

ON SOLUTIONS OF SECOND ORDER COMPLEX DIFFERENTIAL EQUATIONS

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Abstract: In this paper, we establish transcendental entire function $A(z)$ and polynomial $B(z)$ such that the differential equation $f'' + A(z)f' + B(z)f = 0$, has all non-trivial solution of infinite order. We use the notion of *critical rays* of the function $e^{P(z)}$, where $A(z) = d(z)e^{P(z)}$ with some restrictions.

1. Introduction. Consider a second order linear differential equation of the form

$$f'' + A(z)f' + B(z)f = 0; \quad B(z) \not\equiv 0 \quad (1)$$

where $A(z)$ and $B(z)$ are entire functions. It is well known that all solutions of the equation (1) consist of entire functions (Hille, 1969 and Laine, 1993). For an entire function f , we define the order, lower order and exponent of convergence of zeros of f , respectively, as follows

$$\begin{aligned} \rho(f) &= \limsup_{r \rightarrow \infty} \frac{\log^+ \log^+ M(r, f)}{\log r}, \\ \mu(f) &= \liminf_{r \rightarrow \infty} \frac{\log^+ \log^+ M(r, f)}{\log r} \\ \text{and } \lambda(f) &= \limsup_{r \rightarrow \infty} \frac{\log n\left(r, \frac{1}{f}\right)}{\log r} \end{aligned}$$

where $M(r, f) = \max\{|f(z)| : |z| = r\}$ is maximum modulus of f and $n\left(r, \frac{1}{f}\right)$ is the number of zeros of f in the disk $|z| \leq r$. Using Wiman -Valiron theory, it is proved that the equation (1) has all solutions of finite order if and only if $A(z)$ and $B(z)$ are polynomials (Laine, 1993). Therefore, if either $A(z)$ or $B(z)$ is transcendental entire function then almost all solutions of equation (1) are of infinite order. Therefore, it is natural to find conditions on $A(z)$ and $B(z)$ such that all solutions of the equation (1) are of infinite order. Gundersen (1988) proved the following result :

THEOREM A. *A necessary condition for the equation (1) to have a non-trivial solution f of finite order is*

$$\rho(B) \leq \rho(A). \quad (2)$$

We illustrate it with some examples,

EXAMPLE 1. $f(z) = e^z$ satisfies $f'' + e^z f' - (e^z + 1)f = 0$, where $\rho(A) = \rho(B) = 1$.

EXAMPLE 2. With $A(z) = e^z$ and $B(z) = -1$ equation (1) has finite order solution $f(z) = 1 - e^{-z}$, where $\rho(B) < \rho(A)$.

Thus if $\rho(A) < \rho(B)$ then all solutions of the equation (1) are of infinite order. However, given necessary condition is not sufficient, for example

EXAMPLE 3. (Heittokangas, et. al. 2015) *If $A(z) = P(z)e^z + Q(z)e^{-z} + R(z)$, where P, Q, R are polynomials and $B(z)$ is an entire function with $\rho(B) < 1$ then $\rho(f)$ is infinite for all non-trivial solutions of the equation (1).*

Frei (1962) showed that the differential equation

$$f'' + e^{-z}f' + B(z)f = 0 \quad (3)$$

with $B(z)$ a polynomial, then the equation (3) has a non-trivial solution f of finite order if and only if $B(z) = -n^2$, $n \in \mathbb{N}$. Ozawa (1980) proved that the equation (3) with $B(z) = az + b$, $a \neq 0$ has all solutions of infinite order. Amemiya and Ozawa (1981) and Gundersen (1986) studied the equation (3) for $B(z)$ being particular polynomial. Langley (1986) proved that this is true for any non-constant polynomial. Gundersen (1988) proved the following result :

THEOREM 1. *Let f be a non-trivial solution of the equation (1) where either*

$$(1) \rho(B) < \rho(A) < \frac{1}{2} \text{ or}$$

$$(2) A(z) \text{ is transcendental entire function with } \rho(A) = 0 \text{ and } B(z) \text{ is a polynomial}$$

then $\rho(f)$ is infinite.

Hellerstein, Miles and Rossi (1991) proved Theorem 1 for $\rho(B) < \rho(A) = \frac{1}{2}$.

J.R. Long introduced the notion of the deficient value and Borel direction into the studies of the equation (1). For the definition of deficient value, Borel direction and function extremal for Yang's inequality one may refer to (Yang, 1993).

In 2013, J.R. Long et. al. proved that if $A(z)$ is an entire function extremal for Yang’s inequality and $B(z)$ a transcendental entire function with $\rho(B) \neq \rho(A)$, then all solution of the equation (1) are of infinite order.

In 2018, J.R. Long et. al. replaced the condition $\rho(B) \neq \rho(A)$ with the condition that $B(z)$ is an entire function with Fabry gaps.

X.B. Wu et. al., (2015) proved that if $A(z)$ is a non-trivial solution of $w'' + Q(z)w = 0$, where $Q(z) = b_m z^m + \dots + b_0, b_m \neq 0$ and $B(z)$ be an entire function with $\mu(B) < \frac{1}{2} + \frac{1}{2(m+1)}$, then all solutions of equation (1) are of infinite order. J. R. Long (2018) replaced the condition $\mu(B) < \frac{1}{2} + \frac{1}{2(m+1)}$ with $B(z)$ being an entire function with Fabry gaps such that $\rho(B) \neq \rho(A)$.

Furthermore, J.R. Long et. al., (2018) gave partial proof of the question by Gundersen (2017) and proved the following theorem :

THEOREM B. *Let $A(z) = d(z)e^{P(z)}$, where $d(z)(\neq 0)$ is an entire function and $P(z) = a_n z^n + \dots + a_0$ is a polynomial of degree n such that $\rho(d) < n$. Let $B(z) = b_m z^m + \dots + b_0$ be a non-constant polynomial of degree m , then all non-trivial solutions of the equation (1) have infinite order if one of the following condition holds :*

- (1) $m + 2 < 2n$;
- (2) $m + 2 > 2n$ and $m + 2 \neq 2kn$ for all integers k ;
- (3) $m + 2 = 2n$ and $\frac{a_n^2}{b_m}$ is not a negative real.

Motivated by Theorem B, we consider entire function $A(z)$ and $B(z)$ such that $\rho(A) > n$ and $B(z)$ a polynomial. To state and prove our theorem we give some definitions and notations below :

DEFINITION 1. (Long, Shi, Wu and Zhang, 2018) *Let $P(z) = a_n z^n + \dots + a_0, a_n \neq 0$ and $\delta(P, \theta) = \text{Re}(a_n e^{in\theta})$. A ray $\gamma = re^{i\theta}$ is called critical ray of $e^{P(z)}$ if $\delta(P, \theta) = 0$.*

It can be easily seen that there are $2n$ different critical rays of $e^{P(z)}$ which divides the whole complex plane into $2n$ distinct sectors of equal length $\frac{\pi}{n}$. Also $\delta(P, \theta) > 0$ in n sectors and $\delta(P, \theta) < 0$ in remaining n sectors. We note that $\delta(P, \theta)$ is alternatively positive and negative in the $2n$ sectors.

We now fix some notations,

$$E^+ = \{ \theta \in [0, 2\pi] : \delta(P, \theta) \geq 0 \}$$

and

$$E^- = \{ \theta \in [0, 2\pi] : \delta(P, \theta) \leq 0 \}$$

Let $\alpha > 0$ and $\beta > 0$ be such that $0 \leq \alpha < \beta \leq 2\pi$ then

$$S(\alpha, \beta) = \{ z \in \mathbf{C} : \alpha < \arg z < \beta \}$$

We now recall the notion of an entire function to blow up and decay to zero exponentially (Wu, Long, Heittokangas and Qiu, 2015).

DEFINITION 2. *Let $A(z)$ be an entire function with order $\rho(A) \in (0, \infty)$. Then $A(z)$ blows up exponentially in $\overline{S}(\alpha, \beta)$ if for any $\theta \in S(\alpha, \beta)$ we get,*

$$\lim_{r \rightarrow \infty} \frac{\log \log |A(re^{i\theta})|}{\log r} = \rho(A).$$

We say $A(z)$ decays to zero exponentially in $\overline{S}(\alpha, \beta)$ if for any $\theta \in S(\alpha, \beta)$ we have

$$\lim_{r \rightarrow \infty} \frac{\log \log \left| \frac{1}{A(re^{i\theta})} \right|}{\log r} = \rho(A).$$

We illustrate these notions with an example :

EXAMPLE 4. *The function $f(z) = e^z$ has two critical rays namely $\frac{-\pi}{2}$ and $\frac{\pi}{2}$. It is easy to show that $f(z)$ blows up exponentially in $\overline{S}(\frac{-\pi}{2}, \frac{\pi}{2})$ and decays to zero exponentially in $\overline{S}(\frac{\pi}{2}, \frac{3\pi}{2})$.*

We are now able to state our main theorem :

THEOREM 2. *Consider a transcendental entire function $A(z) = d(z)e^{P(z)}$, where $P(z)$ is a non-constant polynomial of degree n and $\rho(d) > n$. Assume that $d(z)$ is bounded away from zero and exponentially blows up in E^+ and E^- respectively and $B(z)$ be a polynomial. Then all non-trivial solutions of the equation (1) are of infinite order.*

We present here some examples to show that the hypothesis of the Theorem 2 can not be relaxed :

EXAMPLE 5. *The function $f(z) = e^{-z} - 1$ satisfies $f'' + e^z f' - f = 0$. The coefficient $A(z)$ satisfies $\lambda(A) < \rho(A)$ and $B(z)$ is a constant polynomial.*

EXAMPLE 6. The differential equation $f'' + (1 + z + ze^z)f' + zf = 0$ possesses finite order solution $f(z) = e^{-z} + 1$, here $A(z) = 1 + z + ze^z = d(z)e^{P(z)}$ satisfies $\rho(A) = \lambda(A) = 1$ and $\lambda(A) = \rho(d) \geq n$, where n is degree of $P(z)$ and $n = 0$ or 1 . The fact that $\lambda(A) = 1$ follows from $\lambda(z + ze^z) = 1$ and $n\left(r, \frac{1}{1+z+ze^z}\right) = n\left(r, \frac{1}{z+ze^z}\right)$.

EXAMPLE 7. Suppose $f'' + ((z + 1)(e^z + 1) + 1)f' + (z + 1)f = 0$ with coefficient $A(z) = (z + 1)(e^z + 1) + 1 = d(z)e^{P(z)}$ satisfying $\lambda(A) = \rho(A) = 1$ and $\lambda(A) = \rho(d) \geq n$, where n is degree of $P(z)$ which is a polynomial of at most degree 1 . The finite order entire function $f(z) = e^{-z} + 1$ satisfies given differential equation.

EXAMPLE 8. Suppose $A(z) = d(z)e^{P(z)}$ where $d(z) = \exp z^2 + 1$ and $e^{P(z)} = e^z$ with $\lambda(A) = \rho(d) = 2$. The function $A(z)$ satisfies that $\rho(d) > n$ but it is not exponentially blows up in E^- of e^z . The differential equation

$$f'' + (e^{z^2} + 1)e^z f' + (e^{z^2+z} + e^z - 1)f = 0$$

possesses finite order solution $f(z) = e^{-z}$. The coefficients $A(z)$ and $B(z)$ both fails to satisfies the hypothesis of the theorem.

The paper is organised as follows: in Section 2, we have stated preliminary lemmas and proved some required results. In Section 3, we have proved Theorem 2.

2. Auxiliary Result. In this section, we present some known results. Next two lemmas are due to Gundersen which has been used extensively over the years.

LEMMA 1. (Gundersen, 1988) Let f be a transcendental meromorphic function with finite order and (k, j) be a pair of integers that satisfies $k > j \geq 0$. Then for $\epsilon > 0$ there exists a set $E \subset [0, 2\pi]$ with linear measure zero such that for $\theta \in [0, 2\pi) \setminus E$ there exists $R(\theta) > 1$ such that

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho(f)-1+\epsilon)} \tag{4}$$

for all $|z| > R(\theta)$ and $\arg z = \theta$.

LEMMA 2. (Gundersen, 1988) Let f be an analytic function on a ray $\gamma = re^{i\theta}$ and suppose that for some constant $\alpha > 1$ we have

$$\left| \frac{f'(z)}{f(z)} \right| = O(|z|^{-\alpha})$$

as $z \rightarrow \infty$ along $\arg z = \theta$. Then there exists a constant $c \neq 0$ such that

$$f(z) \rightarrow c$$

as $z \rightarrow \infty$ along $\arg z = \theta$.

We now prove a result which would be required for proving Theorem 2.

LEMMA 3. *Let $A(z) = d(z)e^{P(z)}$ be an entire function, where $P(z)$ is a non-constant polynomial of degree n and $d(z)$ satisfies the condition of Theorem 2. Then there exists a set $E \subset [0, 2\pi]$ of linear measure zero such that for $\epsilon > 0$ the following holds:*

(i) for $\theta \in E^+ \setminus E$ there exists $R(\theta) > 1$ such that

$$|A(re^{i\theta})| \geq \exp \{ (1 - \epsilon)\delta(P, \theta)r^n \}$$

for $r > R(\theta)$.

(ii) for $\theta \in E^- \setminus E$ there exists $R(\theta) > 1$ such that

$$|A(re^{i\theta})| \geq \exp \{ (1 - \epsilon)\delta(P, \theta)r^n \}$$

for $r > R(\theta)$.

Proof: Here $A(z) = d(z)e^{P(z)} = h(z)e^{a_n z^n}$, where $h(z)$ is also an entire function. Let $E = \{ \theta \in [0, 2\pi] : \delta(P, \theta) = 0 \}$. This means that E is set of critical rays of $e^{P(z)}$, which implies that E has linear measure zero. Let $\epsilon > 0$ then

(i) for $\delta(P, \theta) > 0$, $\exp(-\epsilon\delta(P, \theta)r^n) \rightarrow 0$ as $r \rightarrow \infty$. Thus there exists $R(\theta) > 1$ such that

$$\exp(-\epsilon\delta(P, \theta)r^n) \leq |h(re^{i\theta})| \tag{5}$$

for $r > R(\theta) > 1$. Now

$$|\exp(a_n(re^{i\theta})^n)| = \exp(\delta(P, \theta)r^n) \tag{6}$$

Thus using equations (5) and (6) we have

$$|A(re^{i\theta})| + |h(re^{i\theta})| |\exp(a_n(re^{i\theta})^n)| \geq \exp((1 - \epsilon)\delta(P, \theta)r^n)$$

for $\theta \in E^+ \setminus E$ and $r > R(\theta)$.

(ii) Since $d(z)$ blows up exponentially in E^- therefore, $h(z)$ also blows up exponentially in E^- . Let $\epsilon > 0$ and $\delta(P, \theta) < 0$. Then $\rho(\exp(-\epsilon a_n z^n)) = n < \rho(h)$. Thus, using definition 2, for any $\theta \in E^+ \setminus E$ there exists $R(\theta) > 1$ such that

$$|h(re^{i\theta})| \geq \exp(-\epsilon\delta(P, \theta)r^n) \tag{7}$$

for $r > R(\theta)$. Using equations (6) and (7) we obtain that

$$|A(re^{i\theta})| \geq \exp((1 - \epsilon)\delta(P, \theta)r^n)$$

for $r > R(\theta)$ and $\theta \in E^- \setminus E$.

3. Proof of Main Theorem. In this section we will establish Theorem 2 which is the main result of this paper.

Proof: Let us suppose that there exists a non-trivial solution f of the equation (1) such that $\rho(f) < \infty$. Then from Lemma 1, we have that there exist $E_1 \subset [0, 2\pi]$ of linear measure zero and $m > 0$ such that,

$$\left| \frac{f''(re^{i\theta})}{f(re^{i\theta})} \right| \leq r^m \tag{8}$$

for $\theta \in [0, 2\pi] \setminus E_1$ and $r > R(\theta)$. From part (i) of Lemma 3 we have

$$|A(re^{i\theta})| \geq \exp\left(\frac{1}{2}\delta(P, \theta)r^n\right) \tag{9}$$

for $\theta \in E^+ \setminus E_2$ and $r > R'(\theta)$ where E_2 is set of critical rays of $e^{P(z)}$ of linear measure zero. Using equations (1), (8) and (9), for $\theta \in E^+ \setminus (E_1 \cup E_2)$ we get,

$$\left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| \rightarrow 0$$

as $r \rightarrow \infty$. This implies that for $\theta \in E^+ \setminus (E_1 \cup E_2)$

$$\left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| = O\left(\frac{1}{r^2}\right) \tag{10}$$

as $r \rightarrow \infty$. From Lemma 2,

$$f(re^{i\theta}) \rightarrow a \tag{11}$$

as $r \rightarrow \infty$, for $\theta \in E^+ \setminus (E_1 \cup E_2)$, where a is a non-zero finite constant. Applying Maximum Modulus principle for the function f over the domain E^+ it can be concluded that f is bounded over E^+ . Now using Phragmén-Lindelöf principle,

$$f(re^{i\theta}) \rightarrow a \quad (12)$$

as $r \rightarrow \infty$, for $\theta \in E^+$. From part (ii) of Lemma 3 implies that

$$|A(re^{i\theta})| \geq \exp\left(-\frac{1}{2}\delta(P, \theta)r^n\right) \quad (13)$$

for $\theta \in E^- \setminus E_2$ and for $r > R''(\theta)$. Using equations (1), (8) and (13) we have,

$$\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| \rightarrow 0 \quad (14)$$

as $r \rightarrow \infty$, for $\theta \in E^- \setminus (E_1 \cup E_2)$. From here we can obtain equation (10) for $\theta \in E^- \setminus (E_1 \cup E_2)$. Again using Maximum Modulus principle for the function f over the domain E^- we get f is bounded over E^- . Which further using Phragmén-Lindelöf principle implies,

$$f(re^{i\theta}) \rightarrow b \quad (15)$$

as $r \rightarrow \infty$, for $\theta \in E^-$, where b is a non-zero finite constant. Again using Phragmén-Lindelöf principle we get,

$$f(re^{i\theta}) \rightarrow a \quad (16)$$

as $r \rightarrow \infty$, for all $\theta \in E^+ \cup E^-$, which is a contradiction to the Liouville's theorem.

In our discussion with Professor Gundersen, the following and crucial problem was suggested by him to strengthen the result of the paper. Although we have shown that hypothesis of the theorem are necessary but interestingly we could not find example of function $A(z)$ which satisfies the conditions of the theorem. So Gundersen's question is still valid.

PROBLEM 1. *Construct the function $A(z) = d(z)e^{P(z)}$, where $d(z)$ is an entire function and $P(z)$ is a non-constant polynomial, such that $A(z)$ satisfies the conditions of the Theorem 2.*

All our work is directed to prove the following problem :

PROBLEM 2. *If $A(z)$ is a transcendental entire function such that $\lambda(A) < \rho(A)$ and if $B(z)$ is a non-constant polynomial, then $\rho(f) = \infty$ for all non-trivial solutions f of the equation (1)*

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