Periodic Ordered Permutation Groups and Cyclic Orderings

MANFRED DROSTE

Institut für Algebra, Technische Universität Dresden,
01062 Dresden, Germany

MICHÈLE GIRAUDET

U.R.A. 753 Paris 7 CNRS/University of Le Mans,
32 rue de la Réunion, 75020 Paris, France

AND

DUGALD MACPHERSON

Department of Pure Mathematics, University of Leeds,
Leeds LS2 9JT, England

Received January 22, 1992

1. Introduction

In this paper, we investigate a class of ordered permutation groups known as periodic $o$-groups, and we obtain results on the normal subgroup structure of sufficiently nice periodic $o$-groups. We also investigate their connection with groups of automorphisms of cyclic orders.

Our motivation is the following. First, much work has been done on permutation groups of chains. Homogeneous linear orderings and certain normal subgroups of their automorphism groups have been used, e.g., for the construction of infinite simple torsion-free groups [Hi] or, in the theory of lattice-ordered groups ($l$-groups), for deriving various universal embedding results for arbitrary $l$-groups; see, e.g., [H1, D]. By a result of McCleary [MC1], each transitive primitive $l$-permutation group is of one of three possible types, one being periodic $l$-permutation groups. Results on their normal subgroups have been established by Holland [H2] and Hurd [Hu]; these we intend to sharpen here.

On the other hand, cyclic orders appeared in structure theorems of Cameron [Ca1] for permutation groups with certain transitivity properties, and of Adeleke and Neuman [AN] for infinite Jordan groups; see also
[M] for a survey. The automorphism groups of cyclic orderings are shown here to correspond closely to periodic \( o \)-groups, which generalizes an old result of Rieger [R] (cf. [F]) for cyclically ordered groups.

Let us introduce some notation. An \( o \)-permutation group is just a group of automorphisms of a linear order. Now let \((T, \leq)\) be a chain. The Dedekind-completion of \((T, \leq)\) is denoted by \((\bar{T}, \leq)\). We let \(A(T)\) denote the automorphism group of \((T, \leq)\). It also acts naturally on \((\bar{T}, \leq)\). Also, let

\[
B(T) = \{ g \in A(T) : \exists a < b \text{ in } T \forall t \in T : tg \neq t \Rightarrow a < t < b \}.
\]

This is a normal subgroup of \(A(T)\). Let \(G \leq A(T)\). Then \((G, T)\) is \(o\)-2-transitive, if \(T\) is infinite and for any \(a, b, c, d \in T\) with \(a < b\) and \(c < d\), there is \(g \in G\) with \(ag = c\) and \(bg = d\). We note that then \((T, \leq)\) is dense. Also, \((G, T)\) is said to be periodic (with period \(z\)), if there is \(z \in A(\bar{T})\) such that

(i) for some (or equivalently, any) \(a \in T\), \(a < az\), and \(\{az^n : n \in \mathbb{Z}\}\) is unbounded above and below in \(T\), and

(ii) \(\langle z \rangle = C_{A(\bar{T})}(G)\), the centralizer of \(G\) in \(A(\bar{T})\).

Note that this has no connection with the usual group-theoretic meaning of “periodic.” Periodic groups arise naturally in \(o\)-permutation group theory because of the following theorem of McCleary [MC1] (see also [G, Theorem 4A]): If \((G, T)\) is any transitive primitive \(l\)-permutation group, then either \((G, T)\) is \(o\)-2-transitive, or \((G, T)\) is isomorphic to a subgroup of \((\mathbb{R}, +)\) acting on itself by translation, or \((G, T)\) is periodic. (For unexplained notation and terminology, here and elsewhere, we refer to [G].)

Let \((G, T)\) be periodic with period \(z\). As usual, for \(x \in T\) put \(G_x := \{ g \in G : xg = x \}\). Let \(a \in T\), and put \(E_a := \{ g \mid (a, az) : g \in G_a \}\) (so \(E_a\) is just the group induced on \((a, az)\) by \(G_a\)). Then \((E_a, (a, az))\) is called the \(a\)-extract of \((G, T)\). Also, let \(B(E_a) := E_a \cap B((a, az))\). Put \(G^* := \{ g \in G : \text{for some } a, b \in T \text{ with } a < b, \ g \mid (a, b) = \text{id} \}\). In Section 2 we prove Theorem 1.1 below. It may be regarded as a generalization of a theorem of Hurd [Hu] which (in a special case) says that if \(z\) denotes addition by 1 on the reals, and \(G := C_{A(\mathbb{R})}(z)\), then any proper normal subgroup of \(G\) is contained in \(\langle z \rangle\).

**Theorem 1.1.** Let \((G, T)\) be a periodic \(o\)-permutation group with period \(z\). Assume that

(i) all extracts are \(o\)-2-transitive,

(ii) \(B(E_a) = [B(E_a), B(E_a)] \) for all \(a \in T\),

(iii) whenever \(a, b \in T\) with \(a < b < az\), then \(G_{ab} \leq \langle G^* \rangle\).
Then any proper normal subgroup of $G$ is contained in $\langle z \rangle$. In particular, $G$ is simple if and only if $G \cap \langle z \rangle = 1$.

We consider in Section 3 applications of Theorem 1.1 to natural examples. Among these are the automorphism groups of some $\omega$-categorical digraphs discussed in Cameron [Ca2] and in Section 2 of [DGMS], as well as certain groups of diffeomorphisms of $\mathbb{R}$. These examples indicate implicitly a close relationship between periodic $o$-permutation groups and groups of automorphisms of cyclic orders. This relationship is presented explicitly in Section 4. A cyclic ordering is a structure $(T, C)$ where $C$ is a ternary relation on $T$ and satisfies

(i) for all $a \in T$, if we define a binary relation $<_a$ on $T \setminus \{a\}$ by the rule: $x <_a y$ if and only if $C(a, x, y)$ holds, then $<_a$ is a strict total order,

(ii) for all $x, y, z \in T$, $C(x, y, z) \iff C(y, z, x) \iff C(z, x, y)$.

We also consider coloured cyclic orders (ccos) $(T, C, (P_i)_{i \in I})$ which are just cyclic orders equipped with unary predicates (colours) $(P_i : i \in I)$. Similarly, coloured total orders (ctos) are structures $\mathcal{F} = (T, \leq, (P_i)_{i \in I})$ where $(T, \leq)$ is a chain and the $P_i$ are unary predicates.

We consider in Section 4 a certain correspondence between certain groups of automorphisms of ctos and groups of automorphisms of ccos. Roughly, if $G$ acts on a cto $(T, \leq, (P_i)_{i \in I})$ and has a monotonic increasing central element $z$ such that for any $a \in T$, $\{az^n : n \in \mathbb{Z}\}$ is unbounded above and below in $(T, \leq)$ and $a < az$, then we may form a cco whose elements are the $\langle z \rangle$-orbits on $T$, with $G_i/\langle z \rangle$ acting naturally. The construction can also be reversed. This construction generalizes a theorem of Rieger [R] for cyclically ordered groups.

2. PERIODIC PERMUTATION GROUPS

Our goal in this section is to prove Theorem 1.1, which is stated in the Introduction. Throughout the section, $(G, T)$ denotes a periodic $o$-permutation group with period $z$. Note that if all extracts are $o$-2-transitive then $(T, \leq)$ is dense. We first state without proof an easy and well-known lemma. It shows that a periodic $o$-permutation group is determined by its extracts.

**Lemma 2.1.** Let $g \in G$ and $a \in T$, and let $\hat{g}$ denote the restriction of $g$ to $(a, az)$. Then for any $b \in T$, $bg = bz'\hat{g}z^{-t}$, where $t$ is the unique integer such that $bz' \in [a, az)$.
Lemma 2.2. Suppose that all extracts of \((G, T)\) are o-2-transitive and that whenever \(a, b \in T\) with \(a < b < az\) we have \(G_{ab} \leq \langle G^* \rangle\). Then \(G = \langle G^* \rangle\).

Proof. First we show that \(G = \langle G_a : a \in T \rangle\). Let \(g \in G\). We may assume that \(a < ag\) for some \(a \in T\). Clearly for some \(n \in \mathbb{N}\) there are \(a_i, b_i \in T\) \((i = 1, \ldots, n)\) such that \(a_1 = a\), \(a_n = ag\), and for \(i = 1, \ldots, n - 1\), \(b_i < a_i < a_{i+1} < b_i z\). Choose \(h_i \in G_{b_i}\) with \(a_i h_i = a_{i+1}\) \((i = 1, \ldots, n - 1)\), and put \(h = h_1 \cdots h_{n-1}\). Clearly \(ah = ag\), showing that \(g \in \langle G_a : a \in T \rangle\).

Next, let \(a \in T\) and \(g \in G_a\). Choose any \(b, c \in (a, az)\) such that \(b < c\) and \(b < cg\). By o-2-transitivity of the extracts, there is \(h \in G_{ab}\) with \(ch = cg\). Then \(gh^{-1} \in G_{ac}\), showing that \(g \in \langle G^* \rangle\), as required.

Note that the first part of the above argument shows that if all extracts of \((G, T)\) are transitive, then \((G, T)\) is transitive on \(T\). For the next lemma, recall that if \(H \leq G\) then \(H^G\) denotes the normal closure of \(H\) in \(G\).

Lemma 2.3. Suppose that all extracts of \((G, T)\) are o-2-transitive. Let \(g \in G \setminus \langle z \rangle\). Then for some \(a \in T\), \(G_a \cap \langle g \rangle^G \neq \{1\}\).

Proof. Clearly, replacing \(g\) by \(g^{-1}\) if necessary, we may assume that there are \(a \in T\) and \(n \in \mathbb{N}\) such that \(ag \in (az^n, az^{n+1})\). Choose \(b, c \in T\) such that \(az^{n-1} < b < az^n < c < ag < bz\). By assumption, there is \(h \in G_b\) which fixes \(ag\) such that \(ch < az^n\). Then \(az^n < az^n h^{-1} < az^{n+1}\), so \(k := ghg^{-1} h^{-1}\) satisfies \(ak = ah^{-1} < (a, az)\). Note that \(ak^2 < az^2\). We handle three cases separately.

Case 1. Assume that \(ak^2 < az\). Choose \(f \in G_a\) such that \(akf = ak\) and \(ak^2 f \neq ak^2\). Then \(f^{-1} k f k^{-1}\) lies in \(G_a \cap \langle g \rangle^G\) and moves \(ak\).

Case 2. Assume that \(az < ak^2 < az^2\). We have \(ak^2 < azk = az\), so \(a < ak^2 z < ak\) in \(T\). Choose \(d \in (a, ak^2 z^{-1})\) and \(f \in G_a\) with \(akf = ak\) and \(df > ak^2 z^{-1}\). Then \(f\) moves \(ak^2 z^{-1}\), hence, also \(ak^2\). Thus, \(f^{-1} k f^{-1}\) lies in \(G_a \cap \langle g \rangle^G\) and moves \(ak\).

Case 3. Assume that \(ak^2 = az\). Choose \(f \in G_a\) with \(ak < akf < az\), and put \(k' := f^{-1} k f\), \(k^* := k^{-1} k' \in \langle g \rangle^G\). Then \(akk* = ak' = akf > ak\) and \(ak' k^{-1} < ak^*\). Hence we have \(ak < akk* < ak(k^*)^2 = ak' k^* < a(k'^2) = azf = az < akz\) and now we may argue as in Case 1 for \(k* \in \langle g \rangle^G\) in the interval \((ak, akz)\) to obtain the result.

Lemma 2.4. (Higman [Hi]). Let \(H\) be a permutation group on a set \(\Omega\), and let \(K \leq H\). Assume that for any \(f, g \in K\) and \(h \in H \setminus \{1\}\) there is \(k \in H\) such that \(Ak \cap Akh = \emptyset\), where \(A := \text{supp}(f) \cup \text{supp}(f)\). Then \([K, K]\) is contained in every non-trivial normal subgroup of \(H\).
Proof (sketch). Given \( f, g \in K \) and \( h \in H \setminus \{1\} \), choose \( k \in H \) as indicated and put \( k_1 := k^{-1}f, k_2 := k^{-1}, k_3 := k^{-1}g, k_4 := k^{-1}fg \). Since \((g^{-1})^k\) and \( f^{k_h} \) commute, we obtain \([f, g] = (h^{-1})^{k_2} h^{k_3} (h^{-1})^{k_1} h^{k_4}\).

Remark. The assumptions of this lemma are satisfied if \((H, \Omega)\) is the automorphism group of a chain and is \(o\)-2-transitive, and \(K = H \cap B(\Omega)\). For given \( f, g \in K \) and \( h \in H \setminus \{1\} \), choose \( a \in \Omega \) with \( a \neq ah \) and \( b, c \in \Omega \) with \( \text{supp}(f) \cup \text{supp}(g) \subseteq [b, c] \). Then choose \( k \in H \) to map \([b, c]\) into \([a, ah]\) if \( a < ah \) and into \([ah, a]\) if \( ah < a \).

Now we recall Theorem 1.1 from the Introduction.

**Theorem 1.1.** Let \((G, T)\) be a periodic \(o\)-permutation group with period \( z \). Assume

(i) all extracts are \(o\)-2-transitive,

(ii) \( B(E_a) = [B(E_a), B(E_a)] \) for all \( a \in T \),

(iii) whenever \( a, b \in T \) with \( a < b < az \), then \( G_{ab} \leq \langle G^* \rangle \).

Then any proper normal subgroup of \( G \) is contained in \( \langle z \rangle \). In particular, \( G \) is simple if and only \( G \cap \langle z \rangle = 1 \).

**Proof of Theorem 1.1.** Let \( N \) be any normal subgroup of \( G \) which is not contained in \( \langle z \rangle \). We claim that \( N = G \). By Lemma 2.2 it suffices to show that \( G^* \leq N \). Let \( f \in G^* \). By Lemma 2.3 there are \( a \in T \) and \( k \in N_a \) with \( k \neq 1 \). Since \( G \) is transitive, we may assume that \( f \) fixes pointwise a non-empty interval whose interior contains \( a \). But then \( f|_{(a, az)} \in B(E_a) \) and \( B(E_a) = [B(E_a), B(E_a)] \) by assumption. Hence \( f|_{(a, az)} \) belongs, by Lemma 2.4 and the remark following it, to the normal subgroup generated by \( k|_{(a, az)} \) in \( E_a \). By Lemma 2.1 we get \( f \in \langle k \rangle^G \leq N \).

Let \((G, T)\) be as in Theorem 1.1. Then in general there is no finite number \( m \) such that for any \( f, g \in G \) with \( G = \langle g \rangle^G \) we can write \( f \) as a product of at most \( m \) conjugates of \( g \) or \( g^{-1} \)(for given \( m \in \mathbb{N} \), choose any \( g \in G_a \setminus \{1\} \) \( (a \in T) \) and \( f \in G \) such that \( az^m < af < az^{m+1} \); then \( f \) is not a product of \( \leq m \) conjugates of \( g \) or \( g^{-1} \)). However, we can obtain such a result for the factor group \( H := G / \langle z^n \rangle / \langle z^n \rangle \) under slight additional assumptions as follows.

**Corollary 2.5.** Let \((G, T)\) be a periodic \(o\)-permutation group with period \( z \). Assume that

(i) extracts are \(o\)-2-transitive,

(ii) for \( a \in T \), each element of \( B(E_a) \) is a commutator in \( B(E_a) \),

(iii) whenever \( a, b \in T \) with \( a < b < az \), \( G_{ab} \leq G^* \cdot G^* \).
Let \( n \in \mathbb{N} \) and \( H = G \cdot \langle z^n \rangle / \langle z^n \rangle \). Then \( H \) is simple, and there is \( m \in \mathbb{N} \) such that whenever \( g, h \in H \) with \( h \neq 1 \), \( g \) can be written as a product of at most \( m \) conjugates of \( h \) and \( h^{-1} \).

**Proof.** This follows by inspection of the proof of Theorem 1.1. Note that if \( h \in H \setminus \{1\} \) then \( h = f \langle z^n \rangle \), where \( f \in G \) can be chosen so that for all \( a \in T \), \( az^n < af \leq a \) or \( a \leq af < az^n \). Such an \( f \) can be written as a product of a fixed number (dependent on \( n \) but independent of \( f, h \)) of elements of \( \bigcup (G_a : a \in T) \).

### 3. Some Examples of Periodic \( o \)-Permutation Groups

We consider in this section some applications of Theorem 1.1. Our first lemma is useful for checking whether an \( o \)-permutation group is periodic.

**Lemma 3.1.** Let \((G, T)\) be an \( o \)-permutation group and \( z \in C_{A(T)}(G) \) be such that for some \( x \in T \), \( x < xz \), and \( \{xz^n : n \in \mathbb{Z}\} \) is unbounded above and below in \( \bar{T} \). Assume that for each \( x \in T \) the extracts \( (E_x, (x, xz)) \) are transitive and \( (x, xz) \) is infinite. Then \( C_{A(T)}(G) = \langle z \rangle \), so \((G, T)\) is periodic with period \( z \).

**Proof.** Let \( g \in C_{A(T)}(G) \setminus \{1\} \). Suppose there are \( a \in T \), \( n \in \mathbb{Z} \) such that \( ag \in (az^n, az^{n+1}) \). Choose \( b, c \in (a, az) \) with \( b < agz^{-n} < c \) and then \( h \in G_a \) with \( bh = c \). Then \( az^{-n} < agz^{-n} = ahgz^{-n} = agz^{-n} \), a contradiction. Hence for each \( x \in T \) there is \( n \in \mathbb{Z} \) with \( xg = xz^n \), and in particular we may suppose that \( ag = az^n \) with \( n \neq 0 \). Now \( \{ag^n : m \in \mathbb{Z}\} \) is unbounded above and below in \( T \), and it follows that \( xg = xz^n \) for each \( x \in T \), so \( g = z^n \in \langle z \rangle \).

Next, we consider a condition which is useful for showing that groups satisfy condition (ii) of Theorem 1.1 (in fact, condition (ii) of Corollary 2.5). It appears somewhat technical, but it is often easy to check (cf. Droste and Shortt [DS]).

Let \((G, T)\) be an \( o \)-permutation group, and let \( g \in G \cap B(T) \). We say that there is an \( \omega \)-patching for \( g \) in \( G \) if there are \( a_i \in T \) (i.e., in \( \mathbb{N} \)) with \( a_i < a_{i+1} \) for all \( i \in \mathbb{N} \), and \( h \in G \) such that \( (\text{supp}(g)) \) \( h' \subseteq (a_i, a_{i+1}) \) for all \( i \in \mathbb{N} \), and the mapping \( k : T \rightarrow T \), defined by

\[
xk = \begin{cases} xh^{-1}gh^i & \text{if } x \in (a_i, a_{i+1}) \\
x & \text{otherwise } (i \in \mathbb{N}, x \in T)
\end{cases}
\]

belongs to \( G \). The point here is that if \( k \in G \), then \( g = kh^{-1}k^{-1}h \), so \( g \) is a commutator in \( G \).
Our first examples involve the group \( \text{Diff}(\mathbb{R}) \) of all diffeomorphisms of \( \mathbb{R} \). Note that all elements of \( \text{Diff}(\mathbb{R}) \) are monotonic increasing or monotonic decreasing.

**Corollary 3.2.** Let \( z \in A(\mathbb{R}) \) be such that \( 0 < 0z \) and \( \{0z^n : n \in \mathbb{Z}\} \) is unbounded above and below in \( \mathbb{R} \), and let \( G := \{g \in \text{Diff}(\mathbb{R}) : gz = zg\} \). Then any proper normal subgroup \( N \) of \( G \) is contained in \( \langle z \rangle \). In particular, \( G \) is simple if and only if \( z^n \not\in \text{Diff}(\mathbb{R}) \) for each \( n \in \mathbb{N} \).

**Proof.** First note that \( G \leq A(\mathbb{R}) \). We check that \( (G, \mathbb{R}) \) satisfies the assumptions of Corollary 2.5. By Lemma 3.1, \( (G, \mathbb{R}) \) is periodic with period \( z \). A short calculation as in [DSH; proof of Corollary 1.2] shows that if \( a \in \mathbb{R} \) then any \( g \in B(E_a) \) has an \( \omega \)-patching in \( B(E_a) \). Hence, any \( g \in B(E_a) \) is a commutator in \( B(E_a) \). Now let \( a, b \in \mathbb{R} \) with \( a < b < az \), and let \( f \in G_{ab} \). Choose \( c, d \in \mathbb{R} \) with \( b < c < d < az \) and then \( g \in G_{ab} \) such that \( g \) coincides with \( f \) on \( [a, b] \) and fixes \( [c, d] \) pointwise. Then \( g, fg^{-1} \in G^* \), so \( f \in G^* \cdot G^* \). Now apply Corollary 2.5.

As a further consequence we obtain the following analogue of McCleary [MC2] (cf. Glass [G, Theorem 6.5]).

**Corollary 3.3.** Let \( z \in A(\mathbb{R}) \) be such that \( 0 < 0z \) and \( \{0z^n : n \in \mathbb{Z}\} \) is unbounded above and below in \( \mathbb{R} \), and let \( H := \{h \in \text{Diff}(\mathbb{R}) : \text{there is } n \in \mathbb{N} \setminus \{0\} \text{ with } hz^n = z^n h\} \). Then \( H \) is simple.

**Proof.** Let \( H_n := C_{\text{Diff}(\mathbb{R})}(z^n) (n \in \mathbb{N} \setminus \{0\}) \). Then \( H = \bigcup (H_n : n \in \mathbb{N} \setminus \{0\}) \). Let \( N \) be a non-trivial normal subgroup of \( H \). Let \( m \in \mathbb{N} \); we claim that \( H_m \leq N \), which clearly will yield the result. For some \( n \) there is \( f \in N \cap H_n \) with \( f \neq 1 \), and we may assume that \( m \) divides \( n \). If \( f \not\in \langle z^n \rangle \), we have \( N \cap H_n = H_n \) by Corollary 3.2. On the other hand, if \( f = z^n \) for some \( r \in \mathbb{Z} \setminus \{0\} \), we get \( f \in H_n \setminus \langle z^n \rangle \), so again \( H_n \leq H_n \setminus \langle z^n \rangle \). In any case, \( H_m \leq H_n \leq N \), as claimed.

We conclude this section with three further examples of applications of Theorem 1.1.

**Example 3.4.** Let \( z : \mathbb{R} \to \mathbb{R} \) denote addition of 1, and let \( G \) denote the group of all \( g \in A(\mathbb{R}) \) which commute with \( z \) and satisfy: for any \( a \in \mathbb{R} \), \( g \) coincides on \( [a, az] \) with a finitely piecewise linear order-preserving permutation of \( \mathbb{R} \). By a result of Chehata [Ch], the group of all bounded (finitely) piecewise linear order-preserving permutations of \( \mathbb{R} \) is simple. Hence \( (G, \mathbb{R}) \) satisfies the assumptions of Theorem 1.1 (for assumption (iii) is clear). However, no non-trivial element in \( B(E_a) (a \in \mathbb{R}) \) can have an \( \omega \)-patching.
EXAMPLE 3.5. Let $\Gamma$ be a non-trivial subgroup of the multiplicative group of the reals. We say that $h \in A(\mathbb{R})$ is locally $\Gamma$-linear if, for every non-empty open interval $I$, there are $a < b$ in $I$ and $\lambda \in \Gamma$, $\mu \in \mathbb{R}$ such that $xh = \lambda x + \mu$ for all $x \in [a, b]$. Let $H$ be the group of all locally $\Gamma$-linear elements of $A(\mathbb{R})$, let $\Omega$ denote addition by 1, and put $G := C_{H}(z)$. Then again each $g \in B(E_{a})$ has an $\omega$-patching in $B(E_{a})$ (for $a \in \mathbb{R}$), so $(G, \mathbb{R})$ satisfies the assumptions of Corollary 2.5. Similarly, we can define locally differentiable, locally $C^{k}$ ($1 \leq k \leq \infty$), etc., periodic $o$-permutation groups on $\mathbb{R}$, each satisfying the assumptions of Corollary 2.5.

EXAMPLE 3.6. Let $(\Omega, \leq)$ be a doubly homogeneous chain and assume that $z \in A(\overline{\Omega})$ is such that for $a \in \Omega$, $a < az$, and $\{az^n : n \in \mathbb{Z}\}$ is unbounded above and below in $\overline{\Omega}$. Suppose that either $I = \mathbb{Z}$ or $I = \{1, 2, ..., n - 1\}$ for some $n \in \mathbb{N}$. Let $\Omega_i$ ($i \in I$) be a system of pairwise disjoint subsets (“colours”) of $\overline{\Omega} \setminus \Omega$ satisfying the following two conditions:

(a) $\Omega_i = \Omega z^i$ for each $i \in I$, and $\Omega = \Omega_{n-1} z$ if $I = \{1, ..., n - 1\}$,

(b) the group $H$ of order-preserving permutations of $\Omega$ which preserve each $\Omega_i$ setwise acts $o$-2-transitively on $\Omega$.

Let $G = C_{A(\Omega)}(z)$. Then $G \subseteq H$, and $(G, \Omega)$ satisfies the assumptions of Corollary 2.5.

If in the situation of the last example, $\Omega$ is countable and $I = \{1, 2, ..., n - 1\}$, where $n \in \mathbb{N}$ with $n \geq 2$, then (as we will show in the proof below) the group $G/\langle z^n \rangle$ is isomorphic to the automorphism group of the digraph $S(n)$ described in [DGMS, Section 2]. For $n \geq 2$ the digraph $S(n)$ is built by distributing $\mathbb{R}_0$ points densely around the unit circle, no two making an angle at the centre of $2\pi/n$, and having an arc from $x$ to $y$ if $0 < \arg(x/y) < 2\pi/n$. It is shown in [C] that $S(3)$ is a homogeneous digraph.

THEOREM 3.7. Let $n \geq 2$. Then $\text{Aut}(S(n))$ is simple.

Proof. Fix a vertex $a_0$ in $S(n)$, and for each $i < n$ let $a_i := a_0 \phi^i$, where $\phi$ denotes rotation (clockwise) by $2\pi/n$ (so $a_i$ is in the Dedekind completion of the underlying cyclic order on $S(n)$). Let $C_i$ denote the chain in $S(n)$ from $a_i$ to $a_{i+1}$, and let $C := \bigcup \{C_i : 0 \leq i \leq n - 1\}$, with the ordering induced from the $C_i$ with $C_0 < \cdots < C_{n-1}$. Let $T := \mathbb{Z} \times C$, with the natural lexicographical ordering. There is an automorphism $z$ of $A(\overline{T})$ corresponding to $\phi$, i.e., to shifting by $2\pi/n$, and $z^n \in A(T)$. Now let $K := C_{A(T)}(z)$. Then $K$ is just the same as the group $G$ of the paragraph before the theorem, so $K/\langle z^n \rangle$ is simple. (To see the correspondence, put $\Omega := T$ and let $I := \{1, ..., n - 1\}$ and (for $i \in I$) let $\Omega_i$ be the image of $T$ under $z^i$.)
It is easily checked that there is a homomorphism from \( K \) to \( \text{Aut}(S(n)) \) with kernel \( \langle z^n \rangle \) and that this homomorphism is surjective, so \( \text{Aut}(S(n)) \) is simple.

**Remarks.** 1. The correspondence between the linear order and the cyclic order will be investigated explicitly in Section 4. This correspondence, applied to ctos as in Example 3.6, also enables us to build "\( S(n) \)-like" digraphs of uncountable cardinality with simple automorphism groups.

2. The structure \( S(n) \) has the small index property; that is, if \( H \) is a subgroup of \( \text{Aut}(S(n)) \) of index less than the continuum, then it contains the pointwise stabiliser of a finite set. We may suppose that \( H \) fixes \( a_0 \) (for we may replace \( H \) by a subgroup of \( H \) of countable index fixing \( a_0 \)) and acts faithfully on \( C_0 \). The group induced on \( C_0 \) is essentially that induced on the realisations of \( P_i \) by the automorphism group of \( \mathcal{G} := (\mathbb{Q}, \leq, P_1, \ldots, P_n) \), where all the \( P_i \) are dense in \((\mathbb{Q}, \leq)\). By a theorem of Truss [T], the structure \( \mathcal{G} \) has the small index property. It follows easily that \( S(n) \) has the small index property, noting that points in \( P_i \) correspond to points in \( C_i \).

### 4. A Generalisation of Rieger’s Theorem

We describe in this section a correspondence between certain groups of automorphisms of ctos and groups of automorphisms of ccos (see the introduction for definitions). We first describe the correspondence between the ctos and the ccos themselves. Let \( \mathcal{F} := (T, \leq, (P_i)_{i \in I}) \) denote a cto, where, as usual, the \( P_i \) are unary predicates and suppose that there is \( z \in \text{Aut}(\mathcal{F}) \) (the "looping" element) such that for all \( a \in T \), \( a < az \), and \( \{az^n : n \in \mathbb{Z} \} \) is unbounded above and below in \((T, \leq)\). For each \( a \in T \), let \( \bar{a} := \{az^n : n \in \mathbb{Z} \} \). Form a structure \( \mathcal{F}/\langle z \rangle \) with domain \( \{\bar{a} : a \in T\} \), unary predicates \( \bar{P}_i \) \((i \in I)\), and a ternary relation \( C \) as follows: for \( \bar{a} \in T/\langle z \rangle \), \( \bar{P}_i \bar{a} \) holds if and only if for some (equivalently, all) \( a \in \bar{a} \), \( P_i a \) holds. For \( \bar{a}, \bar{b}, \bar{c} \in T/\langle z \rangle \), \( C(\bar{a}, \bar{b}, \bar{c}) \) holds if and only if there are \( a \in \bar{a} \), \( b \in \bar{b} \), \( c \in \bar{c} \) such that \( a < b < c < az \). With this definition, \( \mathcal{F}/\langle z \rangle = (T/\langle z \rangle, C, (\bar{P}_i)_{i \in I}) \) is clearly a cco.

Suppose now \( \mathcal{F} = (T, \leq, (P_i)_{i \in I}) \) is a cto and \( H \leq \text{Aut}(\mathcal{F}) \). Then an element \( z \in H \) is called a *looping element for \( H \) over \( T \) if it is a looping element of \( A(T) \) and lies in the centre of \( H \). Note that if \( I = \emptyset \) and \( H \) is a periodic \( o \)-permutation group with period \( w \) and \( w^n \in H \) then \( w^n \) is a looping element for \( H \). In general, any power of a looping element is a looping element.
Proposition 4.1. If $\mathcal{T} = (T, \leq, (P_i)_{i \in I})$ is a cto, $H \trianglelefteq \text{Aut}(\mathcal{T})$, and $z$ is a looping element for $H$, then there is a group isomorphism $\psi$ from $H/\langle z \rangle$ onto some subgroup of $\text{Aut}(\mathcal{T}/\langle z \rangle)$ defined as follows (where, for $f \in H$, we let $\bar{f}$ denote its image $f\langle z \rangle$ under the natural map): for all $f \in H$, $\psi(\bar{f})$ is the automorphism of $\mathcal{T}/\langle z \rangle$ such that for all $t \in T$, $(t)\psi(\bar{f}) = \bar{t}$.

Proof. Immediate.

Next, if $M$ is a set cyclically ordered by the relation $C$ and $a \in M$, let $M_a$ denote the totally ordered set $(M, \leq_a)$, where for $x, y \in M$, $x <_a y$ holds if and only if $x = a \neq y$ or $x \neq a, y \neq a$, and $C(a, x, y)$ holds. We now describe how to reverse the procedure of Proposition 4.1. Let $\mathcal{M} = (M, C, (\bar{P}_i)_{i \in I})$ be a cco and let $G$ be a subgroup of $\text{Aut}(\mathcal{M})$. Fix $a \in M$ and let $T = \mathbb{Z} \times M_a$ carry the natural lexicographic product total order $\leq$. Let $\mathcal{T} := (T, \leq, (P_i)_{i \in I})$, where $(k, x) \in T$ satisfies $P_i$ if and only if $x$ satisfies $\bar{P}_i$ in $\mathcal{M}$. For each $f \in G$ and $n \in \mathbb{Z}$ let $(n, f)$ be the map $T \to T$ defined by

$$(k, n)(n, f) = \begin{cases} (n + k, xf) & \text{if either } x = a \text{ or } a = af \text{ or } C(a, x, af^{-1}) \\ (n + k + 1, xf) & \text{otherwise, for all } (k, x) \in T. \end{cases}$$

Also put $H := \{(n, f) : f \in G, n \in \mathbb{Z}\}$.

Theorem 4.2. In this notation we have

(i) $H$ is a subgroup of $\text{Aut}(\mathcal{T})$,

(ii) $(1, e)$ is a looping element for $H$ (where $e$ is the identity of $G$),

(iii) $H/\langle (1, e) \rangle \cong G$, $\mathcal{T}/\langle (1, e) \rangle \cong \mathcal{M}$, and the action of $H/\langle (1, e) \rangle$ on $\mathcal{T}/\langle (1, e) \rangle$ is permutation group equivalent to that of $G$ on $M$.

Proof. Claim 1. $H \subseteq \text{Aut}(\mathcal{T})$.

Proof of Claim 1. To see this, let $(n, f) \in H$. If $a = af$, then $(n, f)$ is an isomorphism of $\{k\} \times M_a$ onto $\{k + n\} \times M_a$ for all $k \in \mathbb{Z}$, and the claim follows. If not, then $(n, f)$ induces order-isomorphisms $[(n + k, a), (n + k, af^{-1})]$ onto $[(n + k, af^{-1}), (n + k + 1, a)]$ and from $[(n + k, af^{-1}), (n + k + 1, a)]$ onto $[(n + k + 1, a), (n + k + 1, af^{-1})]$, which again yields the claim.

Claim 2. (i) $(k, x)(1, e)^n = (n + k, x)$ for all $(k, x) \in T, n \in \mathbb{Z}$

(ii) $(1, e)(n, f) = (n + 1, f) = (n, f)(1, e)$ for all $(n, f) \in H$.

Proof of Claim 2. Immediate.

Claim 3. If $f \in \text{Aut}(\mathcal{T})$ and there are $(k, x), (k, x') \in T$ and $y, y' \in M_a$ such that $(k, x)f = (n, y)$ and $(k, x')f = (n + 2, y')$, then $f$ does not commute with $(1, e)$.
Proof of Claim 3. Note that \((k + 1, x) - 1 f(1, e) = (n + 1, y) < (n + 2, y') = (k, x') f\).

Claim 4. \(H\) is a subgroup of \(\text{Aut}(\mathcal{F})\).

Proof of Claim 4. It is easily checked from the definitions that \((0, e) = \text{id}_{\text{Aut}(\mathcal{F})}\) and that for all \((n, f) \in H\),

\[
(n, f)^{-1} = \begin{cases} (-n, f^{-1}) & \text{if } a = af, \\ (-n - 1, f^{-1}) & \text{otherwise.} \end{cases}
\]

We now check that \(H\) is closed under products. Let \((n, f), (n', f') \in H\). Then if \((k, x) \in T\), we have \((k, x)(n, f)(n', f') = (k + n + n' + \varepsilon_x, xff'')\), where \(\varepsilon_x \in \{0, 1, 2\}\) and is independent of \(k\). We may assume that \(a \neq af^{-1}\) and \(a \neq af''^{-1}\), as otherwise the result is trivial. Also, we may assume that \(aff'' \neq a\), since otherwise \((n, f)(n', f') = (n + n' + 1, ff'').\) By Claim 3, either \(\varepsilon_x\) only takes values 0 and 1 (as \(x\) ranges through \(M_a\)), or it only takes values 1 and 2. By monotonicity, it takes its smallest value on an interval \((a, b)\) of \(M_a\) and the other value on \((b, \infty)\) (where possibly one of these intervals is empty). Also, the jump occurs when \(b = af^{-1}\) or \(bf = af''^{-1}\). It is easy to see that the jump in fact occurs at \(b = af''^{-1}f^{-1}\), and that \(\varepsilon_x \in \{0, 1\}\) if \(C(a, af''^{-1}f^{-1}, af^{-1})\) and \(\varepsilon_x \in \{1, 2\}\) if \(C(a, af^{-1}, af''^{-1}f^{-1})\). It follows that \((n, f)(n', f')\) is of the form \((m, f'f)\) for some \(m \in \mathbb{Z}\), so it belongs to \(H\).

We have now proved parts (i) and (ii) of the theorem. For the third part, note that the projection map \(p: \mathbb{Z} \times H \to G\) given by \((n, f) \mapsto f\) is a group homomorphism with kernel \(\langle (1, e) \rangle\). The rest is immediate.

Remarks. 1. Proposition 4.1 and Theorem 4.2 generalize the theorem of Rieger [R] (see also Fuchs [F, Section IV, Theorem 21]). Rieger's theorem deals just with ordered and cyclically ordered groups (that is, groups which carry such an order and preserve it by left and right multiplication).

2. The above bijection can easily be extended to one between

(a) groups of order preserving and reversing permutations of a total order which are periodic (in the natural sense), and

(b) groups consisting of automorphisms of the quaternary separation relation induced on a circular order, that is, automorphisms which preserve or reverse the circular order.

Acknowledgments

The first author thanks the SERC and the DAAD (Procope) for financial support. The second author also thanks DAAD (Procope). The third author was supported by SERC Advanced Fellowship B/AF 931.
REFERENCES


