CONVEX HILBERT CUBES IN SUPEREXTENSIONS

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If $Z$ is a metric continuum, then every nondegenerate compact convex subset of the superextension $\lambda(Z)$ is homeomorphic to the Hilbert cube.

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1. Introduction

The superextension $\lambda(Z)$ of a normal space $Z$ is the set of all maximal linked families of closed subsets of $Z$, equipped with a Wallman-type topology. This construction was first devised by De Groot [4] as a part of his program to characterize complete regularity in terms of closed subbases. Since then, superextensions have been studied from other viewpoints by a variety of authors, and several attractive results have come out: Van Mill's result [10] that $\lambda(Z)$ is homeomorphic to the Hilbert cube if $Z$ is a nondegenerate metric continuum; the author's result [14] that $\lambda(Z)$ has the Lefschetz fixed point property if $Z$ has finitely many components; a result of Bell-Ginsberg-Teodorčević [2] that the equivalence of the statements '\(\lambda(Z)\) is first countable' and '\(\lambda(Z)\) is metrizable' is undecidable in ZFC; and Ivanov's results [6, 7] on characterizing compacta $Z$ for which $\lambda(Z)$ is an AR(compact) or an AR(dim 0).

Many proofs in superextensions involve some sort of 'convex structure' which is naturally available on $\lambda(Z)$. We will briefly describe the axiomatics and some relevant properties of convex structures below. These structures were also used in [15] to define pseudo-boundaries and pseudo-interiors on certain topological spaces, and the resulting theory led to a proof that many convex sets in $\lambda(Z)$ ($Z$ a metric continuum) are homeomorphic to the Hilbert cube. It is therefore natural to ask whether all nontrivial compact convex sets are Hilbert cubes. In the present paper,
we settle this problem in the affirmative. The main ingredient of the proof is a selection theorem for convex structures [16]. Some related open problems are described.

2. Preliminaries

2.1. Abstract and uniform convex structure. A *convex structure* consists of a set $X$ and a family $\mathcal{C}$ of ‘convex’ subsets of $X$, subject to the following conditions:

1. $\emptyset$ and $X$ are convex;
2. the intersection of a family of convex sets is convex;
3. the union of an updirected family of convex sets is convex.

The family $\mathcal{C}$ is also called a *convexity* on $X$. Reference to $\mathcal{C}$ (or to an eventual topology on $X$) is suppressed. It is assumed throughout that all singleton sets are convex. The *convex hull* $\text{co}(A)$ of a set $A \subseteq X$ is defined the obvious way. If the set $A$ is finite, then $\text{co}(A)$ is also called a *polytope*, and if $A$ consists of two points $u$, $v$, then $\text{co}\{u, v\}$ is called the *interval* between $u$ and $v$. A *half-space* of $X$ is a convex set with a convex complement. The convex structure $X$ satisfies the *separation axiom* $S_4$ provided for every two disjoint convex subsets $C$, $D$ there is a half-space $H$ with $C \subseteq H$, $D \subseteq X \setminus H$.

A function from $X$ to another convex structure $Y$ is *convexity preserving* if it inverts convex subsets of $Y$ into convex subsets of $X$. Other concepts, like *relative convexity* on a subset, are self-defined.

In many cases, $X$ also carries a topology making all polytopes compact. Assuming this to be the case, $X$ is called *uniformizable* (metrizable) as a ‘topological convex structure’ provided there is a (metric) uniformity inducing the $X$-topology and being *compatible* with the $X$-convexity in the following sense. For each uniform entourage $U$ there is an ‘associated’ uniform entourage $V$ such that $V$-close sets have $U$-close convex hulls. A uniformizable convex structure is *closure-stable* (the closure of each convex set is convex), and it has an open base of convex sets. For instance, a topological vector space with its ordinary convex sets is uniformizable iff it is locally convex. For completely different examples, see [16].

General references for abstract convexity are [8, 15]. Information on uniform convexity can be found in [16].

2.2. Selection Theorem. Let $X$ be a metrizable $S_4$ convex structure with connected convex sets. Let $Y$ be a paracompact space, and let $G : Y \to X$ be a lower semi-continuous multifunction with compact convex point-values. Then $F$ admits a continuous selection.

The above formulation is somewhat more restricted than the one in [16, 4.3], but it suffices for our present purposes.

2.3. Superextensions and their convexity. Let $Z$ be a normal $T_1$ space. A *maximal linked system* (mls) on $Z$ is a family of closed, pairwise intersecting subsets of $Z$,
maximal with these properties. Put
\[ \lambda(Z) = \{ m: \text{an mls on } Z \}; \]
\[ A^+ = \{ m: \text{for some } M \in m, M \subset A \} \quad \text{where } A \subset Z; \]
\[ \mathcal{F} = \{ A^+: A \subset Z \text{ closed} \}. \]

With the topology, generated by the closed subbase \( \mathcal{F} \), the space \( \lambda(Z) \) is called the superextension of \( Z \). It is compact and Hausdorff, and there is a natural embedding of \( Z \) into \( \lambda(Z) \). See [9, 18] for details.

The family of all updirected unions of intersections of subfamilies in \( \mathcal{F} \) is the convexity generated by \( \mathcal{F} \). It is regarded to be the canonical convexity of \( \lambda(Z) \). In general, a convexity is determined by its polytopes in the sense that a set is convex iff it includes the hull of each of its finite subsets. The \( \lambda(Z) \)-convexity can be described alternatively as follows. For \( m, m_1, \ldots, m_k \) in \( \lambda(Z) \),
\[ m \in co\{m_1, \ldots, m_k\} \iff m \subset m_1 \cup \cdots \cup m_k. \]

The convexity of \( \lambda(Z) \) is always \( S_d \), and it has the following binarity property: if \( \mathcal{D} \) is a finite collection of pairwise intersecting convex sets, then \( \bigcap \mathcal{D} \neq \emptyset \). This is perhaps the most important property of superextensions; a more detailed account will be given below.

Let us list a few other properties needed later.

1. If \( Z \) is connected, then so is \( \lambda(Z) \) [18, III.4.1];
2. If \( Z \) is compact metrizable, then so is \( \lambda(Z) \) [18, IV.2.4];
3. The canonical uniformity and convexity on \( \lambda(Z) \) are compatible [16, 3.4].

By (2) and (3), \( \lambda(Z) \) is a metrizable convex structure if \( Z \) is compact metric. In [10] Van Mill proved that \( \lambda(Z) \) is homeomorphic to the Hilbert cube if \( Z \) is a nondegenerate metric continuum. The author proved in [15, 7.6] that, in the same circumstances, every nondegenerate closed half-space of \( \lambda(Z) \) is a Hilbert cube. We note that the points which form singleton half-spaces of \( \lambda(Z) \) (the extreme points) correspond to the points of \( Z \), as we observed in [15].

2.4. Binary convexity. A convex structure, satisfying the above quoted binarity property, is called a binary convex structure. For examples and background information, see [17].

Due to the complicated nature of the points of \( \lambda(Z) \), computations in \( \lambda(Z) \) tend to be cumbersome. It is more economical to work on the general level of binary convexity, and to fill in specific properties of superextensions where they are needed.

The following is a frequently used tool. For a moment, let \( X \) merely have the separation axiom \( S_d \), and let \( b \in X \). The relation \( \preceq_b \), defined by
\[ u \preceq_b v \iff u \in co\{b, v\} \]
determines a partial order on \( X \), called the base-point order of \( b \). On \( co\{b, a\} \), the orders \( \preceq_b \) and \( \preceq_a \) are mutually inverse. If \( X \) is also binary, then each interval
co\{b, a\} becomes a (distributive) lattice under \(\leq_b\). Every convex set is downdirected under \(\leq_b\). Certain convex sets \(C\) have a minimal—hence a smallest—element for \(\leq_a\). Such a point is called the nearest-point of \(b\) in \(C\), and we will denote it by \(p(b, C)\) (note that the term ‘nearest’ refers to an ordering; for metric information, see [11]). The following facts will be used several times:

1. If \(S \neq \emptyset\) is finite and if \(b \in X\), then \(p(b, \co(S))\) exists and it equals the infimum of \(S\) in \(\leq_b\) [17, 3.2];
2. If \(C, D\) are intersecting convex sets and if \(b \in D\), then \(p(b, C)\in D\) whenever \(p(b, C)\) exists [17, 2.2].

In a uniformizable, binary and \(S_4\)-convex structure \(X\), the nonempty convex closed sets are exactly the ones which have a smallest element in each base-point order. For such a set \(C\), we have a nearest-point function

\[ p = p(-, C) : X \to C \]

with the following property:

3. \(p(c) = c\) for each \(c \in C\), \(p\) is convexity preserving, and \(p\) is continuous if \(C\) is compact (in general, \(p\) is known only to be ‘weakly’ continuous).

See [17, sections 2–3] for a detailed treatment of these topics. We finally mention a result—implicit in [15, 2.9]—which is also needed below:

4. If \(C, D\) are intersecting convex subsets, then \(\cl_X C \cap D = \cl_D(C \cap D)\) (this is known as the relative closure property). Here, \(X\) is metrizable.

3. The main result

3.1. Proposition. Let \(Z\) be a metric continuum, and let \(m_1, m_2 \in \lambda(Z)\) be distinct. Then the interval \(I = \co\{m_1, m_2\}\) admits a relative half-space \(H\) such that both \(H\) and \(I\setminus H\) are dense in \(I\).

Proof. As \(m_1 \neq m_2\), there exist \(M_1 \in m_1\) and \(M_2 \in m_2\) with \(M_1 \cap M_2 = \emptyset\). Note that each \(m_i\) is a closed subset of the hyperspace of \(Z\). Hence we may assume that \(M_i\) is minimal in \(m_i\). No point of the nonempty open set \(Z \setminus (M_1 \cup M_2)\) is isolated, and we can find two dense subsets \(F_1, F_2\) with

\[ F_1 \cap F_2 = \emptyset, \quad F_1 \cup F_2 = Z \setminus (M_1 \cup M_2). \]

Let \(H = (M_1 \cup F_1)^+\), and note that

\[ \lambda(Z) \setminus H = \{m: \text{all } M \in m \text{ meet } M_2 \cup F_2\}. \]

Both \(H\) and \(\lambda(Z) \setminus H\) are convex, as the reader can easily verify, and \(m_1 \in H\), \(m_2 \in I \setminus H\).

Almost by definition, the sets of type \(O^+\), \(O \subset Z\) open, form an open subbase for the \(\lambda(Z)\)-topology. Let \(\bigcap_{i=1}^n O_i^+\) (with \(O_i \subset Z\) open) be a basic neighborhood of \(m_2\). Then there exist \(N_i \in m_2\) \((i = 1, \ldots, n)\) with \(N_i \subset O_i\) for all \(i\). Hence \(O_i \not\subset M_2\) by
the minimality of the latter, and it follows that \( O_i \) meets \( M \cup F_1 \). We find that the convex sets

\[
O_i^+(i=1, \ldots, n), \quad H,
\]

meet two by two, and by binarity,

\[
\bigcap_{i=1}^n O_i^+ \cap H \neq \emptyset,
\]

showing that \( m_2 \in \text{cl} \, H \). One similarly shows that \( m_1 \in \text{cl} \, (\lambda(Z) \setminus H) \). By the relative closure property, 2.4(4),

\[
m_2 \in \text{cl} \,(H \cap I), \quad m_1 \in \text{cl} \,(I \setminus H),
\]

whereas

\[
m_1 \in H \cap I, \quad m_2 \in I \setminus H.
\]

As \( \text{cl} \,(H \cap I) \) is a convex set (closure-stability, see 2.1) containing \( m_1 \) and \( m_2 \), we conclude that \( H \cap I \) is dense in \( I \). Similarly, \( I \setminus H \) is dense in \( I \). \( \square \)

For convenience, let us say that a subset of a topological space is codense if its complement is dense. The relevance of the above fact lies in the following result of [16], which is a consequence of the selection theorem:

**3.2. Theorem.** Let \( X \) be a metrizable \( S_4 \) convex structure with connected convex sets and with a locally compact underlying space. If \( X \) admits a half-space which is both dense and codense, then \( X \) is a Hilbert cube manifold.

By way of example, it is easy to see that every infinite dimensional, compact metric convex subset \( C \) of a topological vector space admits a dense and codense relative half-space. The above theorem applies if \( C \) is locally convex (then \( C \) is metrizable as a convex structure).

Unfortunately, we have been unable to extend Proposition 3.1 to more complicated convex subsets of a superextension. We will therefore combine the restricted information of Proposition 3.1 with some infinite dimensional topology.

From now on, \( X \) denotes a metrizable, binary and \( S_4 \) convex structure such that \( X \) is connected and each nontrivial interval in \( X \) has a dense and codense relative half-space. Note by 2.4 (1)-(3) that each polytope of \( X \), and hence each convex subset of \( X \), will be connected. An interval \( \text{co} \{u, v\} \) will be denoted more conveniently by \( [u, v] \).

**3.3. Lemma.** Let \( S \) be a finite nonempty subset of \( X \), and let \( b \in X \setminus \text{co} \,(S) \). Then the subspace \( Y = \bigcup_{u \in S} [b, u] \) is homeomorphic to the Hilbert cube.
Let us first note that the condition \( b \notin \text{co}(S) \) is essential. For, if \( b \) is in \( [u, v] \setminus \{u, v\} \), then
\[
[u, b] \cap [b, v] = \{b\}
\]
by [17, 2.3], and hence \( b \) is a cutpoint of \([b, u] \cup [b, v]\).

**Proof of Lemma 3.3.** For each \( u \in S \) we fix a relative half-space \( H_u \) of \([b, u] \) which is dense and codense in \([b, u] \). We may assume that \( u \in H_u \). We first collect some notation and some elementary facts.

If \( T \subseteq S \) is nonempty, then the infimum (for \( \leq_b \)) of \( T \) will be denoted by \( \wedge T \). Note that \( b \neq \wedge T \) since \( b \notin \text{co}(T) \subseteq \text{co}(S) \). By [17, 3.2],
\[
[b, \wedge T] = \bigcap_{u \in T} [b, u].
\]
If \( A \subseteq X \) and \( T \subseteq S \), then we put
\[
T(A) = \{ u \in T : A \text{ meets } [b, u] \}.
\]
Using binarity and (1), it follows that
\[
T(A) = \{ u \in T : A \text{ meets } [b, u] \}.
\]

For \( u \in S \), we let \( p_u : X \to [b, u] \) denote the nearest-point function. We have:

- If \( T \subseteq p_u^{-1}H_u \) is nonempty, then \( p_u^{-1}H_u \cap [b, \wedge T] \) is a relatively dense half-space of \([b, \wedge T] \) containing \( \wedge T \) (it is also codense if \( u \in T \)).

Indeed, \( p_u^{-1}H_u \) is a half-space of \( X \) since \( p_u \) is convexity preserving, see 2.4(3).

As \( T \subseteq p_u^{-1}H_u \), we have
\[
\wedge T \subseteq \text{co}(T) \subseteq p_u^{-1}H_u.
\]
On the other hand, \( b \) is adherent to \( H_u \subseteq p_u^{-1}H_u \). Hence by the relative closure

property 2.4(4) and by closure stability, it follows that \( p_u^{-1}H_u \cap [b, \wedge T] \) is relatively dense in \([b, \wedge T] \). Finally, let \( u \in T \) and \( K_u = [b, u] \setminus H_u \). Then \( b \in K_u \), and \( K_u \) is dense in \([b, u] \) by assumption. In particular, \( \wedge T \) is adherent to \( K_u \subseteq p_u^{-1}K_u \), and the desired result follows as above.

The following technical result will be used twice. For \( T \subseteq S \) we let
\[
Q_T = \bigcup_{v \in T} [b, v].
\]

**Statement.** Let \( T \) be a nonempty subset of \( S \). Then for each \( \varepsilon > 0 \) there is a map \( f : Q_T \to Q_T \) such that

1. \( f(Q_T) \subseteq Q_T \cap p_u^{-1}H_u \) for each \( u \in T \) with \( T \subseteq p_u^{-1}H_u \);
2. \( f \) is \( \varepsilon \)-close to identity.

Note how essential the condition \( T \subseteq p_u^{-1}H_u \) is for this result. If \( v \in T \setminus p_u^{-1}H_u \) then the entire interval \([b, v] \) remains outside of \( p_u^{-1}H_u \), and in regard of (i), the condition (ii) can no longer be guaranteed.
Proof of Statement. Let $\varepsilon > 0$, and consider two open (in $X$) covers $\mathcal{U}$, $\mathcal{V}$, of $Q_T$ with the following properties.

1. Each $U \in \mathcal{U}$ is convex and has diameter $< \varepsilon$; (4)
2. Each $V \in \mathcal{V}$ is convex and meets $Q_T$; (5)
3. The closed covering $\{V^\ominus : V \in \mathcal{V}\}$ is a finite barycentric refinement of $\mathcal{U}$. (6)

For each $V \in \mathcal{V}$ we note that $T(V)$ is nonempty by (5). We let

$$c_V = p(\wedge T(V), V^-),$$

and we construct a multifunction $F : Q_T \to X$ by

$$F(y) = \operatorname{co}\{c_V : y \in V \in \mathcal{V}\} \quad \text{where } y \in Q_T.$$

For a fixed $y \in Q_T$ note that $\bigcap \{V : y \in V \in \mathcal{V}\}$ is a neighborhood of $y$, and for each member $y'$ of it, at least the same $V$'s are involved, showing that $F(y) \subseteq F(y')$. Lower semi-continuity of $F$ easily follows.

If $y \in Q_T$, say, $y \in [b, v]$ for some $v \in T$, and if $y \in V \in \mathcal{V}$, then $v \in T(V)$ by definition of the latter. Hence $[b, v]$ is a convex set containing $\wedge T(V)$ and meeting $V^\ominus$ (in $y$, for instance), whence by 2.4(2),

$$c_V = p(\wedge T(V), V^-) \subseteq [b, v].$$

By (3), $p_u^{-1}H_u$ is another convex set containing $\wedge T(V)$ and meeting $V^\ominus$, and we have, similarly, that

$$c_V \in p_u^{-1}H_u \quad u \in T, \quad T \subseteq p_u^{-1}H_u.$$

This shows that for each $y \in Q_T$ (with a corresponding $v \in T$ as above),

$$F(y) \subseteq [b, v] \cap p_u^{-1}H_u \subseteq Q_T \cap p_u^{-1}H_u.$$ (7)

By the selection theorem, there is a continuous $f : Q_T \to Q_T$ selecting from $F$. By (7), $f$ maps into $p_u^{-1}H_u$. For $y \in Q_T$, fix a set $U \in \mathcal{U}$ with

$$\bigcup \{V^\ominus : y \in V \in \mathcal{V}\} \subseteq U,$$

see (6). If $y \in V$, then $c_V \in U$, and as $U$ is convex, we conclude that $F(y) \subseteq U$. Now diam $U < \varepsilon$ by (4), and hence $f$ is $\varepsilon$-close to identity. □

The main line of our proof of Lemma 3.3 is an induction on the cardinality $n$ of $S$. For $n = 1$, use the assumptions and Theorem 3.2. Suppose $n > 1$, and assume the result to be valid for unions $Y$ of less than $n$ intervals $[b, u]$.

First case. There is a point $u \in S$ with $S \cap p_u^{-1}H_u$ a proper subset of $S$.

Proof. We put $S_0 = S \cap p_u^{-1}H_u$ and $S_1 = S \setminus S_0$. Define $Q_0 = Q_{S_0}$, i.e., $Q_0 = \bigcup_{v \in S_0} [b, v]$. For each $v \neq u$ in $S_0$ we construct in $[b, v]$ (a lattice—see 2.4) a point

$$v' = \bigvee_{w \in S_1} (w \wedge v),$$
where $\vee$ denotes supremum. Note that $w$ need not be in $[b, v]$, but $w \wedge v$ does. Then put

$$Q_1 = \bigcup_{w \in S_1} [b, w] \cup \bigcup_{v \in S_0, v \neq u} [b, v']\].$$

Note that $v' \in [b, v]$, whence

$$Y = \bigcup_{v \in S} [b, v] = Q_0 \cup Q_1.$$

Each of $Q_0, Q_1$ is a Hilbert cube by inductive assumption. By (1),

$$Q_0 \cap Q_1 = \bigcup_{w \in S_1} [b, u \wedge w] \cup \bigcup_{v \in S_0, v \neq u} [b, u \wedge v']
\cup \bigcup_{v \in S_0, v \neq u} [b, v \wedge w] \cup \bigcup_{v \in S_0, v \neq u} [b, v \wedge v'].$$

We enumerate the four main expressions on the right by I, . . . , IV. In the expression IV there is for each $v \neq u$ in $S_0$ a largest term, namely $[b, v']$ (just take $v_0 = v$). Hence,

$$\text{IV} = \bigcup_{v \in S_0, v \neq u} [b, v'].$$

Fix a $v \neq u$ in $S_0$. Then for each $w \in S_1$ we have $v \wedge w \leq b \vee v'$ by the construction of $v'$, and hence $[b, v \wedge w] \subseteq [b, v']$. This allows us to drop III entirely. We also have a term $[b, u \wedge v']$ in II, which is obviously included in $[b, v']$. Hence, the expression II is also redundant. We obtain

$$Q_0 \cap Q_1 = \bigcup_{w \in S_1} [b, u \wedge w] \cup \bigcup_{v \in S_0, v \neq u} [b, v'], \quad (8)$$

a union of $n-1$ intervals, and hence a Hilbert cube.

Note that $b \not\in p_u^{-1}H_u$, that $u \wedge w \not\in p_u^{-1}H_u$ for each $w \in S_1$, and that $v' \not\in p_u^{-1}H_u$ for each $v \in S_0$, $v \neq u$. Hence by (8)

$$Q_0 \cap Q_1 \subseteq X \setminus p_u^{-1}H_u.$$

Apply the Statement with $T = S_0$: for each $\varepsilon > 0$ there is an $f: Q_0 \to Q_0$ with $f(Q_0) \subseteq p_u^{-1}H_u$ (hence $f(Q_0)$ is disjoint with $Q_0 \cap Q_1$) and $f$ is $\varepsilon$-close to identity. It follows that $Q_0 \cap Q_1$ is a $Z$-set in $Q_0$. By a result of Handel, [5, thm. 1], $Q_0 \cup Q_1$ is a Hilbert cube.

**Second case.** $S \subseteq p_u^{-1}H_u$ for each $u \in S$.

**Proof.** This situation is treated outside of the induction loop. Let $\varepsilon > 0$. By the Statement (with $T = S$) we obtain a map $f: Y \to Y$ with the following properties:

(i) $f(Y) \subseteq p_u^{-1}H_u$ for all $u \in S$;

(ii) $f$ is $\varepsilon$-close to identity.

We construct a second map as follows. Take covers $\mathcal{U}, \mathcal{V}$, as in (4) to (6) (with $T = S$). This time we construct points

$$d_v = p(b, V^-), \quad V \in \mathcal{V}.$$
For each \( y \in Y \) we put
\[
G(y) = \text{co}\{d_{V}: y \in V \in \mathcal{V}\}.
\]
Lower semi-continuity of \( G \) is again easy to prove. If \( y \in V \in \mathcal{V} \), then \([b, y]\) is a convex set meeting \( V^-\), whence by 2.4(2), \( d_{V} \in [b, y] \). This gives us
\[
G(y) \subseteq [b, y] \subseteq Y.
\]
Let \( y \in [b, u] \) for some \( u \in S \). Then the set
\[
K_{u} = [b, u] \setminus H_{u}
\]
is also dense in \([b, u]\) and hence it meets each \( V^-\) with \( y \in V \). By 2.4(2) again, we have \( d_{V} \in K_{u} \) for each \( V \) as above. Hence:
\[
G(Y) = K_{u}.
\]
Let \( g \) be a continuous selection from \( G \). Then \( g \) is \( \varepsilon \)-close to identity (argue as in the Statement) and if \( y \in [b, u] \) for some \( u \in S \) then
\[
g(y) \in K_{u}
\]
whereas \( f \) maps the whole of \( Y \) within \( p_{\varepsilon}^{-1}H_{u} \) by (ii) above. We conclude that \( f \) and \( g \) have disjoint images.

Now \( Y \) is a union of finitely many intervals \([b, u]\), \( u \in S \), and for each nonempty \( T \subseteq S \), the convex set \( \bigcap_{u \in T} [b, u] \) is an AR, [16, 5.1]. Induction on the cardinality of \( S \) then shows that \( Y \) is an AR. By Toruńczyk's Q-manifold (\( Q = \text{Hilbert cube} \)) characterization [13, section 2], we conclude that \( Y \) is a Hilbert cube. \( \square \)

The above result may also illustrate how much an abstract convexity with quite reasonable properties can differ from the ordinary vector space convexity.

3.4. Main Theorem. Each compact convex set in \( X \) with more than one point is homeomorphic to the Hilbert cube.

Proof. By [16, 2.5], every two nonproximate (relative to a compatible uniformity) convex sets in \( X \) can be separated with a convexity preserving map \( X \to \mathbb{R} \). Starting with two distinct points in a compact convex \( C \subseteq X \), we therefore can find an open half-space \( O \) of \( X \) with
\[
O \cap C \neq \emptyset, \quad C \not\subseteq O.
\]
After maximizing \( O \) with these properties, we find, in addition, that \( O \cap C \) is dense in \( C \).

Fix a point \( b \in C \setminus O \), and let \( \varepsilon > 0 \). Choose \( \delta, \delta_{1} > 0 \) such that two \( \delta_{1} \)-close (resp. \( \delta \)-close) sets have \( \delta \)-close (resp. \( \frac{1}{\varepsilon} \)-close) hulls. Then take a finite set \( S \subseteq O \cap C \) which is \( \frac{1}{\varepsilon} \delta_{1} \)-dense in \( C \). Put \( Y = \bigcup_{u \in S} [b, u] \), and note that \( b \notin \text{co}(S) \) since \( b \notin O \).

We construct a multifunction \( F : C \to C \) as follows. For \( x \in C \), let
\[
S_{x} = \{ u \in S : d(u, x) \leq \frac{1}{\varepsilon} \delta_{1} \}
\]
(d is some compatible metric on X). Then put
\[ F(x) = [b, \wedge S_x] \]
(where, as in Lemma 3.3, \( \wedge S_x \) is the infimum of \( S_x \) in the base-point order of \( b \)). Note that, for some \( u \in S_x, [b, \wedge S_x] \subseteq [b, u] \), whence \( F(x) \subseteq Y \).

To see that \( F \) is lower semi-continuous at \( x \in C \), let \( \mu \) be the minimum of all \( d(x, u) \) with \( u \in S \setminus S_x \). Then \( \mu > \frac{1}{2} \delta_1 \) and for each \( x' \) with \( d(x, x') < \mu - \frac{1}{2} \delta_1 \) we have \( S_x \cap S_{x'} \). Hence \( \wedge S_x \approx \wedge S_{x'} \) and it follows that \( F(x) \cap F(x') \). Lower semi-continuity follows immediately.

Henceforth, we will write
\[ B(x, r) = \{ x' : d(x, x') < r \}, \quad x \in \mathcal{C}, \quad r > 0. \]

We now show that for \( x \in C \), the point \( x \) is at less than \( \delta \) from \( F(x) \). Indeed, as \( S_x \) and \( \{ x \} \) are \( \delta_1 \)-close, we find that \( \text{co}(S_x) \) and \( x \) are \( \delta \)-close. In particular,
\[ \wedge S_x \subset B(x, \delta) \cap F(x). \]

As was shown in [16], the multifunction \( G \) with
\[ G(x) = \text{cl co}(B(x, \delta) \cap F(x)), \quad x \in \mathcal{C}, \]
is again lower semi-continuous. Take a continuous selection \( g \) of \( G \). Note that \( G(C) \subseteq F(C) \subseteq Y \), whence \( g(C) \subseteq Y \). Also, \( g(x) \) is in \( \text{cl co}(B(x, \delta)) \), and as \( B(x, \delta) \) is \( \delta \)-close to \( x \), we have
\[ \text{cl co } B(x, \delta) \subseteq \text{cl } B(x, \frac{1}{2} \epsilon) \subseteq B(x, \epsilon), \]
showing that \( g \) is \( \epsilon \)-close to identity.

Now the space \( Y \subseteq C \) is a Hilbert cube by Lemma 3.3. Hence there exist two maps \( f_1, f_2 : Y \to Y \) which are \( \epsilon \)-close to identity, and which have disjoint images. Then \( f_1 g, f_2 g : C \to Y \subseteq C \) are \( 2 \epsilon \)-close to identity, and have disjoint images. By Toruńczyk's result again, \( C \) is homeomorphic to the Hilbert cube. \( \square \)

### 3.5. Corollary

Let \( Z \) be a metric continuum. Then each nondegenerate compact convex subset of \( \lambda(Z) \) is homeomorphic to the Hilbert cube.

### 3.6. Questions

1. In [12, 3.2], Quinn and Wong proved that a union of compact convex sets \( C_1, \ldots, C_n \) in a metric locally convex vectorspace is homeomorphic to the Hilbert cube \( Q \) provided each \( C_i \), and each intersection of two or more \( C_i \)'s is homeomorphic to \( Q \). In regard of Lemma 3.3, we are led to ask the following.

   Let \( X \) be a metrizable \( S_4 \) convex structure with connected convex sets. Let \( C_1, \ldots, C_n \) be compact convex subsets of \( X \), each intersection of which is homeomorphic to \( Q \). Must \( \bigcup_{i=1}^{n} C_i \) be homeomorphic to \( Q \)?

   Even if \( X \) is binary, the question remains unsolved, since in Lemma 3.3 we used some additional condition concerning dense and codense half-spaces.
Let $Z$ be a metric continuum, and let $C \subset \lambda(Z)$ be convex, topologically complete, and nowhere locally compact. The main result of this paper may suggest that $C$ will then be homeomorphic to the separable Hilbert space. The following example (which I owe to Jan van Mill) shows that the situation is more complicated.

Let $Z = Q$, and fix a sequence $X_n$ of closed pairwise disjoint subsets, such that for each $n$,

$$\text{int } X_n = \emptyset; \quad X_n \text{ is } (1/n)\text{-dense in } Q.$$  

Now each $X_n^+$ is a closed half-space of $\lambda(Q)$, and hence the set

$$C = \lambda(Q) \setminus \bigcup_n X_n^+ = \bigcap_n \lambda(Q) \setminus X_n^+$$

is convex and $G_\delta$. Both $C$ and $\lambda(Q) \setminus C$ are dense in $\lambda(Q)$. Hence $C$ is nowhere locally compact and complete. Its complement, $\bigcup_n X_n^+$, is easily seen to be a $\sigma$-$Z$-set which is not continuum-wise connected, being the union of countably many pairwise disjoint continua. Hence, by a result of Curtis [3], $C$ cannot be homeomorphic to $l_2$. We note that by [1, Theorem 3.5], the convex set $C$ satisfies $C \times C = l_2$.

(3) Let $X$ be a $Q$-manifold with a metrizable $S_4$ convexity, where convex sets are connected. Does there exist a dense and codense half-space in $X$ (compare Theorem 3.2)?

References


