Note
Upper bounds for the $k$-subdomination number of graphs

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Abstract

For a positive integer $k$, a $k$-subdominating function of $G=(V,E)$ is a function $f: V \rightarrow \{-1,1\}$ such that the sum of the function values, taken over closed neighborhoods of vertices, is at least one for at least $k$ vertices of $G$. The sum of the function values taken over all vertices is called the aggregate of $f$ and the minimum aggregate among all $k$-subdominating functions of $G$ is the $k$-subdomination number $\gamma_k(G)$. In this paper, we solve a conjecture proposed in (Ars. Combin 43 (1996) 235), which determines a sharp upper bound on $\gamma_k(G)$ for trees if $k \geq |V|/2$ and give an upper bound on $\gamma_k(G)$ for connected graphs. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

All graphs under consideration are simple. For a graph $G=(V,E)$ and vertex $v \in V$, let $N(v) = \{u \in V : uv \in E\}$ and $N[v] = \{v\} \cup N(v)$ be the open and closed neighborhoods of $v$ in $G$, respectively. For a subset $A$ of $V$, we set $N_A(v) = N(v) \cap A$ and $d_A(v) = |N_A(v)|$. For $k \in Z^+$, a $k$-subdominating function (kSF) of $G$ is a function $f: V \rightarrow \{-1,1\}$ such that $f[v] = \sum_{u \in N[v]} f(u) \geq 1$ for at least $k$ vertices $v$ of $G$. The aggregate $ag(f)$ of such a function is defined by $ag(f) = \sum_{v \in V} f(v)$ and the $k$-subdomination number $\gamma_k(G)$ by $\gamma_k(G) = \min \{ag(f) : f \text{ is kSF of } G\}$. The concept of $k$-subdominatoin number was introduced and first studied by Cockayne and Mynhardt [2]. In the special cases where $k = |V|$ and $k = \lceil |V|/2 \rceil$, $\gamma_k$ is respectively the signed domination number $\gamma_s$ [3] and the majority domination number $\gamma_{maj}$ [1].

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In [2], Cockayne et al. established a sharp lower bound on $\gamma_{ks}$ for trees. Moreover, they also gave a sharp upper bound on $\gamma_{ks}$ for trees if $k \leq |V|/2$, and proposed the following two conjectures:

**Conjecture 1.** For any $n$-vertex tree $T$ and any $k$ with $n/2 < k \leq n$, $\gamma_{ks}(T) \leq 2k - n$.

**Conjecture 2.** For any connected graph $G$ of order $n$ and any $k$ with $n/2 < k \leq n$, $\gamma_{ks}(G) \leq 2k - n$.

In this paper, we show that Conjecture 1 is true and Conjecture 2 is incorrect, and give an upper bound for the $k$-subdomination number of graphs.

### 2. An upper bound on the $k$-subdomination number for trees

Alon (mentioned in [2]) established the following upper bounds on $\gamma_{ks}$ for a connected graph.

**Theorem A** (Cockayne and Mynhardt [2]).

For any connected graph $G$ of order $n$,

$$\gamma_{maj}(G) \leq \begin{cases} 1 & \text{if } n \text{ is odd}, \\ 2 & \text{if } n \text{ is even}. \end{cases}$$

And Henning and Hind [4] proved the following:

**Theorem B** (Henning and Hind [4]).

If $T$ is a tree of order $n$, $k = \lceil (n + 1)/2 \rceil$, then $\gamma_{ks}(T) \leq 2$.

In order to prove Conjecture 1, we need some definitions from [2]. A *leaf* of a tree is a vertex of degree one and a *remote vertex* of a tree is a vertex having exactly one non-leaf neighbor. We write $L$ and $R$ for the sets of leaves and remote vertices of $T$, respectively.

**Theorem 1.** For any $n$-vertex tree $T$ and $k$ with $n/2 < k \leq n$, $\gamma_{ks}(T) \leq 2k - n$.

**Proof.** To prove the theorem, by the definition of $\gamma_{ks}$, it suffices to show that there exists a $k$-subdominating function $f$ with $ag(f) \leq 2k - n$.

First we may suppose $k > \lceil (n + 1)/2 \rceil$ by Theorem B and $k < n$ for $f(v) = 1$ for all $v \in V$ is an $n$-subdominating function with $ag(f) = n$.

Now we apply induction on the number of vertices of $T$. Note that the assertion is true for $n \leq 4$, so suppose that $n \geq 5$ and the theorem holds for smaller values of $n$. Suppose furthermore $T$ is not a star since for star $T$ with leaves $v_1, v_2, \ldots, v_{n-1}$ there
exists a $k$-subdominating function
\[
f(x) = \begin{cases} 
-1 & \text{if } x = v_i, \ i = 1, \ldots, n - k, \\
1 & \text{otherwise}
\end{cases}
\]
with $\text{ag}(f) = 2k - n$. Thus $|R| \geq 2$.

Suppose that there exists a vertex $u \in R$ such that $d(u)$ is even, and that $v'$ is a leaf adjacent to $u$. Then for the subtree $T_1 = T - v'$ and $k$ with $(n - 1)/2 < k \leq n - 1$, by the induction hypothesis, there exists a $k$-subdominating function $f_1$ on $T_1$ with $\text{ag}(f_1) \leq 2k - (n - 1) = 2k - n + 1$. We define
\[
f(x) = \begin{cases} 
-1 & \text{if } x = v', \\
f_1(x) & \text{otherwise.}
\end{cases}
\]

Then $f$ is a $k$-subdominating function on $T$. Indeed, if $f_1[u] < 1$ in $T_1$, $f$ is a $k$-subdominating function on $T$. And if $f_1[u] \geq 1$ in $T_1$, then $f_1[u] \geq 2$ as $d(u)$ is even, hence $f$ is also a $k$-subdominating function on $T$. Clearly, $\text{ag}(f) = \text{ag}(f_1) - 1 \leq 2k - n$.

Hence we may suppose that $d(u)$ is odd for all $u \in R$, hence $d(u) \geq 3$ as $u$ is not a leaf of $T$. Take a $u' \in R$, write $d(u') = 2s + 1$ and $N(u') = \{v_1, v_2, \ldots, v_{2s+1}\}$, where $v_{2s+1}$ is the unique non-leaf neighbor of $u'$. We separate three cases according to the values of $k$.

**Case 1:** $n - s \leq k \leq n - 1$.

Define
\[
f(x) = \begin{cases} 
-1 & \text{if } x = v_i, \ i = 1, 2, \ldots, n - k, \\
1 & \text{otherwise.}
\end{cases}
\]

Then it is easily seen that $f[x] \geq 1$ if $x \neq v_i, i = 1, 2, \ldots, n - k$, thus $f$ is a $k$-subdominating function on $T$ with $\text{ag}(f) = 2k - n$.

**Case 2:** $(n + 3)/2 < k \leq n - s - 1$ ($\Rightarrow n \geq 8$ as $s \geq 1$).

Put $k_1 = k - s - 2$ and $n_1 = n - 2s - 1$, then $\frac{1}{2}n_1 < k_1 < n_1$. Now consider the subtree $T_1 = T - (N[u'] \setminus \{v_{2s+1}\})$ of order $n_1 < n$. By the induction hypothesis, there exists a $k_1$-subdominating function $f_1$ on $T_1$ with $\text{ag}(f_1) \leq 2k_1 - n_1 = 2k - n - 3$. Define
\[
f(x) = \begin{cases} 
 f(x) & \text{if } x \in V(T_1), \\
-1 & \text{if } x = v_i, \ i = 1, 2, \ldots, s - 1, \\
1 & \text{otherwise.}
\end{cases}
\]

Clearly, $f$ is a $k$-subdominating function on $T$ with $\text{ag}(f) = \text{ag}(f_1) + 3 \leq 2k - n$.

**Case 3:** $\lfloor (n + 1)/2 \rfloor < k \leq (n + 3)/2$.

Then $n = 2k - 3$. To complete the proof, it suffices to show that there exists a $k$-subdominating function $f$ on $T$ with $\text{ag}(f) \leq 2k - n = 3$. For this purpose, among all partitions $\{W_1, W_2\}$ of $V$ with $|W_1| - |W_2| \leq 1$, called equipartitions, choose one such that the number of edges between $W_1$ and $W_2$ is minimum, assume $|W_2| = k - 1$ and $|W_1| = k - 2$. Define a function $\delta(v) = d_{W_i}(v) - d_{W_i \setminus \{v\}}$ for every $v \in W_i$, let $G_i$ denote the subgraph induced by $W_i$, and let $L_i$ and $S_i$ denote the sets of vertices $v \in W_i$ satisfying $d_{W_i}(v) = 1$ and $|N(v) \cap L_i| \geq \lceil \delta(v)/2 \rceil$, respectively, $i = 1, 2$. 


Claim 1. \( \delta(v) > 0 \) for all \( v \in V \) except at most one \( v^* \in W_2 \) with \( \delta(v^*) = 0 \).

First \( \delta(v) \geq 0 \) for all \( v \in W_2 \). Otherwise, moving a \( v \in W_2 \) with \( \delta(v) < 0 \) to \( W_1 \), we obtain a new equipartition with fewer edges between its parts. Also, \( \delta(v) > 0 \) for all \( v \in W_1 \). Otherwise, taking \( u \in W_1 \) with \( \delta(u) \leq 0 \), we obtain a \( k \)-subdominating function of \( ag(f) = 2k - n \) by making \( u \) and all of \( W_2 \) positive, all remaining vertices negative, as \( f[u] = 1 - \delta(u) \geq 1 \).

Furthermore, if there exist two distinct vertices \( v_1, v_2 \in W_2 \) with \( \delta(v_1) = \delta(v_2) = 0 \), then we have a \( k \)-subdominating function of \( ag(f) = 3 \) by letting the positive set of \( f \) consist of \( v_1, v_2 \) and all of \( W_1 \).

Claim 2. (a) \( d_{W_i}(v) \geq 1 \) for all \( v \in W_i \), \( i = 1, 2 \).

(b) \( v \in L \) for all \( v \in L_i \), \( i = 1, 2 \), except at most one \( v^* \in W_2 \) with \( \delta(v^*) = 0 \) \( (d_{W_i}(v^*) = 1, d(v^*) = 2) \).

(c) \( |L_i| \geq 2, i = 1, 2 \).

Indeed, \( d_{W_i}(v) \geq [d(v)/2] \geq 1 \) and by Claim 1, for all \( v \in W_i \) with \( d_{W_i}(v) = 1 \) except \( v^* \), \( d(v) = 2d_{W_i}(v) - \delta(v) \leq 1 \), yielding (a) and (b). (c) follows from \( G_i \) being acyclic.

Claim 3. \( S_i \neq \emptyset, i = 1, 2 \).

To see this, let \( P = v_1v_2 \cdots v_{l+1} \) be a longest path in \( G_i \). Then obviously \( l \geq 1 \) by Claim 2(a). Moreover, \( v_l \in S_i \). Otherwise, there exists a path \( v_lv'v'' \) in \( G_i \) with \( v' \neq v_{l-1} \), and \( P' = v_1v_2 \cdots v_lv'' \) is a path longer than \( P \).

If \( \lceil \delta(u)/2 \rceil \leq \lceil \delta(v)/2 \rceil \) for some \( u \in S_1 \) and some \( v \in S_2 \), then \( \lceil \delta(u)/2 \rceil \leq |N(u) \cap L_1| \) and \( \lceil \delta(v)/2 \rceil \leq |N(v) \cap L_2| \) by the definition of \( S_i \). Let \( Q_1 \subseteq N(u) \cap L_1 \) and \( Q_2 \subseteq N(v) \cap L_2 \) be sets of \( \lceil \delta(u)/2 \rceil \) vertices, respectively. By Claim 2(b), \( w \in L \) for all vertices \( w \in Q_1 \cup (Q_2 - \{v^*\}). \)

Define
\[
 f(x) = \begin{cases} 
-1 & \text{if } x \in Q_1 \cup W_1 \setminus (\{u\} \cup Q_1), \\
1 & \text{otherwise.} 
\end{cases}
\]

Clearly, \( f \) is a \( k \)-subdominating function on \( T \) with \( ag(f) = 3 \) if \( f[u] \geq 1 \). And if \( f[u] \leq 0 \), then the exceptional vertex \( v^* \in N(u) \cap Q_2 \), implying \( f[v^*] = f(u) + f(v) - 1 = 1 \) by Claim 2(b), which guarantees that \( f \) is still a \( k \)-subdominating function with \( ag(f) = 3 \).

So, suppose \( \lceil \delta(u)/2 \rceil > \lceil \delta(v)/2 \rceil \) for all \( u \in S_1 \) and all \( v \in S_2 \). Thus, for all \( u \in S_1 \) and all \( v \in S_2 \), \( \lceil \delta(u)/2 \rceil \geq \lceil \delta(v)/2 \rceil + 1 \), so that \( \lceil \delta(u)/2 \rceil \geq \lceil \delta(v)/2 \rceil \). Let \( u \in S_1 \) and \( v \in S_2 \). Then \( |N(u) \cap L_1| \geq \lceil \delta(v)/2 \rceil > \lceil \delta(v)/2 \rceil - 1 \) and \( |N(v) \cap L_2| \geq \lceil \delta(v)/2 \rceil \). Let \( Q_1 \subseteq N(u) \cap L_1 \) and \( Q_2 \subseteq N(v) \cap L_2 \) be sets of \( \lceil \delta(v)/2 \rceil - 1 \) and \( \lceil \delta(v)/2 \rceil \) vertices, respectively, and define
\[
 f(x) = \begin{cases} 
-1 & \text{if } x \in Q_1 \cup W_2 \setminus (\{v\} \cup Q_2), \\
1 & \text{otherwise.} 
\end{cases}
\]
As before, it follows that $f$ is a $k$-subdominating function with $\text{ag}(f) = 3$. Theorem 1 is proved. 

Note that $\gamma_{ks}(K_{1,n-1}) = 2k - n$ if $k > \frac{1}{2}n$. The bound established in Theorem 1 is sharp indeed.

3. An upper bound on the $k$-subdomination number for graphs

Conjecture 2 is shown in [4] to be false in the special case when $k = \lceil (n + 1)/2 \rceil$. The conjecture has yet to be settled when $\lceil (n + 1)/2 \rceil < k \leq n$. In this section, we prove the conjecture in the special case when $n - k + 1$ divides $k$. For this purpose, we shall need the following result.

**Theorem 2.** For any connected graph $G$ of order $n$ and any $k$ with $\frac{1}{2}n < k \leq n$,

$$\gamma_{ks}(G) \leq 2 \left\lfloor \frac{k}{n - k + 1} \right\rfloor (n - k + 1) - n.$$  

**Proof.** Among all partitions $\{A'_{11}, A'_{12}\}$ of $V(G)$ with $|A'_{11}| = k$ and $|A'_{12}| = n - k$, let $\{A_{11}, A_{12}\}$ be one such that the number of edges between $A_{11}$ and $A_{12}$ is minimum. Note that for any $u \in A_{11}$ and $v \in A_{12}$, if $uv \notin E(G)$, then

$$d_{A_{11}}(u) + d_{A_{12}}(v) \geq d_{A_{12}}(u) + d_{A_{11}}(v).$$  

(1)

And if $uv \in E(G)$, then

$$d_{A_{11}}(u) + d_{A_{12}}(v) \geq d_{A_{12}}(u) + d_{A_{11}}(v) - 2.$$  

(2)

Otherwise the exchange of $u$ and $v$ yields a partition with fewer edges between its parts.

If $d_{A_{11}}(u) \geq d_{A_{12}}(u)$ for each $u \in A_{11}$, we define

$$f(x) = \begin{cases} 1 & \text{if } x \in A_{11}, \\ -1 & \text{if } x \in A_{12}. \end{cases}$$

Then clearly $f$ is a $k$-subdominating function on $G$ with $\text{ag}(f) \leq 2k - n$. Thus we may assume there exists a vertex $u_1 \in A_{11}$ with $d_{A_{11}}(u_1) < d_{A_{12}}(u_1)$. Then for any $v \in A_{12}$, using (1) and (2), we have

$$d_{A_{12}}(v) > d_{A_{11}}(v) \quad \text{if } v \notin N(u_1),$$

$$d_{A_{12}}(v) \geq d_{A_{11}}(v) - 1 \quad \text{if } v \in N(u_1).$$

Among all partitions $\{A'_{21}, A'_{22}\}$ of $A_{11} - \{u_1\}$ with $|A'_{21}| = 2k - n - 1$ and $|A'_{22}| = n - k$, let $\{A_{21}, A_{22}\}$ be one such that the number of edges joining vertices in $A_{21}$ to vertices in $A_{22}$ is minimum. If $d_{A_{21}}(u) \geq d_{A_{22}}(u)$ for each $u \in A_{21}$, define

$$f(x) = \begin{cases} 1 & \text{if } x \in A_{21} \cup A_{12} \cup \{u_1\}, \\ -1 & \text{if } x \in A_{22}. \end{cases}$$
It is easily seen that $f$ is a $k$-subdominating function of $ag(f) \leq 2k - n$, hence $\gamma_{ks}(G) \leq 2k - n$. So we may assume there exists $u_2 \in A_{21}$ such that $d_{A_{21}}(u_2) < d_{A_{22}}(u_2)$. For any $v \in A_{22}$, by the choice of $\{A_{21}, A_{22}\}$, similarly we have

$$d_{A_{22}}(v) > d_{A_{21}}(v) \quad \text{if} \quad v \notin N(u_2),$$

$$d_{A_{22}}(v) \geq d_{A_{21}}(v) - 1 \quad \text{if} \quad v \in N(u_2).$$

For $A_{21} - \{u_2\}$, a similar argument shows that either $\gamma_{ks}(G) \leq 2k - n$ or there exists $u_i \in A_{1i}$, $i = 1, 2, \ldots, \lceil k/(n-k+1) \rceil$, such that $d_{A_{1i}}(u_i) < d_{A_{2i}}(u_i)$ and

$$d_{A_{22}}(v) > d_{A_{21}}(v) \quad \text{if} \quad v \notin N(u_i),$$

$$d_{A_{22}}(v) \geq d_{A_{21}}(v) - 1 \quad \text{if} \quad v \in N(u_i).$$

Define

$$f(u) = \begin{cases} 
1 & \text{if} \quad u \in A_{12} \cup A_{22} \cup \ldots \cup A_{\lceil k/(n-k+1) \rceil 2} \cup \{u_1, u_2, \ldots, u_{\lceil k/(n-k+1) \rceil}\}, \\
-1 & \text{otherwise}. \end{cases}$$

$f$ is a $k$-subdominating function on $G$ with

$$ag(f) \leq 2\lceil k/(n-k+1) \rceil(n-k+1) - n.$$

The proof of Theorem 2 is complete. \qed

**Corollary 1.** Let $G$ be a connected graph of order $n$ and $k$ an integer with $n/2 < k \leq n$. If $n - k + 1 | k$, then $\gamma_{ks}(G) \leq 2k - n$.

Thus Conjecture 2 is true if $n - k + 1 | k$.

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**References**


