equal to:

\[
\Delta \hat{\eta}_{\text{uniform}} = \frac{1}{2^n}
\]  
(24)

As follows from (23), the real resolution of quantization for small signal values is in the \(n\)-bit ASD-ADC\(^{\text{A}}\) higher than for \(n\)-bit uniform quantization. The resolution increase is proportional to the \(p\) value. If \(p \to \infty\), the resolution grows by one bit because:

\[
\lim_{p \to \infty} \Delta \hat{\eta}_s(0) = \frac{2^p}{2^{n+p+1} - 2^p + 1} = \frac{1}{2^n}
\]  
(25)

For finite \(p\) values, the resolution increase is lower than one bit because \(1/2^{n+p} < \Delta \hat{\eta}_s(0) < 1/2^n\) on the basis of (23).

On the other hand, the relative size \(\Delta \hat{\eta}_s(2^n - 1)\) of the last quantization step for large analog signal values in the ASD-ADC\(^{\text{A}}\) can be found as:

\[
\Delta \hat{\eta}_s(2^n - 1) = \frac{2^n(2^p + 1) - (2^n - 1)(2^p + 1)}{2^{n+p+1} - 2^n(2^p - 1) - (2^n - 1)(2^p - 1)}
\]  
(26)

As follows from (26), the resolution of quantization in the ASD-ADC\(^{\text{A}}\) for large signal values is lower than \(n\) bits. The resolution decrease is proportional to \(p\). If \(p \to \infty\), the resolution is reduced by one bit because on the basis of (26):

\[
\lim_{p \to \infty} \Delta \hat{\eta}_s(2^n - 1) = \frac{1}{2^n - 1}
\]  
(27)

For finite \(p\) values, the resolution decrease is lower than one bit since \(1/2^n < \Delta \hat{\eta}_s(2^n - 1) < 1/2^{n-1}\).

**V. ASD-ADC\(^{\text{A}}\) QUANTIZATION**

As stated, the ASD-ADC\(^{\text{A}}\) pulse width representation \(d_i\) are quantized uniformly with \(n+p\) bits of precision. We will examine how the resolution transformed to the signal domain varies with the signal value in the ASD-ADC\(^{\text{A}}\).

**A. Quantization Characteristics in ASD-ADC\(^{\text{A}}\)**

The digital value \(k_i\) assigned to the long pulse width representation \(d_i\) in a result of the quantization process in the ASD-ADC\(^{\text{A}}\) is stated by the proportion:

\[
k_i = \left[ \frac{2^{n+p} \cdot d_i}{d_{\text{max}}} \right]
\]  
(28)
where $d_{\text{max}}$ is defined by (6) and $\lfloor x \rfloor$ denotes the floor function of $x$.

By setting (16) to (15), the $d_{\text{max}}$ is defined with respect to $p$:

$$d_{\text{max}} = \frac{2^p - 1}{2}$$

(29)

On the other hand, by setting (29) and (11) to (28), the digital value $k_i$ can be expressed as a function of the analog absolute signal value $|\eta|$:  

$$k_i(|\eta|) = \frac{2^{n+p} + |\eta|}{(2^p - 1)(1 - |\eta|)}$$

(30)

The relationship of $k_i$ versus $|\eta|$ according to (30) defines the quantization characteristics in the ASD-ADC$^{(+)}$ and is illustrated in Fig. 8 for $n=4$ and $p=\{1, 2, 3\}$. The plots correspond to the set of optimal modulation depth $\eta_{\text{max, opt}} = \{1/3, 3/5, 7/9\}$ for each $p$ value.

The shape of the quantization characteristics in the ASD-ADC$^{(+)}$ is opposite to the shape of the relevant characteristics in the ASD-ADC$^{(-)}$ presented in Fig. 7. As seen in Fig. 8, the small signal absolute values $|\eta|$ are quantized in the ASD-ADC$^{(+)}$ with lower resolution than large signal values $|\eta|$ which signifies that the ASD-ADC$^{(+)}$ provides expansion capability. Thus, the ASD-ADC$^{(+)}$ do not match the classical compounded quantization rules.

### B. Conversion Resolution in ASD-ADC$^{(+)}$

In the present Section, we investigate how the resolution in the signal value domain varies over the signal range in the ASD-ADC$^{(+)}$.

The absolute value of the analog signal $|\eta_i|$ corresponding to the digital state $k_i$ in the ASD-ADC$^{(+)}$ can be found on the basis of (30):

$$|\eta_i(k_i)| = \frac{k_i(2^p - 1)}{2^{n+p+1} + k_i(2^p - 1)}$$

(31)

The relative size $\Delta\hat{\eta}_i(k_i)$ of the quantization step is defined as:

$$\Delta\hat{\eta}_i(k_i) = \frac{|\eta_i(k_i + 1)| - |\eta_i(k_i)|}{\eta_{\text{max}}}$$

(32)

After setting (31) to (32):

$$\Delta\hat{\eta}_i(k_i) = \frac{(k_i + 1)(2^p + 1)}{2^{n+p+1} + (k_i + 1)(2^p - 1)} - \frac{k_i(2^p + 1)}{2^{n+p+1} + k_i(2^p - 1)}$$

(33)

The relative size of the first quantization step $\Delta\hat{\eta}_i(0)$ for small absolute signal values in the ASD-ADC$^{(+)}$ is stated as:

$$\Delta\hat{\eta}_i(0) = \frac{2^p + 1}{2^{n+p+1} + 2^p - 1}$$

(34)

The approximation of the $\Delta\hat{\eta}_i(0)$ expressed by (34) is the same as the approximation of the $\Delta\hat{\eta}_i(0)$ defined by (23) because non-linearity of the ASD-ADC$^{(+)}$ characteristics goes over smoothly into non-linearity of the ASD-ADC$^{(-)}$ for small $|\eta|$. As shown in [3], the non-linearity of characteristics both for short and long pulses is approximated well by linear function if absolute signal values are small.

The relative size of the last quantization step $\Delta\hat{\eta}_i(2^n+1)$ in the ASD-ADC$^{(+)}$ can be found as:

$$\Delta\hat{\eta}_i(2^n+1) = \frac{2^{n+p+1} + 2^n}{2^{n+p+1} + 2^{n+p} + 2^n}$$

(35)

As follows from (35), the resolution increase for large absolute signal values in the ASD-ADC$^{(+)}$ depends on the $p$ value. For example, if $p=1$, then $1/2^{n+2} < \Delta\hat{\eta}_i(2^n+1) < 1/2^{n+1}$ so the resolution of quantizing large signal values is higher by almost one bit in relation to the resolution for small absolute signal values (compare $\Delta\hat{\eta}_i(0)$ for $p=1$ on the basis of (34)). If $p=2$, the resolution is increased by more than one bit ($1/2^{n+4} = \Delta\hat{\eta}_i(2^n+1) < 1/2^{n+3}$, and respectively by more than 2 bits if $p=3$ ($1/2^{n+6} = \Delta\hat{\eta}_i(2^n+1) < 1/2^{n+5}$).

Summing up, whereas the shape of the ASD-ADC$^{(+)}$ quantization harmonize with the compressor characteristics, the ASD-ADC$^{(+)}$ displays expansion capability which is inconsistent with the well-known principles of companding. As a result, the signal-to-noise ratio for small signal values in ASD-ADC$^{(+)}$ is much worse than for large signal values. In order to balance the signal-to-noise ratio in ASD-ADC$^{(+)}$ and ASD-ADC$^{(-)}$, the undesirable effect of signal expansion provided by ASDM$^{(+)}$ should be compensated, for example, by digital compression of ASD-ADC$^{(+)}$ output signal.

### VI. CONCLUSIONS

The paper analyzes the impact of non-linearity of asynchronous Sigma-Delta analog-to-digital converter characteristics on the quantization process. The ASD-ADC was considered as time-interleaved multi-resolution converter that employs two different characteristics for signal processing alternately. As shown in the paper, the classical uniform quantization of the asynchronous Sigma-Delta modulator...
output by time-to-digital converter is ineffective. This is due to the different shapes of ASDM characteristics that provide disparity of converter resolution transformed into the signal value domain. Further research should be focused on compensation techniques needed to cope with the problem of balancing the ASD-ADC resolution and shaping the characteristics of signal-to-noise ratio.

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REFERENCES


