DUAL GAMES ON COMBINATORIAL STRUCTURES: THE SHAPLEY AND BANZHAF VALUES

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Abstract

The intention of this article is to introduce a new structure, that we will call decreasing system, which generalize the well-known convex geometries introduced by (PH Edelman, RE Jamison (1985)). We will introduce these structures through the dualization another known structure, augmenting system (Bilbao, 2003). In this article we axiomatize the Shapley and Banzhaf values on this new structure.

Key words: Combinatorial structures, dual games, Shapley and Banzhaf values.

Introduction

The paper is devoted to introduce a new combinatorial structure which it is dual of augmenting systems and to determine the Shapley and Banzhaf value via an isomorphims beetwen both structures.

In this framework of the cooperative games only certain coalitions are allowed to form. We consider allowable coalitions using the theory of augmenting systems, a notion derived from combinational abstract theory and introduced by Bilbao (2003). The first model in which the feasible coalitions are defined by the connected subgraphs of a graph is introduced by Myerson (1997). Contributions on graph-restricted games include Owen (1986), Borm, Owen and Tijs (1992) and Hamiache (1999). In these models the possibilities of coalition formation are determined by a communication graph between the players. Another type of combinatorial structure introduced by Gilles, Owen and van den Brink (1997) is equivalent to a subclass of antimatroids. This line of research focuses on the possibilities of coalitons formation determined by the positions of the players in the permission structures.
This paper is organized as follows. In section 1, augmenting systems and decreasing systems, such as their properties, are treated. We interpret their properties in the framework of partial cooperation and explain the duality relation between them. In section 2, this duality is translated to vector spaces of corresponding games. The Shapley value are studied in sections 3. In Section 4 we axiomatize firsts the Banzhaf value for an augmenting systems, and below, the same value for a decreasing system, again using duality.

1 Augmenting systems and decreasing systems

Antimatroids were introduced by Dilworth [4] as particular examples of semi-modular lattices. Since then, several authors have obtained the same concept by abstracting various combinatorial situations (see Korte, Lovasz, and Schrader [9]). In this section, a general cooperation structure is introduced, which is a weakening of the antimatroid structure.

Let \( N \) be a finite set. A set system over \( N \) is a pair \((N, \mathcal{F})\) where \( \mathcal{F} \subseteq 2^N \) is a family of subsets. The sets belonging to \( \mathcal{F} \) are called feasibles. We will write \( S \subseteq_i T \) instead of \( S \subseteq \{i\} \) and \( S \supset \{i\} \), respectively. For all \( S \in \mathcal{F} \), we denote \( au(S) = \{i \in N \setminus S, \text{ such that } S \cup i \in \mathcal{F}\} \) and \( ext(S) = \{i \in S, \text{ such that } S \setminus i \in \mathcal{F}\} \).

**Definition 1** A set system \((N, \mathcal{A})\) is an antimatroid if

A1. \( \emptyset \in \mathcal{A} \)
A2. for \( S, T \in \mathcal{A} \), we have \( S \cup T \in \mathcal{A} \)
A3. for \( S \in \mathcal{A} \) with \( S \neq \emptyset \), there exists \( i \in S \) such that \( S \setminus i \in \mathcal{A} \).

The definition of antimatroid implies the following augmentation property: If \( S, T \in \mathcal{A} \) with \( |T| > |S| \), then there exists \( i \in T \setminus S \) such that \( S \cup i \in \mathcal{A} \) where \( |T| \) means the cardinal of set \( T \).

We call a set system \((N, \mathcal{F})\) normal if \( N = \bigcup_{S \in \mathcal{F}} S \).

**Definition 2** A normal augmenting system is a normal set system \((N, \mathcal{F})\) with the following properties:

P1. \( \emptyset \in \mathcal{F} \),
P2. for \( S, T \in \mathcal{F} \) with \( S \cap T \neq \emptyset \), we have \( S \cup T \in \mathcal{F} \),
P3. for \( S, T \in \mathcal{F} \) with \( S \subset T \), there exists \( i \in T \setminus S \) such that \( S \cup i \in \mathcal{F} \).

**Remark 3** It follows from the definition that normal antimatroids are always augmenting systems.
Proposition 4  An augmenting system $(N, \mathcal{F})$ is an antimatroid if and only if $\mathcal{F}$ is closed under union.

**PROOF.** The necessary condition follows from A2. Conversely, we only have to prove A3. Let $S \in \mathcal{F}$ with $S \neq \emptyset$. By property $P3$ there exists a chain of feasible subsets $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_{s-1} \subset S_s = S$ such that $S_k \in \mathcal{F}$ and $|S_k| = k$ for $0 \leq k \leq s$. Hence there exists an element $i \in S$ such that $S \setminus i = S_{s-1} \in \mathcal{F}$.

**Definition 5** Let $(N, \mathcal{F})$ be an augmenting system, we call filter of $\{i\} \in \mathcal{F}$ to set $\mathcal{F}_i = \{ S \in \mathcal{F} \text{ such that } i \in S \}$

**Proposition 6** Let $(N, \mathcal{F})$ be an augmenting system. Then their filters are antimatroids.

**PROOF.** To prove A2, let $S, T \in \mathcal{F}_i$, then $S \cap T \neq \emptyset$ and by property $P2$ we have that $S \cup T \in \mathcal{F}_i$. A3 is consequence of $P3$.

**Example 7** The following collections of subsets of $N = \{1, \ldots, n\}$, given by $\mathcal{F} = 2^N$ and $\mathcal{F} = \{\emptyset, \{1\}, \ldots, \{n\}\}$, are the maximum augmenting system and a minimal augmenting system over $N$, respectively.

**Example 8** Bilbao [2] In a communication graph $\mathcal{G} = (N, \mathcal{E})$, the set system $(N, \mathcal{F})$ given by $\mathcal{F} = \{ S \subseteq N : (S, E(S)) \text{ is a connected subgraph of } \mathcal{G} \}$ is an augmenting system.

Convex geometries were introduced by Edelman [7] as a combinatorial abstraction of convex sets. Let $N$ be a finite set. A set system $(N, \mathcal{G})$ is a convex geometry if it satisfies the following properties:

**Definition 9** A set system $(N, \mathcal{G})$ is a convex geometry if

G1. $\emptyset \in \mathcal{G}$
G2. for $S, T \in \mathcal{G}$, we have $S \cap T \in \mathcal{G}$
G3. for $S \in \mathcal{G}$ with $S \neq N$, there exists $i \in N \setminus S$ such that $S \cup i \in \mathcal{G}$.

We will introduce a new combinatorial structure as follows.

**Definition 10** A decreasing system is a set system $(N, \mathcal{D})$ with the following properties:

D1. $N \in \mathcal{D}$
D2. for $S, T \in \mathcal{D}$, with $S \cup T \neq N$, we have $S \cap T \in \mathcal{D}$
D3. for $S, T \in \mathcal{D}$ with $S \subset T$, there exists $i \in T \setminus S$ such that $T \setminus i \in \mathcal{D}$.

**Definition 11** A decreasing system $(N, \mathcal{D})$ is called d-normal if $\emptyset \in \mathcal{D}$.
The relationships between the combinatorial structures above mentioned is given in next proposition.

**Proposition 12** A d-normal decreasing system \((N, \mathcal{D})\) is a convex geometry if and only if \(\mathcal{D}\) is closed under intersection.

**Proof.** G2) property is involved in the D2) property. Let now \(A \in \mathcal{D}, A \subseteq N\). Then for D3) it exist a chain \(i_1, i_2, ..., i_{|N\setminus A|}\) such that \(A \subset A \cup i_{|N\setminus A|} \subset ... \subset A \cup \{i_{|N\setminus A|}, ..., i_1\}\). Then G3) holds.

**Proposition 13** Let \((N, \mathcal{D})\) a d-normal decreasing system, then for every \(\{i\} \in \mathcal{D}\), the set system \((N \setminus i, \mathcal{D}_i)\) is a convex geometry where \(\mathcal{D}_i = \{S \in \mathcal{D}, i \notin S\}\)

**Proof.** Let \(\mathcal{D}_i = \{S \in \mathcal{D}, i \notin S\}\). Then \(N \setminus i, \emptyset \in \mathcal{D}_i\). To prove G2) let \(S, T \in \mathcal{D}_i\). Because \(S \cup T \neq N\) we have \(S \cap T \in \mathcal{D}\) then \(S \cap T \in \mathcal{D}_i\). The property G3) is a consequence of D3) applied to \(\mathcal{D}_i\).

**Example 14** The following collection of subsets of \(N = \{1, ..., n\}\), given by \(\mathcal{D} = 2^N, \mathcal{D} = \{N, N \setminus i\}, \mathcal{D} = \{N\} \cup \{S \subseteq N : |S| = n - 1\}\), are decreasing systems over \(N\).

**Example 15** In a communication graph \(G = (N, E)\), the set system \((N, \mathcal{D})\) given by \(\mathcal{D} = \{S \subseteq N : (N \setminus S, E(N \setminus S))\) is a connected subgraph of \(G\)\} is a decreasing system.

**Theorem 16** A family \(\mathcal{F}\) of subsets of \(N\), whit \(N \in \mathcal{F}\), is an augmenting system if and only if the system of its complements \(\mathcal{D}\) is a d-normal decreasing system.

**Proof.** We suppose that \((N, \mathcal{F})\) is a normal augmenting system, then \(N, \emptyset \in \mathcal{D}\). Let now \(S, T \in \mathcal{D}\) such that \(S \cup T \neq N\), then \((N \setminus S) \cap (N \setminus T) \neq \emptyset\) and for property A2) we have that \((N \setminus S) \cap (N \setminus T) \in \mathcal{F}\). This implies that \(S \cap T \in \mathcal{D}\). In order to prove D3) let \(S, T \in \mathcal{D}\) such that \(S \subseteq T\). Then \((N \setminus T) \subseteq (N \setminus S)\) and for property A3) it exists \(i \in (N \setminus S) \setminus (N \setminus T)\) such that \((N \setminus T) \cup \{i\} \in \mathcal{F}\). Then \(T \setminus i \in \mathcal{D}\). The reciprocal is identical.

**Example 17** Normal augmenting system and its dual decreasing system.
2 Dual games

Let \( \mathcal{F} \subseteq 2^N \) be a collection of feasible coalitions such that \( \emptyset \in \mathcal{F} \) and for each \( i \in N \) there exists \( S \in \mathcal{F} \) with \( i \in S \). We consider the vector space \( \Gamma(\mathcal{F}) \) of games \( v : \mathcal{F} \to \mathbb{R} \) on the combinatorial structure \( \mathcal{F} \). For every nonempty coalition \( T \in \mathcal{F} \), the unanimity game \( \varsigma_T \) and the identity game \( \delta_T \), are defined as follows:

\[
\varsigma_T(S) = \begin{cases} 
1 & \text{if } T \subseteq S \\
0 & \text{otherwise}
\end{cases}, \quad \delta_T(S) = \begin{cases} 
1 & \text{if } T = S \\
0 & \text{otherwise}
\end{cases}
\]

for all \( S \in \mathcal{F} \).

**Remark 18** The collections \( \{ \varsigma_T : T \in \mathcal{F}\backslash\{\emptyset\} \} \) and \( \{ \delta_T : T \in \mathcal{F}\backslash\{\emptyset\} \} \) are two bases of the vector space \( \Gamma(\mathcal{F}) \).

In this section we construct a dual operator between the sets \( \Gamma(\mathcal{F}) \) and \( \Gamma(\mathcal{D}) \), that is \( \Phi : \Gamma(\mathcal{F}) \to \Gamma(\mathcal{D}) \), where \( \mathcal{F} \) is a normal augmenting system and \( \mathcal{D} \) is its dual \( d \)-decreasing system. For all \( v \in \Gamma(\mathcal{F}) \), we have \( \Phi(v) = v^* \), where the dual game \( v^* \) of \( v \) is defined, for each \( S \in \mathcal{F} \), by

\[
v^*(S) = v(N) - v(N\backslash S)
\]

Since the families \( \Gamma(\mathcal{F}) \) and \( \Gamma(\mathcal{D}) \) have the same cardinal, we deduce from the above remark that the vector spaces \( \Gamma(\mathcal{F}) \) and \( \Gamma(\mathcal{D}) \) have the same dimension. Moreover, \( \Phi \) is an isomorphism such that \( (v^*)^* = v \), for all \( v \in \Gamma(\mathcal{F}) \).

**Proposition 19** Let \( \mathcal{F} \) be an augmenting system, whit \( N \in \mathcal{F} \), and let \( \mathcal{D} \) be its dual \( d \)-decreasing system. For every \( S \in \mathcal{F} \) such that \( S \neq N \), it holds:

1. The dual of the unanimity game \( \varsigma_{N\backslash S} \in \Gamma(\mathcal{D}) \) is the game \( \mu_S \in \Gamma(\mathcal{F}) \) defined, for each \( T \in \mathcal{D} \), by

   \[
   \mu_S(T) = \begin{cases} 
0 & \text{if } T \subseteq S \\
1 & \text{otherwise}
\end{cases}
\]

2. The dual of the identity game \( \delta_{N\backslash S} \in \Gamma(\mathcal{D}) \), where \( S \neq \emptyset \), is the game \( \rho_S \in \Gamma(\mathcal{F}) \) defined, for each \( T \in \mathcal{F} \), by

   \[
   \rho_S(T) = \begin{cases} 
-1 & \text{if } T = S \\
0 & \text{otherwise}
\end{cases}
\]

3. The dual of the game \( \delta_N \in \Gamma(\mathcal{D}) \) is \( \rho_\emptyset \in \Gamma(\mathcal{F}) \) defined by \( \rho_\emptyset(T) = 1 \) for all nonempty \( T \in \mathcal{F} \).
PROOF. Let $T \in A$. The dual of $\zeta_{N \setminus S}$ is determined by

$$
\zeta^*_{N \setminus S}(T) = \zeta_{N \setminus S}(N) - \zeta_{N \setminus S}(N \setminus T) = 1 - \begin{cases} 
1, & \text{if } N \setminus T \supseteq N \setminus S \\
0, & \text{otherwise}
\end{cases}
$$

Thus, $\zeta^*_{N \setminus S}(T) = \mu_S(T)$. For the identity game $\delta_{N \setminus S}$, where $S \neq \emptyset$, we obtain

$$
\delta^*_{N \setminus S}(T) = \delta_{N \setminus S}(N) - \delta_{N \setminus S}(N \setminus T) = 1 - \begin{cases} 
1, & \text{if } N \setminus T = N \setminus S \\
0, & \text{otherwise}
\end{cases}
$$

Therefore $\delta^*_{N \setminus S}(T) = \rho_S(T)$. If $S = \emptyset$ then

$$
\delta^*_N(T) = \delta_N(N) - \delta_N(N \setminus T) = 1 - \begin{cases} 
1, & \text{if } N \setminus T = N \\
0, & \text{otherwise}
\end{cases}
$$

and hence $\delta^*_N(T) = \rho_\emptyset(T)$.

**Definition 20** A game $v : F \to \mathbb{R}$ is monotone if for all $S, T \in F$ with $S \subseteq T$, it holds $v(S) \leq v(T)$. Now, we introduce the concept of convexity for games on augmenting systems and decreasing systems.

**Definition 21** Let $F$ be an augmenting system. A game $v \in \Gamma(F)$ is said to be convex on $F$ if for all $S, T \in F$, such that $S \cup T \in F$ then

$$
v(S \cup T) + \sum_{C \in \pi^F(S \cap T)} v(C) \geq v(S) + v(T),
$$

where $\pi^F(S \cap T) = \{C \in F, \text{maximal feasibles, such that } C \subseteq S \cap T\}$. $v$ is called concave on an augmenting system if the reverse inequality holds.

**Definition 22** Let $D$ be a $d$-decreasing system. A game $v \in \Gamma(D)$ is said to be convex on $D$ if for all $S, T \in D$, such that $S \cup T \neq N$ then

$$
v(S \cap T) + \sum_{C \in \pi^D(S \cup T)} v(C) \geq v(S) + v(T),
$$

where $\pi^D(S \cup T) = \{C \in D, \text{maximal feasibles, such that } C \subseteq S \cup T\}$. $v$ is called concave on an augmenting system if the reverse inequality holds.

**Proposition 23** Let $D$ be a $d$-decreasing system and let $F$ be its dual augmenting system. Then:
1. $v \in \Gamma(\mathcal{D})$ is monotone if and only if $v^* \in \Gamma(\mathcal{F})$ is monotone.

2. $v \in \Gamma(\mathcal{D})$ is convex if and only if $v^* \in \Gamma(\mathcal{F})$ is concave.

**PROOF.** Let $v \in \Gamma(\mathcal{D})$.

1. We suppose that $v$ is monotone. If $S, T \in \mathcal{D}$, with $S \subseteq T$, then $N \setminus T \subseteq N \setminus S$ and we obtain $v(N \setminus T) \leq v(N \setminus S)$. Therefore, $v^*(S) = v(N) - v(N \setminus S) \leq v(N) - v(N \setminus T) = v^*(T)$.

2. We suppose that $v$ is convex. Thus, for any $S, T \in \mathcal{F}$, such that $S \cup T \in \mathcal{F}$, we have

$$v^*(S \cup T) + \sum_{C \in \pi(S \cap T)} v^*(C) = v(N) - v(N \setminus (S \cup T)) + \sum_{C \in \pi(S \cap T)} [v(N) - v(N \setminus C)] = v(N) - v(N \setminus (S \cup T)) + \sum_{N \setminus C \in \pi^D(N \setminus (S \cap T))} [v(N) - v(N \setminus C)] \geq v^*(S) + v^*(T)$$

where $\pi^D(N \setminus (S \cap T)) = \{\text{dual components of } (S \cap T)\}$

3 **The Shapley Value**

A group value on $\Gamma(\mathcal{F})$ is a function $\Psi : \Gamma(\mathcal{F}) \to \mathbb{R}^N$. Each coordinate of $\Psi$ is understood as the payment of the corresponding player. Let $\mathcal{F}$ be a normal augmenting system and let $\Psi = (\Psi_i)_{i \in N}$ be a group value on $\Gamma(\mathcal{F})$, where $\Psi_i : \Gamma(\mathcal{F}) \to R$ for all $i \in N$. We consider the following axioms:

(1) **Linearity axiom:** If $v_1, v_2 \in \Gamma(\mathcal{F})$ and $\alpha_1, \alpha_2 \in \mathbb{R}$, then

$$\Psi(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 \Psi(v_1) + \alpha_2 \Psi(v_2).$$

**Definition 24** Let $\mathcal{F}$ be a normal augmenting system. For any player $i \in N$, her marginal contribution to $S \in \mathcal{F}$ such that $S \cup i \in \mathcal{F}$ in the game $v \in \Gamma(\mathcal{F})$ is given by $v(S, i) = v(S \cup i) - v(S)$. If $\mathcal{D}$ is a $d$-decreasing system and $v \in \Gamma(\mathcal{D})$, the marginal contribution of $i$ to $S \in \mathcal{D}$ such that $S \setminus i \in D$ is given by $v(S, i) = v(S) - v(S \setminus i)$. 7
Let $\mathcal{F}$ be an augmenting system with $N \in \mathcal{F}$ and let $D$ be its dual d-decreasing system. If $v \in \Gamma(\mathcal{F})$ and $i \in au(S)$ then:

$$v^* (N \setminus (S \cup i),i) = v^* ((N \setminus (S \cup i)) \cup i) - v^* ((N \setminus (S \cup i))) =$$

$$v(N) - v(S) - v(N) + v(S \cup i) =$$

$$v(S \cup i) - v(S) = v(S,i).$$

Then for each $S \in \mathcal{F}$ with $i \in au(S)$,

$$v(S,i) = v^* (N \setminus (S \cup i),i)$$

A player $i$ is dummy in the game $v \in \Gamma(\mathcal{F})$ if, for each $S \in \mathcal{F}$ such that $i \in au(S)$,

$$v(S,i) = \begin{cases} v(i), & \text{if } \{i\} \in \mathcal{F} \\ 0, & \text{otherwise} \end{cases}$$

2. **Dummy axiom:** If $i \in N$ is a dummy player in $v \in \Gamma(\mathcal{F})$ then

$$\Psi_i(v) = \begin{cases} v(i), & \text{if } \{i\} \in \mathcal{F} \\ 0, & \text{otherwise.} \end{cases}$$

3. **Efficiency axiom:** For all $v \in \Gamma(F)$,

$$\sum_{i \in N} \Psi_i(v) = v(N)$$

Let $T, S \in \mathcal{F}$ with $T \subseteq S$. The number of maximal chains for the interval $[T,S] = \{C \in F, T \subseteq C \subseteq S\}$, is denoted by $Ch[T,S]$. In the particular case of $T = \emptyset$, this number is $Ch(S)$, and $Ch(\mathcal{F})$ represent the total number of maximal chains in $F$.

4. **Chain axiom:** For each $S \in F$ and $i, j \in au(S)$,

$$Ch(S \cup i, N)\Psi_j(\delta_T) = Ch(S \cup j, N)\Psi_i(\delta_S)$$

The next theorem characterizes the Shapley value $\Phi^F$ on $\Gamma(\mathcal{F})$

**Theorem 25** Bilbao and Ordóñez [1]. The Shapley value is the unique value that satisfies linearity, dummy, efficiency and chain axioms. Moreover, for all $i \in N$ and each game $v \in \Gamma(\mathcal{F})$,

$$\Phi_i(v) = \sum_{\{S \in F: i \in S^*\}} \frac{Ch(S)Ch(S \cup i, N)}{Ch(N)} v(S,i).$$
We propose a new concept of dummy player on augmenting systems and decreasing systems.

**Definition 26** Let $\mathcal{F}$ be an normal augmenting system. A player $i \in N$ is a $c-$dummy player if for all $S, T \in \mathcal{F}$ with $i \in au(S) \cap au(T)$, we have $v(S, i) = v(T, i)$.

If $i$ is a $c$-dummy player then his marginal contributions are equal, and we denote this value by $v(i)$. In particular, when $\{i\} \in \mathcal{F}$ and $i$ is a $c$-dummy player we have $v(i) = v(i)$. So, we write the following axiom for a value $\Psi$ on $\Gamma(\mathcal{F})$.

**a-dummy axiom:** If $i \in N$ is a $c-$dummy player in the game $v \in \Gamma(\mathcal{F})$ then

$$\Psi_i(v) = v(i)$$

In a normal augmenting system, a player is $c-$dummy if all the marginal contributions are equal. The Shapley value obtained by Bilbao and Ordóñez [1] satisfies the $a$-dummy axiom for games on normal augmenting systems.

**Proposition 27** Let $\mathcal{F}$ be an augmenting system. A group value $\Psi$ on $\Gamma(\mathcal{F})$ satisfies linearity, $c$-dummy, efficiency and chain axioms if and only if $\Psi = \Phi^F$.

**Proof.** We only need to prove that the $c$-dummy axiom implies the dummy axiom. If $i$ is a dummy player then

$$v(i) = \begin{cases} v(i) & \text{if } \{i\} \in \mathcal{F} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, using the $c$-dummy axiom, $\Psi_i(v) = v(i)$.

**Proposition 28** Let $\mathcal{F}$ be a normal augmenting system and let $D$ be its dual $d$-decreasing system. For any group value $\Psi^F : \Gamma(\mathcal{F}) \rightarrow R^N$ we define the group value on $\Gamma(D)$ by $\Psi^D = \Psi^F \circ \Theta$. Then the following statements hold.

1. $\Psi^F$ satisfies the linearity axiom if and only if $\Psi^D$ satisfies the linearity axiom.
2. $\Psi^F$ satisfies the dummy player axiom if and only if $\Psi^D$ satisfies the dummy player axiom.
3. $\Psi^F$ satisfies the efficiency axiom if and only if $\Psi^D$ satisfies the efficiency axiom.
PROOF. 1. Since $\Theta$ is an isomorphism and $\Psi^F$ is lineal then $\Psi^D$ is lineal too.

2. In fact, we claim that if $i$ is a dummy player in $v \in \Gamma(F)$ then $i$ is a dummy player in $v^* \in \Gamma(D)$. Let $S,T \in D$ and $i \in ex(S) \cap ex(T)$. Thus, by duality, we obtain $i \in au(N \backslash S) \cap au(N \backslash T)$ where $N \backslash S$ and $N \backslash T \in A$. Then

$$v^*(S,i) = v^*(S) - v^*(S \backslash i) = v((N \backslash S) \cup i) - v(N \backslash S) = v((N \backslash T) \cup i) - v(N \backslash T) = v^*(T,i)$$

because $i$ is dummy in $v$. Moreover $v^* \langle i \rangle = v \langle i \rangle$. Finally, if $i$ is a dummy player in $v$ then

$$\Theta^F(v) = \Theta^D(v^*) = v^* \langle i \rangle = v \langle i \rangle$$

3. For each $v \in \Gamma(F)$ we have that $v^*(N) = v(N) - v(\emptyset) = v(N)$. If $\Psi^F$ is efficient then $\Psi^F(\omega) = \omega(N)$ for all $\omega \in \Gamma(F)$. Hence, we have, for all $v \in \Gamma(F)$,

$$\Theta^F(v) = \Theta^D(v^*) = v^*(N) = v(N)$$

As $(\ast)$ is an involutive operator we have the reverse implications.

It remains to analyze as the chain axiom is translated to decreasing system.

**Proposition 29** Let $F$ a normal augmenting system and let $D$ its dual decreasing system. Then $\Theta^F$ is a group value on $\Gamma(F)$ satisfying the chain axiom if and only if the value $\Theta^D = \Theta^F \circ \Theta$ satisfies that, for all $S \in D$, $S \neq \emptyset$, and $i, j \in ext(S)$,

$$Ch([S \backslash i,N]) \Theta^D_j(\delta_S) = Ch([S \backslash j,N]) \Theta^D_i(\delta_S)$$

**PROOF.** Let $S \in D$, $S \neq \emptyset$. We say that $\rho_{N \backslash S}^* = \delta_S$ where $N \backslash S \in F$ and $N \backslash S \neq N$. Therefore $\Theta^D_i(\delta_S) = \Theta^F_i(\rho_{N \backslash S}^*)$. If $i, j \in ext(S)$ then $i, j \in au(N \backslash S)$ and we can substitute these numbers in the chain axiom.

The other implication is due to $(\ast)$ is an involutive operator.

The above proposition allows us to define a new axiom for games on decreasing system

**D-chain axiom:** for all $S \in D$, $S \neq \emptyset$, and $i, j \in ext(S)$,

$$Ch([S \backslash i,N]) \Theta^D_j(\delta_S) = Ch([S \backslash j,N]) \Theta^D_i(\delta_S)$$

The last axiom can be interpreted as follows: the player that obtain positive marginal contributions in the game $\delta_S$ are those that are meet to $S$ immediately.
and their payments are in proportion to the number of feasible orders. The above axioms characterizes the Shapley value on a d-decreasing systems.

**Theorem 30** Let $D$ a d-decreasing system. There exists a unique group value $\Phi^D$ on $\Gamma(D)$ that satisfies linearity, dummy, efficiency and $D$-chain axioms. Moreover, for all $i \in N$, and each game $v \in \Gamma(D)$,

$$\Phi^D_i(v) = \sum_{\{S \in D : i \in \text{ext}(S)\}} \frac{\text{Ch}[S \setminus i] \text{Ch}[S, N]}{\text{Ch}[N]} v(S, i)$$

This value is called the Shapley value of $v \in \Gamma(D)$

**PROOF.** We first prove the existence and uniqueness of $\Phi^D$. The value $\Phi^D = \Phi^F \circ \Theta$, where $\Phi^F$ is the Shapley value on $\Gamma(F)$, is the only value that satisfies linearity, dummy, efficiency and $F$-chain axiom.

We will get now the formula that represents the value $\Phi^D$. If $v \in \Gamma(D)$ then

$$\Phi^D_i(v) = (\Phi^F_i \circ \Theta)(v) = \Phi^F_i(v^*)$$

Then for theorem (25),

$$\Phi^D_i(v) = \sum_{\{T \in F : i \in \text{au}(T)\}} \frac{\text{Ch}(T) \text{Ch}(T \cup i, N)}{\text{Ch}(N)} v(T, i) =$$

$$\cdot \sum_{\{T \in F : i \in \text{au}(T)\}} \frac{\text{Ch}(T) \text{Ch}(T \cup i, N)}{\text{Ch}(N)} v(N \setminus (T \cup i), i)$$

Note that there exists an isomorphism between the collections $\{T \in F : i \in \text{au}(T)\}$ and $\{S \in D : i \in \text{ext}(S)\}$, identifying with $T = N \setminus S$. Hence,

$$\Phi^D_i(v) = \sum_{\{S \in F : i \in \text{ext}(S)\}} \frac{\text{Ch}(S \setminus i) \text{Ch}(S, N)}{\text{Ch}(N)} v(S, i)$$

because the isomorphisms between the maximal chain in $[\emptyset, T]$ and the maximal chains in $[N \setminus T, N]$.

### 4 The Banzhaf value

The classical Banzhaf value has been defined by Owen (1975) for the player $i \in N$ as

$$\beta_i(N, v) = \frac{1}{2^{n-1}} \sum_{\{S \in 2^{N \setminus i} : S \neq S\}} [v(S \cup i) - v(S)]$$

where $n = |N|$
First we axiomatize the Banzhaf value for augmenting systems. We suppose that \((N, v, \mathcal{F})\) is an augmenting system. We will assume throughout that 
\(N = \bigcup_{S \in \mathcal{F}} S\). The property \(P3\) implies that for each \(i \in N\) there exists a feasible set \(S \in \mathcal{F}\) such that \(i \in S^*\).

**Definition 31** Let \(v : \mathcal{F} \rightarrow \mathbb{R}\) be a game on an augmenting system \((N, \mathcal{F})\). The Banzhaf value for the player \(i \in N\) is given by

\[
\beta_i(N, v, F) = \frac{1}{A(i)} \sum_{S \in \mathcal{F} : i \in au(S)} [v(S \cup i) - v(S)]
\]

where \(A(i) = |\{S \in \mathcal{F} : i \in au(S)\}|\)

Note that if \(\mathcal{F} = 2^N\) then

\[A(i) = |\{S \in \mathcal{F} : i \in au(S)\}| = |\{S \subseteq N : i \in au(S)\}| = |\{S \subseteq N : i \notin S\}| = 2^{n-1}.
\]

and our value coincides with the traditional Banzhaf value.

Let \(\Gamma(\mathcal{F})\) be the real vector space of the games on the augmenting system \(\mathcal{F} \subseteq 2^N\). We will follow the work of Weber (1988) to obtain an axiomatic development of the Banzhaf value for games on augmenting systems. For this, we consider the following game on \(\mathcal{F}\). For any \(T \in \mathcal{F}, T \neq \emptyset\), the identity game \(\delta_T : \mathcal{F} \rightarrow \mathbb{R}\) is defined by:

\[
\delta_T(S) := \begin{cases} 
1 & \text{if } S = T \\
0 & \text{otherwise}
\end{cases}
\]

Let \(\Phi : \Gamma(\mathcal{F}) \rightarrow \mathbb{R}^N\) a map such that \(\Phi(v) = (\Phi_1(v), \ldots, \Phi_1(v))\). The meaning of this function is to give the expected payoffs to the players of a game. We introduce several axioms that give rise to a unique function for games on augmenting systems. If \(\mathcal{F} = 2^N\) then this function is equal to the classical Banzhaf value. First, we consider the linearity property.

**Linearity axiom:** For all \(\alpha, \beta \in \mathbb{R}\), and \(v, w \in \Gamma(\mathcal{F})\) we have

\[
\Phi_i(\alpha v + \beta w) = \alpha \Phi_i(v) + \beta \Phi_i(w) \text{ for every } i \in N.
\]

**Theorem 32** Let \(\Phi_i : \Gamma(\mathcal{F}) \rightarrow \mathbb{R}^N\) a value for \(i\) which satisfies the linearity axiom. Then exists an unique ser of coefficients \(\{a^i_S : S \in \mathcal{F}, S \neq \emptyset\}\) such that

\[
\Phi_i(v) = \sum_{\{S \in \mathcal{F}, S \neq \emptyset\}} a^i_S v(S).
\]

for every \(v \in \Gamma(\mathcal{F})\).
PROOF. The collection \( \{ \delta_S : S \in \mathcal{F}, S \neq \emptyset \} \) is a basis of the vector space \( \Gamma(\mathcal{F}) \). Then each game \( v \in \Gamma(\mathcal{F}) \) satisfies \( v = \sum_{\{S \in \mathcal{F}, S \neq \emptyset\}} v(S) \delta_S \). Let \( a'_i = \Phi_i(\delta_S) \) for every \( i \in N \), and every nonempty \( S \in \mathcal{F} \). Applying the linearity axiom we obtain the result.

We will now introduce the concept of dummy player.

**Definition 33** Let \( v : \mathcal{F} \to \mathbb{R} \) be a game on an augmenting system \((N, \mathcal{F})\). The player \( i \in N \) is a dummy player for the game \( v \) if for all \( S \in \mathcal{F} \) such that \( i \in au(S) \), we have

\[
v(S \cup i) - v(S) = \begin{cases} v(\{i\}) & \text{if } \{i\} \in \mathcal{F} \\ 0 & \text{otherwise.} \end{cases}
\]

We need a preparatory lemma about some properties of the dummy player in the identity game.

**Lemma 34** Let \( v : \mathcal{F} \to \mathbb{R} \) be a game on an augmenting system \((N, \mathcal{F})\) and let a nonempty \( S \in \mathcal{F} \). Then:

(i) If \( i \in S \setminus ex(S) \) then \( i \) is dummy player in the identity game \( \delta_S \).

(ii) If \( i \notin S \cup au(S) \) then the player \( i \) is dummy in the identity game \( \delta_S \).

(iii) If \( i \in au(S) \) then the player \( i \) is dummy in the game \( \delta_S + \delta_{S \setminus i} \).

**PROOF.** The proof of this lema can be wanted in [1]

The following axiom gives the payoff received for a dummy player.

**Dummy axiom:** If the player \( i \in N \) is a dummy player for \( v \in \Gamma(\mathcal{F}) \), then

\[
\Phi_i(v) = \begin{cases} v(\{i\}) & \text{if } \{i\} \in \mathcal{F} \\ 0 & \text{otherwise.} \end{cases}
\]

Let us consider a dummy player \( i \in N \) in the game \( v \in \Gamma(\mathcal{F}) \). Note that

\[
\beta_i(N, v, F) = \begin{cases} \frac{1}{A(\emptyset)} \sum_{\{S \in F : i \in au(S)\}} v(\{i\}) & \text{if } \{i\} \in \mathcal{F} \\ 0 & \text{otherwise.} \end{cases}
\]

This implies that the Banzhaf value satisfies the dummy axiom.
Theorem 35  Let \( \Phi_i : \Gamma(\mathcal{F}) \rightarrow \mathbb{R} \) be a value for player \( i \in N \) that satisfies linearity and dummy axioms. Then, for every game \( v \in \Gamma(\mathcal{F}) \),
\[
\Phi_i(v) = \sum_{\{S \in F : i \in au(S)\}} a^i_{S \cup i} [v(S \cup i) - v(S)].
\]
Moreover, if \( \{i\} \in \mathcal{F} \) then
\[
\sum_{\{S \in F : i \in au(S)\}} a^i_{S \cup i} = 1.
\]

**PROOF.** The proof of this Theorem can be wanted in [1].

The following axiom gives an anonymity property with respect to its enlarged coalitions of the payoff received for a player.

**Augmenting axiom:** Let \( i \in N \). for each \( S, T \in \mathcal{F} \) such that \( i \in au(S) \cap au(T) \) it holds that \( \Phi_i(\delta_{S \cup i}) = \Phi_i(\delta_{T \cup i}) \).

Moreover, for every game \( v \in \Gamma(\mathcal{F}) \) and every \( S \in \mathcal{F} \) such that \( i \in au(S) \), we compute
\[
\beta_i(N, \delta_{S \cup i}, \mathcal{F}) = \frac{1}{A(i)} \sum_{R \in F : i \in au(R)} [\delta_{S \cup i}(R) - \delta_{S \cup i}(R)].
\]
Since \( i \in au(R) \) we have \( i \notin R \) and hence \( S \cup i \neq R \), which implies \( \delta_{S \cup i}(R) = 0 \). Notice also that \( \delta_{S \cup i}(R \cup i) = 1 \iff S = R \), and hence \( i \in au(R) \iff i \in au(R) \). Then
\[
\beta_i(N, \delta_{S \cup i}, \mathcal{F}) = \begin{cases} 
\frac{1}{A(i)} & \text{if } i \in au(S) \\
0 & \text{otherwise}
\end{cases}
\]
which implies the augmenting axiom.

By using the previous results we prove the following characterization of the Banzhaf value for games on augmenting systems defined by the connected subgraphs of a graph.

**Theorem 36**  Let \((N, \mathcal{F})\) be an augmenting system, the Banzhaf value is the unique value \( \Phi : \Gamma(\mathcal{F}) \rightarrow \mathbb{R}^n \) that satisfies linearity, dummy and augmenting axioms.

**PROOF.** We know that the Banzhaf value satisfies the three axioms. Conversely, let \( \Phi \) be a value that satisfies linearity, dummy and augmenting axioms. If follows by above theorems that for every game \( v \in \Gamma(\mathcal{F}) \) and every
player $i \in N$,
\[ \Phi_i(v) = \sum_{\{T \in \mathcal{F} : i \in au(T)\}} a^i_{S \cup i} [v(S \cup i) - v(S)] \]

It follow from Theorem (35) that for $\{i\} \in \mathcal{F}$, \[ \sum_{\{S \in \mathcal{F} : i \in au(S)\}} a^i_{S \cup i} = 1 \]. Now let $S, T \in \mathcal{F}$ such that $i \in au(S) \cap au(T)$. The augmenting axiom implies that $\Phi_i(\delta_{S \cup i}) = \Phi_i(\delta_{T \cup i})$, and hence $a^i_{S \cup i} = a^i_{T \cup i}$. As consequence
\[ \sum_{\{S \in \mathcal{F} : i \in au(S)\}} a^i_{S \cup i} = a^i_{S \cup i} |\{T \in F : i \in au(T)\}| = 1 \]

Then we obtain
\[ a^i_{S \cup i} = \frac{1}{|\{T \in F : i \in au(T)\}|} = \frac{1}{A(i)} \]
for all $S \in F$ and $i \in au(S)$. Therefore, $\Phi_i(v) = \beta_i(N, v, F)$ for all $v \in \Gamma(\mathcal{F})$ and every $i \in N$

We introduce the following axiom:

**Decreasing axiom:** Let $i \in N$. For each $S, T \in \mathcal{F}$ such that $i \in ext(T) \cap ext(S)$ its holds that $\Phi_i(\delta_{S \setminus i}) = \Phi_i(\delta_{T \setminus i})$ where $\delta_S$ is the identity game over $S$.

**Proposition 37** Let $F$ be a normal augmenting system and let $D$ be its dual decreasing system. For any group value $\Psi^F : \Gamma(\mathcal{F}) \rightarrow \mathbb{R}^N$ we define the group value on $\Gamma(\mathcal{D})$ by $\Psi^D = \Psi^F \circ \Theta$. Then the following statements hold.

1. $\Psi^F$ satisfies the linearity axiom if and only if $\Psi^D$ satisfies the linearity axiom.
2. $\Psi^F$ satisfies the dummy player axiom if and only if $\Psi^D$ satisfies the dummy player axiom.
3. $\Psi^F$ satisfies the augmenting axiom if and only if $\Psi^D$ satisfies the decreasing axiom.

**PROOF.** 1. and 2. This proof is similar to the equivalente obtained in [1]. By proposition (37) we have that $\Psi^D(\delta_{S \setminus i}) = \Psi^F(\delta^*_S) = \Psi^F(\rho_{N \setminus S \cup i}) = \Psi^F(\rho_{N \setminus T \cup i}) = \Psi^D(\delta_{T \setminus i})$

**Theorem 38** Let $D$ a d-decreasing system. There exists a unique group value $\Phi^D$ on $\Gamma(\mathcal{D})$ that satisfies linearity, dummy and decreasing axioms. Moreover, for all $i \in N$, and each game $v \in \Gamma(\mathcal{D})$,
\[ \Phi^D_i(v) = \frac{1}{A^1(i)} \sum_{\{S \in \mathcal{F} : i \in ext(S)\}} [v(S) - v(S \setminus i)] \]
where \( A'(i) = |\{ S \in D : i \in ext(S)\}| \). This value is called the Banzhaf value of \( v \in \Gamma(D) \).

**Proof.** We first prove the existence and uniqueness of \( \Phi^D \). The value \( \Phi^D = \Phi^F \circ \Theta \), where \( \Phi^F \) is the Banzhaf value on \( \Gamma(F) \), is the only value that satisfies linearity, dummy and augmenting axioms. We will get now the formula that represents the value \( \Phi^D \). If \( v \in \Gamma(D) \) then

\[
\Phi^D_i(v) = (\Phi^F_i \circ \Theta)(v) = \Phi^F_i(v^*)
\]

Then for theorem (35),

\[
\Phi^D_i(v) = \frac{1}{A(i)} \sum_{\{ T \in F : i \in au(T) \}} v^* \langle T, i \rangle = \frac{1}{A(i)} \sum_{\{ T \in F : i \in au(T) \}} v \langle N \setminus (T \cup i) , i \rangle
\]

Note that the numbers \( A(i) \) on \( F \) and \( A'(i) \) on \( D \) are equals. Then

\[
\Phi^D_i(v) = \frac{1}{A'(i)} \sum_{\{ S \in D : i \in ext(S) \}} v \langle S, i \rangle
\]

**References**


