Generalization and Partial Demonstration of an Entanglement Based Deutsch-Jozsa Like Algorithm Using a 5-Qubit Quantum Computer

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This paper demonstrates the use of entanglement resources in quantum speedup by presenting two algorithms which are generalizations of an algorithm recently proposed by Goswami and Panigrahi [arXiv:1706.09489 (2017)]. Our first algorithm provides deterministic solutions having an advantage over classical algorithms, whereas the second algorithm yields probabilistic results. The former one has been experimentally verified by using IBM's five-qubit quantum computer with a high fidelity.

PACS numbers:
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I. INTRODUCTION

Deutsch-Jozsa algorithm [1] is one of the first examples of quantum algorithm which is exponentially faster than any possible deterministic classical algorithm that solves the same problem. A special case of Deutsch-Jozsa problem is Deutsch problem [2]; given that a function $f : \{0,1\} \rightarrow \{0,1\}$; is either constant or balanced, the task is to determine whether the function is constant or balanced. Goswami and Panigrahi recently provided a quantum algorithm [3] which uses entanglement as a resource for quantum speedup. They have shown for two black boxes f and g, with a promise that either both are constant or both are balanced, one can solve the following two problems in a single use of each function:

- They are constant or balanced.
- They are equal or unequal.

In this paper, we consider two possible generalizations of this algorithm. Firstly, we take n black boxes instead of two. Secondly, we use black boxes which execute functions $f : \{0,1\}^n \rightarrow \{0,1\}$; and show explicitly how entanglement helps in reducing the number of queries required to solve the aforementioned problem. We show that our first algorithm gives deterministic result and reduces the number of queries by one. The second algorithm says deterministically whether the functions are constant or balanced, and whether the functions are equal or not, with a possibility of error. We find the probability with which one can conclude erroneously whether the functions are equal or not, in the worst case for our second algorithm. We also prove a special property of balanced functions, and use it in our calculations.

Recently, IBM has developed a 5-qubit quantum processor, ibmqx2, which is the world's first commercial quantum computing service provided by IBM via. a free web based interface called *IBM Quantum Experience* (IBM QE) [4]. Researchers have taken a proper advantage of it by demonstrating and running a variety of quantum computing experiments, e.g., [5–17]. Hence, we have implemented different cases of the first proposed algorithm using the IBM quantum computer.

The rest of the paper is organized as follows. Section II describes some preliminary concepts about Deutsch-Jozsa Algorithm. Section III proposes our first algorithm, following which the second algorithm has also been explicated. Section V demonstrates the experimental verification of the first algorithm through IBM quantum experience. Finally, Section VI concludes the paper by summarizing as well as providing future directions of our work.

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II. DEUTSCH-JOZSA ALGORITHM: SOME PRELIMINARIES

We have a function $f : \{0,1\}^n \rightarrow \{0,1\}$, where $n \in \mathbb{N}$; such that f satisfies one of the following two possibilities.

- f is constant i.e, $f(x) = 0$ or $f(x) = 1$, $\forall x \in \{0,1\}^n$
- f is balanced i.e, $|\{x \in \{0,1\}^n : f(x) = 0\}| = |\{x \in \{0,1\}^n : f(x) = 1\}| = 2^{n-1}$

Here, the task is to determine which of the two above properties, is satisfied by f. Classically, it takes $2^{n-1} - 1$ queries to solve this task in the worst case, whereas Deutsch-Jozsa Algorithm solves this problem in only 1 query. We assume that the function f is calculated using unitary $U_f : |x\rangle |y\rangle \rightarrow |x\rangle |y \oplus f(x)\rangle$. The Deutsch-Jozsa Algorithm has been explicated in the following circuit.

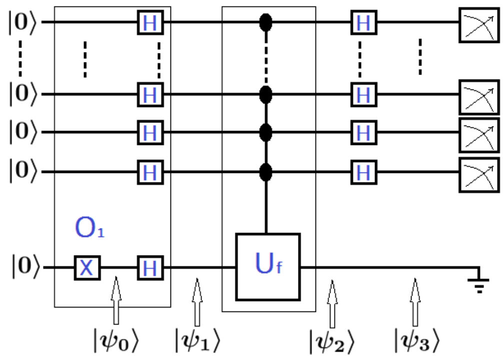

Figure 1: Circuit implementing the general Deutsch–Jozsa Algorithm.

The input state $|\psi_0\rangle = |0\rangle^{\otimes n} |1\rangle$, after application of Hadamard gates we get,

$$|\psi_1\rangle = \sum_{x \in \{0,1\}^n} \frac{|x\rangle}{\sqrt{2^n}} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right] \tag{1}$$

In Figure 1, the first n wires represent n-qubit query register and the last wire represents the answer register. Now the function is evaluated using unitary U_f operations on query register as well as on answer register. The obtained state,

$$|\psi_2\rangle = \sum_{x \in \{0,1\}^n} \frac{(-1)^{f(x)} |x\rangle}{\sqrt{2^n}} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right] \tag{2}$$

contains the result of the evaluation of a function in a superposed state. Eventually, application of Hadamard gates to query registers reveals whether f is constant or balanced as described below.

$$|\psi_2\rangle \xrightarrow{H^{\otimes n} \otimes I} |\psi_3\rangle = \sum_{x \in \{0,1\}^n} \sum_{z \in \{0,1\}^n} \frac{(-1)^{x.z \oplus f(x)} |x\rangle}{2^n} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right] \tag{3}$$

when $z = |0\rangle^{\otimes n}$ and f is constant,

$$|\psi_3\rangle = \sum_{x \in \{0,1\}^n} \frac{(-1)^{f(x)}}{2^n} |x\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right] \tag{4}$$

Hence, the probability of obtaining $|0\rangle^{\otimes n}$ in query register is given as,

$$\left| \sum_{x \in \{0,1\}^n} \frac{(-1)^{f(x)}}{2^n} \right|^2 = \begin{cases} 1, & \text{if } f \text{ is constant} \\ 0, & \text{if } f \text{ is balanced} \end{cases} \tag{5}$$

It is found that, if one gets $|0\rangle$ by measuring each of the qubits in the query register, then the function is constant otherwise balanced.

III. FIRST PROPOSED ALGORITHM

Problem: Given that there are n functions, $f_i : \{0,1\} \rightarrow \{0,1\}$; such that either f_i is constant or balanced $\forall i \in [1,n]$. One needs to determine whether the functions are constant or balanced and whether they are equal or unequal (even if one function is different from the rest, the conclusion should be 'unequal').

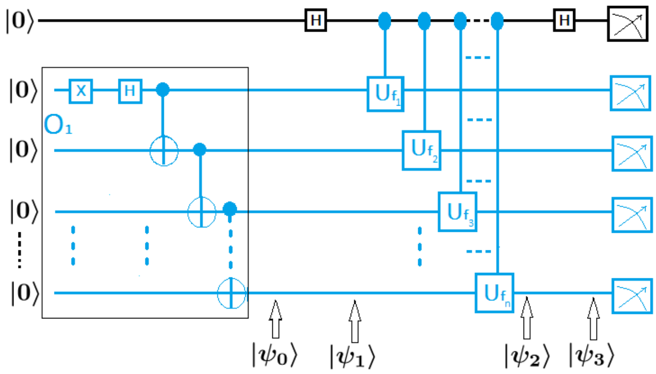

Figure 2: Circuit depicting the first proposed Algorithm. The black wire and the blue wires signify the register qubit and the answer qubits respectively.

Suppose, there are n functions and n entangled qubits of the form $\frac{|0\rangle^{\otimes n} - |1\rangle^{\otimes n}}{\sqrt{2}}$, then the initial state,

$$|\psi_0\rangle = |0\rangle \left\{ \frac{|0\rangle^{\otimes n} - |1\rangle^{\otimes n}}{\sqrt{2}} \right\} \tag{6}$$

$$\Big\downarrow H \otimes I^{\otimes n}$$

$$|\psi_1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \left\{ \frac{|0\rangle^{\otimes n} - |1\rangle^{\otimes n}}{\sqrt{2}} \right\} \tag{7}$$

$$\Big\downarrow U_{c_{f_1}} \otimes U_{c_{f_2}} \otimes ... \otimes U_{c_{f_n}}$$

$$|\psi_2\rangle = \frac{|0\rangle + (-1)^{f_1(0)\oplus f_1(1)} |1\rangle}{2} \{ |a_1 a_2 ... a_n\rangle - |\bar{a_1} \bar{a_2} ... \bar{a_n}\rangle \}, \tag{8}$$

where $a_i \in \{0,1\}$, $U_{c_{f_i}}$ represents a Controlled-Unitary function applied on the i^{th} ancilla qubit. If $a_i = 0$ or 1 $\forall i$, then $f_1(0) = f_2(0) = ... = f_n(0)$ and $f_1(1) = f_2(1) = ... = f_n(1)$. Hence all the functions are said to be equal. If $f_1(0) \oplus f_1(1) = 0$, then $f_1(0) = f_1(1)$. On applying Hadamard gate on the first qubit one measures $|0\rangle$, which implies that f_1 is constant. Since either all functions are balanced or all functions are constant, we can conclude that f_i, $\forall i \in [1,n]$, is constant or otherwise balanced. To summarize,

- If the first qubit is in the state $|0\rangle$, we conclude that all the functions are constant, otherwise balanced.
- If all the answer qubits are correlated (either in the form of $|0\rangle^{\otimes n}$ or $|1\rangle^{\otimes n}$), we conclude that the functions are equal. Otherwise, at least one of them is different.

Total number of queries required by this algorithm is n. Classically, we would require at least the following information to solve this task, $f_1(0)$, $f_1(1)$, $f_i \oplus f_{i+1}$ $\forall i \in [1, n-1]$. This amounts to a total of $n+1$ queries. Hence, one query can be lessened by using entanglement as the main resource. We also propose that following a similar approach we cannot bring the number of queries down any further. The rationale behind this belief is that, each function must be queried at least once to be able to conclude whether it is constant or balanced deterministically. Since there are n functions we can not have an algorithm which does the above task in less than n queries. This proposal may be subject to further investigation.

One may think of generalizing this algorithm for functions $f_i : \{0,1\}^n \rightarrow \{0,1\}$. Hence, we propose the second algorithm.

IV. SECOND PROPOSED ALGORITHM

Now we consider two functions f and g with n inputs ($f : \{0,1\}^n \rightarrow \{0,1\}$; where $n \in \mathbb{N}$ and $g : \{0,1\}^n \rightarrow \{0,1\}$; where $n \in \mathbb{N}$ and two entangled ancillas. The problem is to determine whether f and g are constant or balanced (given that either both are constant or both are balanced) and whether f and g are equal.

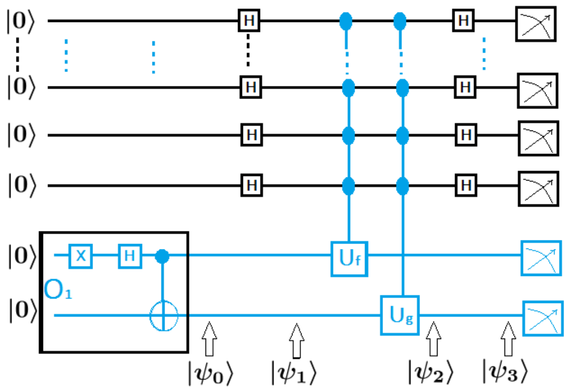

Figure 3: Circuit illustrating the second proposed Algorithm: Here, the black and blue wires signify register and answer qubits respectively.

$$|\psi_0\rangle = |0\rangle^{\otimes n}\left(\frac{|00\rangle - |11\rangle}{\sqrt{2}}\right) \tag{9}$$

By applying Hadamard gates,

$$|\psi_1\rangle = \frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} |x\rangle \left(\frac{|00\rangle - |11\rangle}{\sqrt{2}}\right) \tag{10}$$

After application of unitaries,

$$|\psi_2\rangle = \frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} |x\rangle \left(\frac{|0 \oplus f(x)\rangle |0 \oplus g(x)\rangle - |1 \oplus f(x)\rangle |1 \oplus g(x)\rangle}{\sqrt{2}}\right) \tag{11}$$

$$= \frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} |x\rangle \left[\frac{(-1)^{f(x)}}{\sqrt{2}}\left(|0\rangle |0 \oplus f(x) \oplus g(x)\rangle - |1\rangle |1 \oplus f(x) \oplus g(x)\rangle\right)\right] \tag{12}$$

Application of Hadamard gates on n register qubits results,

$$|\psi_3\rangle = \frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} \left(\sum_{y \in \{0,1\}^n} (-1)^{x.y \oplus f(x)} \left[\frac{|y\rangle}{2^{n/2}}\right]\right) \left[\frac{1}{\sqrt{2}}\left(|0\rangle |0 \oplus f(x) \oplus g(x)\rangle - |1\rangle |1 \oplus f(x) \oplus g(x)\rangle\right)\right] \tag{13}$$

for $y = |0\rangle^{\otimes n}$;

$$|\psi_3\rangle = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |0\rangle^{\otimes n} \frac{1}{\sqrt{2}}\left(|0\rangle |0 \oplus f(x) \oplus g(x)\rangle - |1\rangle |1 \oplus f(x) \oplus g(x)\rangle\right) \tag{14}$$

When function is constant, $f(x) = 0$ or 1, $\forall x$. Then,

$$|\psi_3\rangle = \frac{1}{2^n} 2^n |0\rangle^{\otimes n} \frac{1}{\sqrt{2}}\left(|0\rangle |0 \oplus f(x) \oplus g(x)\rangle - |1\rangle |1 \oplus f(x) \oplus g(x)\rangle\right) \tag{15}$$

Probability of getting $|0\rangle^{\otimes n}$ by the measurement on first n qubits is $|\frac{2^n}{2^n}|^2 = 1$.

When the functions are balanced , we need the following mathematical machinery to calculate the probability of getting $|0\rangle^n$. From equation (14), we state that the ancilla qubits can assume one of the two forms,

- $|E_1\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$
- $|E_2\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$

Let A be a set defined as $A = \{x \in \{0,1\}^n : f(x) \oplus g(x) = 0\}$. Then we define $p = |A|$. Let B be a set defined as $B = \{x \in A : f(x) = 0\}$. Then we define $m = |B|$. It is evident that,

- In equation (14), the number of terms in which ancilla qubits take form $|E_1\rangle$ is p where total number of terms is 2^n.
- Total number of terms containing $|E_1\rangle$ for which $f(x) = 0$ is m.

The following conditions must be satisfied if the functions are balanced:

- $m \leqslant 2^{n-1}$
- $p \leqslant 2^n$
- $m \leqslant p$
- $2^{n-1} - m \leqslant 2^n - p$

When functions are balanced, $|\psi_3\rangle$ takes the following from

$$|\psi_3\rangle = m(-1)^0 |E_1\rangle + (-1)^1(p-m)|E_1\rangle + (2^{n-1} - m)(-1)^0 |E_2\rangle + (2^{n-1} - p + m)(-1)^1 |E_2\rangle \tag{16}$$

$$\implies |\psi_3\rangle = (2m - p)|E_1\rangle + (p - 2m)|E_2\rangle \tag{17}$$

Therefore, probability of getting $|0\rangle^{\otimes n}$ on measurement for balanced function $= 2(2m-p)^2$.

Now we prove that $2m$ is always equal to p. We define the sets C, D, E, F as follows:

$C = \{x \in \{0,1\}^n : f(x) = 0\}$

$D = \{x \in \{0,1\}^n : g(x) = 0\}$

$E = \{x \in \{0,1\}^n : f(x) = 1\}$

$F = \{x \in \{0,1\}^n : g(x) = 1\}$.

Since $m = |x \in \{0,1\}^n : f(x) = g(x) = 0|$ and $(2^{n-1} - m) = |x \in \{0,1\}^n : f(x) = 0, g(x) = 1|$,

We have,

$$|C| + |D| = 2m + 2(2^{n-1} - m) \tag{18}$$

Since $(p - m) = |x \in \{0,1\}^n : f(x) = g(x) = 1|$ and $(2^{n-1} - m) = |x \in \{0,1\}^n : f(x) = 0, g(x) = 1|$,
We have,

$$|E| + |F| = 2(p-m) + 2(2^{n-1} - m) \tag{19}$$

. since $f(x)$ and $g(x)$ are balanced,

$$|E| + |F| = 2^n \tag{20}$$

$$\implies 2(p-m) + 2(2^{n-1} - m) = 2^n \tag{21}$$

$$\implies 2m = p \tag{22}$$

Probability that we get $|0\rangle^{\otimes n}$ in the register qubit for balanced functions is $2(2m-p)^2 = 0$. Hence, we can deterministically say that the functions are constant when the register qubits are measured to be $|0\rangle^{\otimes n}$,otherwise balanced.

If $f(x) = g(x)\ \forall x \in \{0,1\}^n$ the probability of getting the ancilla state $|E_1\rangle$ is 1. But probability of getting $|E_1\rangle$ i.e., correlated output (either both $|0\rangle$ or $|1\rangle$) in the ancilla qubits despite the functions being unequal in the worst case is equal to $\frac{2^n - 2}{2^n} = 1 - 2^{1-n}$. Hence, this algorithm does not deterministically say whether the functions are equal or unequal for $n > 1$.

V. EXPERIMENTAL DEMONSTRATION OF THE FIRST PROPOSED ALGORITHM IN IBM'S 5-QUBIT QUANTUM COMPUTER

We consider the simplest case in which two functions are used.

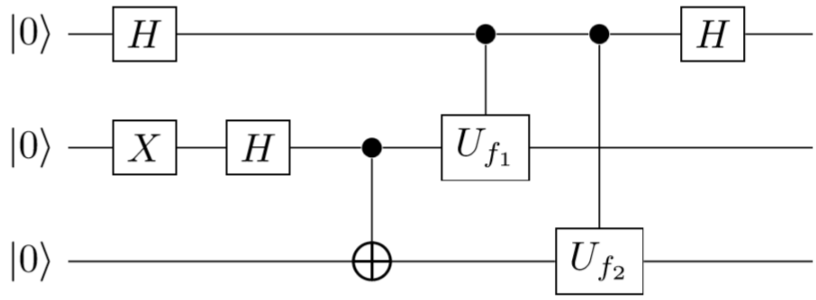

Figure 4: Quantum circuit implementing First Proposed Algorithm for two functions.

The above circuit can be modified as shown below, so that on measurement, last two qubits will be in state $|01\rangle$ if the functions are unequal, and in state $|00\rangle$ if the functions are equal.

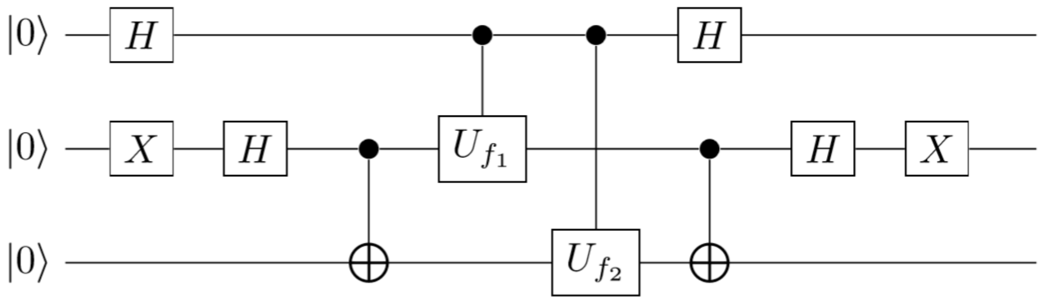

Figure 5: Modified quantum circuit implementing First Proposed Algorithm for two functions for the experiment.

Fidelity calculation is done by determining the experimental density matrix ρ_E and theoretical density matrix ρ_T, followed by determination of the quantity $Tr(\sqrt{\rho_E \rho_T \rho_E})$.
Experimental density matrix is calculated using the formula

$$\rho_E = \frac{1}{2^N} \sum_{i_1,i_2,i_3=0}^{3} T_{i_1 i_2 i_3} (\sigma_{i_1} \otimes \sigma_{i_2} \otimes \sigma_{i_3}) \tag{23}$$

where σ_i's are Pauli's matrices.

$$T_{i_j i_k i_l} = S_{i_j} \otimes S_{i_k} \otimes S_{i_l} \tag{24}$$

where,
$S_0 = P_{|0\rangle} + P_{|1\rangle}$, $S_1 = P_{|0_x\rangle} - P_{|1_x\rangle}$, $S_2 = P_{|0_y\rangle} - P_{|1_y\rangle}$, and $S_3 = P_{|0_z\rangle} - P_{|1_z\rangle}$
Theoretical density matrix is given by,

$$\rho_T = |\psi\rangle \langle\psi| \tag{25}$$

Here, Our first proposed algorithm has been implemented in IBM's 5-qubit quantum computer by considering the following possible cases,

Case -1: Functions are equal and balanced

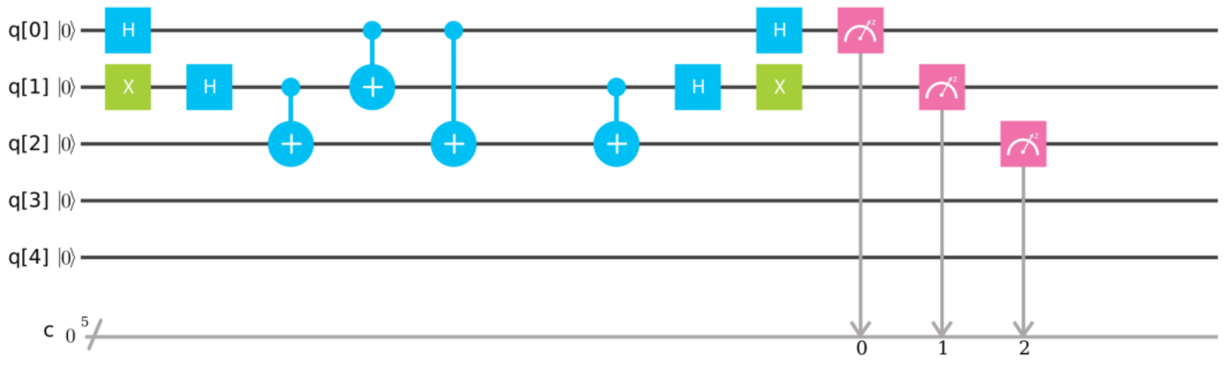

Figure 6: IBM quantum circuit illustrating first proposed algorithm where functions are both balanced and equal.

Results:

- Fidelity of run (8192 shots): 0.7737
- Fidelity of simulation: 1.0

Case -2: Functions are unequal and balanced

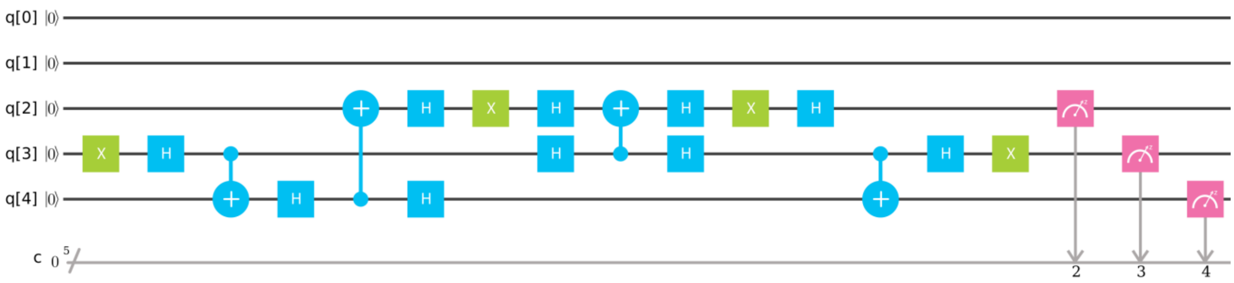

Figure 7: IBM quantum circuit illustrating first proposed algorithm where functions are balanced and unequal.

Results:

- Fidelity of run (8192 shots): 0.57031
- Fidelity of simulation: 1.0

Case -3: Functions are equal and constant

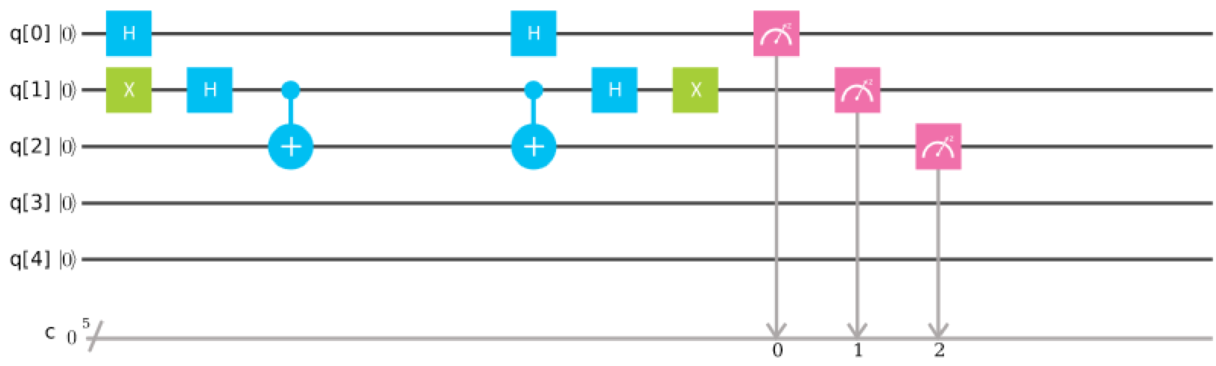

Figure 8: IBM quantum circuit illustrating first proposed algorithm where functions are constant and equal.

Results:

- Fidelity of run (8192 shots): 0.97538
- Fidelity of simulation: 1

Case -4: Functions are unequal and constant

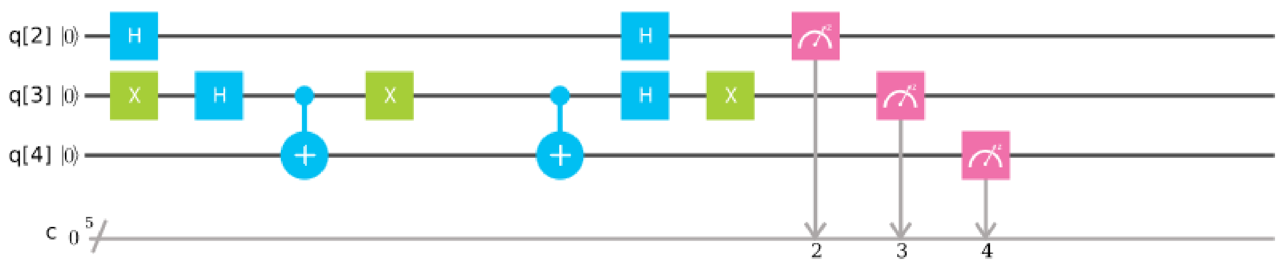

Figure 9: IBM quantum circuit implementing first proposed algorithm where functions are constant and unequal.

Results:

- Fidelity of run (8192 shots): 0.8107
- Fidelity of simulation: 1.0

Simulation results confirm with theoretical prediction. Error in run results can be attributed to error in quantum gates. We can see that fidelity in lowest in the Case-2 where the number of gates used is maximum and least in Case-3 where number of gates used is minimum of the four cases shown.

VI. CONCLUSION

We have proposed an algorithm which uses one less query than it's corresponding classical or quantum (without entanglement) algorithm which decides whether n functions (given that either all the functions are constant or all are balanced) are: (i) equal or unequal (ii) constant or balanced.

We have also made a conjecture that the number of queries cannot be reduced further using any quantum algorithm. We have tried to generalize this algorithm for functions that take n-qubit input and return 0 or 1 and came to a conclusion that though it is possible to say deterministically whether they are constant or balanced, the algorithm tells us whether the functions are equal or not with a certain probability of going wrong, which is equal to $\frac{2^n-2}{2^n} = 1 - 2^{1-n}$. The first proposed algorithm has been experimentally realized using IBM's 5-qubit quantum computer, by taking different cases into account, and the desired results with a high fidelity have been produced.

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