General form of some rational recursive sequences

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\section*{A B S T R A C T}

In this note, we study the general form of some rational recursive sequences. By some modification of the methods and ideas, as well as the transformation from the paper [K. S. Berenhaut, J. D. Foley and S. Stević, The global attractivity of the rational difference equation \( y_n = \frac{y_{n-k} + y_{n-l}}{1 + y_{n-k}y_{n-l}} \), Appl. Math. Lett. 20 (2007), 54–58], we give a new proof for the conjectures posed therein.

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1. Introduction

In 2003, X. Li and D. Zhu [1] investigated the qualitative behavior of the equation

\[ x_n = \frac{1 + x_{n-1}x_{n-2}}{x_{n-1} + x_{n-2}}, \quad n = 0, 1, \ldots \]  

(1.1)

with \( x_{-2}, x_{-1} \in (0, \infty) \), and in 2005, X. Li [2,3] investigated the behaviors of some particular third-order difference equations related to Eq. (1.1) by using semi-cycle analysis similar to that in [4]. The problem concerning periodicity of semi-cycles of difference equations was solved in very general settings by L. Berg and S. Stević in [5], partially motivated also by [6].

K. Berenhaut, J. Foley and S. Stević [7] investigated a rational difference equation, and put forward two conjectures. And, motivated by paper [8], they started with the investigation of the following difference equation \( y_n = A + \left( \frac{y_{n-k} + y_{n-l}}{y_{n-m}} \right)^p \) for \( p > 0 \) (see, [9–11] and [12]). Among others, in [9] they used a transformation method, which has turned out to be very useful in studying equations

\[ x_n = \frac{x_{n-k} + x_{n-l} + x_{n-m} + x_{n-k}x_{n-l}x_{n-m}}{1 + x_{n-k}x_{n-l} + x_{n-l}x_{n-m} + x_{n-m}x_{n-k}}, \quad n = 0, 1, 2, \ldots \]  

(1.2)

where \( x_{-m}, x_{-m+1}, \ldots, x_{-1} \in (0, \infty) \) and \( 1 \leq k < l < m \), as well as in confirming Conjecture 1 from [7] (see, [13]).

In the meantime, it turned out that the method used in [14] by C. Cinar, S. Stević and I. Yalcinkaya, can be used in confirming Conjecture 2 from [7] (see also [15]). More precisely, papers [14] and [15] use Corollary 3 from [16] in solving similar problems. For example, C. Cinar, S. Stević and I. Yalcinkaya shown, in an elegant way, that the main result in [1] is a consequence of Corollary 3 in [16]. With some calculations it can be also shown that Conjecture 2 from [7] can be confirmed in this way (see, [17]).

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It is clear that (1.1) and (1.2) can be rewritten as

\[ x_n = \frac{(x_{n-1} + 1)(x_{n-2} + 1) + (x_{n-1} - 1)(x_{n-2} - 1)}{(x_{n-1} + 1)(x_{n-2} + 1) - (x_{n-1} - 1)(x_{n-2} - 1)}, \quad n = 0, 1, \ldots, \]  

(1.3)

and

\[ x_n = \frac{(x_{n-1} + 1)(x_{n-2} + 1) + (x_{n-1} - 1)(x_{n-2} - 1)}{(x_{n-1} + 1)(x_{n-2} + 1) - (x_{n-1} - 1)(x_{n-2} - 1)}, \quad n = 0, 1, \ldots, \]  

(1.4)

respectively.

Generalizing (1.3) and (1.4), in this note, we consider the global attractivity for the following general rational recursive sequences

\[ x_n = \frac{\prod_{i=1}^{v} (x_{n-k_i} + 1) + \prod_{i=1}^{v} (x_{n-k_i} - 1)}{\prod_{i=1}^{v} (x_{n-k_i} + 1) - \prod_{i=1}^{v} (x_{n-k_i} - 1)}, \quad n = 0, 1, \ldots, \]  

(1.5)

where \( v > 1, x_{-k_i}, x_{-k_i+1}, \ldots, x_1 \in (0, \infty) \) and \( 1 \leq k_1 < k_2 < \cdots < k_v \).

With respect to the equilibrium point \( \overline{x} \) of Eq. (1.5), it should satisfy

\[ \overline{x} = \frac{(\overline{x} + 1)^v + (\overline{x} - 1)^v}{(\overline{x} + 1)^v - (\overline{x} - 1)^v}, \]

from which we may get a unique positive equilibrium point \( \overline{x} = 1 \).

It is worth noting that when \( v \) is only odd in (1.5), the main result in [18] confirms Conjecture 2 from [7].

Some other related results can be found in [19–28].

In this note, by employing the transformation method suggested by Berenhaut and Stević and following the lines in their paper [13] (see also [7]) we give a new proof for the conjectures therein.

2. Main result and its proof

The main purpose in this note is to prove the following result.

**Theorem 2.1.** Every solution of Eq. (1.5) converges to the unique equilibrium \( \overline{x} = 1 \) as \( n \) tends to infinity.

First, we consider the following transformed sequence \( \{x_n^v\} \) defined by

\[ x_n^v = \begin{cases} x_n, & \text{if } x_n \geq 1, \\ 1, & \text{otherwise}. \end{cases} \]  

(2.1)

The next lemma is a slight extension of Lemmas 2 and 3 in [13].

**Lemma 2.1.** Suppose that \( \{x_n\} \) satisfies (1.5), and that \( \{x_n^v\} \) is obtained from \( \{x_n\} \) via (2.1). Then we have

\[ x_n^v = \frac{\prod_{i=1}^{v} (x_{n-k_i}^v + 1) + \prod_{i=1}^{v} (x_{n-k_i}^v - 1)}{\prod_{i=1}^{v} (x_{n-k_i}^v + 1) - \prod_{i=1}^{v} (x_{n-k_i}^v - 1)}, \quad n = 0, 1, \ldots, \]  

(2.2)

**Proof.** Let \( \mathcal{N}_n = \{ i \in \{1, 2, \ldots, v\} : x_{n-k_i} < 1 \} \) for a given \( n \), and \( ||\mathcal{N}_n|| \) denotes the cardinality of \( \mathcal{N}_n \). Then, by (1.5) and (2.1), we have the following five cases.

(1) If \( ||\mathcal{N}_n|| = 0 \), then

\[ x_n = \frac{\prod_{i=1}^{v} (x_{n-k_i}^v + 1) + \prod_{i=1}^{v} (x_{n-k_i}^v - 1)}{\prod_{i=1}^{v} (x_{n-k_i}^v + 1) - \prod_{i=1}^{v} (x_{n-k_i}^v - 1)} \geq 1. \]  

(2.3)

(2) If \( 0 < ||\mathcal{N}_n|| = j < v \), and \( j \) is odd. Let

\[ x_n = G(x_{n-k_1}, x_{n-k_2}, \ldots, x_{n-k_j}). \]
Since $G(x_{n-k_1}, x_{n-k_2}, \ldots, x_{n-k_v})$ is symmetric in $x_{n-k_1}, x_{n-k_2}, \ldots, x_{n-k_v}$, we may assume, without loss of generality, that 
\[ x_{n-k_1}, x_{n-k_2}, \ldots, x_{n-k_j} < 1; x_{n-k_{j+1}}, x_{n-k_{j+2}}, \ldots, x_{n-k_v} \geq 1, \quad j = 1, \ldots, v - 1. \]

Then
\[
x_n = \prod_{i=1}^{j} \left( \frac{1}{x_{n-k_i}} + 1 \right) \cdot \prod_{i=j+1}^{v} \left( x_{n-k_i}^* + 1 \right) + \prod_{i=1}^{j} \left( \frac{1}{x_{n-k_i}} - 1 \right) \cdot \prod_{i=j+1}^{v} \left( x_{n-k_i}^* - 1 \right) \\
\leq 1. \quad (2.4)
\]

(3) If $0 < \| \mathcal{N}_n \| = j < v$, and $j$ is even. Similarly, we may assume that 
\[ x_{n-k_1}, x_{n-k_2}, \ldots, x_{n-k}_j < 1; x_{n-k_{j+1}}, x_{n-k_{j+2}}, \ldots, x_{n-k_v} \geq 1, \quad j = 2, \ldots, v - 1. \]

Then
\[
x_n = \prod_{i=1}^{j} \left( \frac{1}{x_{n-k_i}} + 1 \right) \cdot \prod_{i=j+1}^{v} \left( x_{n-k_i}^* + 1 \right) + \prod_{i=1}^{j} \left( \frac{1}{x_{n-k_i}} - 1 \right) \cdot \prod_{i=j+1}^{v} \left( x_{n-k_i}^* - 1 \right) \\
\geq 1. \quad (2.5)
\]

(4) If $\| \mathcal{N}_n \| = v$ is odd, then
\[
x_n = \prod_{i=1}^{v} \left( \frac{1}{x_{n-k_i}} + 1 \right) + \prod_{i=1}^{v} \left( \frac{1}{x_{n-k_i}} - 1 \right) = \prod_{i=1}^{v} \left( x_{n-k_i}^* + 1 \right) - \prod_{i=1}^{v} \left( x_{n-k_i}^* - 1 \right) \leq 1. \quad (2.6)
\]

(5) If $\| \mathcal{N}_n \| = v$ is even, then
\[
x_n = \prod_{i=1}^{v} \left( \frac{1}{x_{n-k_i}} + 1 \right) + \prod_{i=1}^{v} \left( \frac{1}{x_{n-k_i}} - 1 \right) = \prod_{i=1}^{v} \left( x_{n-k_i}^* + 1 \right) + \prod_{i=1}^{v} \left( x_{n-k_i}^* - 1 \right) \geq 1. \quad (2.7)
\]

From (2.1), (2.3)–(2.7) and (2.2) follows. The proof is complete. \hfill \square

**Lemma 2.2.** Suppose $f$ is defined by
\[
f(y_1, y_2, \ldots, y_v) = \prod_{i=1}^{v} (y_i + 1) + \prod_{i=1}^{v} (y_i - 1) \quad \text{if } n \equiv 0, 1, \ldots \quad (2.8)
\]

and $y_1, \ldots, y_v \in (1, +\infty)$. Then $f$ is increasing in $y_1, \ldots, y_v$, respectively.
Proof. It follows from

$$\frac{\partial f}{\partial y_i} = 4 \prod_{j=1, j \neq i}^v (y_j^2 + 1) \prod_{j=1}^n (y_j + 1) - \prod_{j=1}^n (y_j - 1)$$

that Lemma 2.2 holds. □

Lemma 2.3. Let \( \{x_n\} \) be a positive solution, and let \( \{x_n^*\} \) be defined by (2.1). Then

$$\max_{1 \leq i \leq v} x_{n+k_i}^* \geq x_n^* \geq 1$$

(2.9)

for all \( n \geq k_v \).

Proof. Let

$$M = \max_{1 \leq i \leq v} x_{n-1}^*,$$

and applying Lemma 2.2 \( v \) times to (2.2), we obtain

$$x_n^* \leq \frac{(M + 1)^v + (M - 1)^v}{(M + 1)^v - (M - 1)^v}.$$

Clearly,

$$(M - 1)^{v-1} \leq (M + 1)^{v-1}.$$ 

So,

$$(M + 1)(M - 1)^v \leq (M - 1)(M + 1)^v,$$

that is,

$$(M + 1)^v + (M - 1)^v \leq M [(M + 1)^v - (M - 1)^v].$$

Hence,

$$x_n^* \leq \frac{(M + 1)^v + (M - 1)^v}{(M + 1)^v - (M - 1)^v} \leq M.$$ 

(2.10)

By (2.1), (2.10) and (2.9) follows. The proof is complete. □

Now, set

$$D_n = \max_{n-k_v \leq m \leq n-1} \{x_m^*\}$$

(2.11)

for all \( n \geq k_v \).

The following lemma is the direct corollary of Lemma 2.3 and (2.11).

Lemma 2.4. The sequence \( D_n \) is monotonically non-increasing in \( n \geq k_v \).

Since \( D_n \geq 1 \) for \( n \geq k_v \), Lemma 2.4 implies that, as \( n \) tends to infinity, the sequence \( D_n \) converges to a limit \( D \), where \( D \geq 1 \).

Proof of Theorem 2.1. It suffices to prove that every positive solution \( \{x_n\}_{n=k_v}^{\infty} \) of Eq. (1.5) converges to \( x_1 \) as \( n \to \infty \). Namely, we need to prove

$$\lim_{n \to \infty} x_n^* = 1.$$ 

(2.12)

By (2.11), the values of \( D_n \) are taken on by entries in the sequence \( \{x_n^*\} \), and as well, by Lemma 2.3, \( x_n^* \in [1, D_n] \) for \( n \geq k_v \).

For any \( \epsilon \in (0, D) \), we can always find an \( N \) such that \( x_N^* \in [D, D + \epsilon] \) and \( D_n \leq D + \epsilon \) for \( n \geq N - k_v \), and so

$$x_n^* \in [1, D + \epsilon], \quad n \geq N - k_v.$$ 

Since \( x_n^* \geq 1 \) for all \( n \), employing Lemmas 2.1 and 2.2, it follows from (2.2) that

$$D \leq x_n^* \leq \frac{(D + \epsilon + 1)^v + (D + \epsilon - 1)^v}{(D + \epsilon + 1)^v - (D + \epsilon - 1)^v}.$$
namely,
\[(D - 1)(D + \varepsilon + 1)^v \leq (D + 1)(D + \varepsilon - 1)^v.\]

It follows that
\[\varepsilon \sum_{i=1}^{n} \sum_{i=1}^{n} C_i e^{i-1} (D - 1)^{v-i} - (D - 1)e \sum_{i=1}^{n} C_i e^{i-1} (D + 1)^{v-i},\]

which can be rewritten as
\[(D^2 - 1) \left( (D + 1)^{v-1} - (D - 1)^{v-1} \right) \leq \varepsilon \sum_{i=1}^{n} C_i e^{i-1} \left( (D + 1)(D - 1)^{v-1} - (D - 1)(D + 1)^{v-1} \right).\]

Since \(v > 1\) and \(\varepsilon > 0\) is arbitrary, it follows that \(D = 1\), and so (2.12) holds. The proof of Theorem 2.1 is complete. \(\square\)

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