THE POWER OF NON-UNIFORM WIRELESS POWER

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Abstract. We study a fundamental measure for wireless interference in the SINR model known as (weighted) inductive independence. This measure characterizes the effectiveness of using oblivious power — when the power used by a transmitter only depends on the distance to the receiver — as a mechanism for improving wireless capacity.

We prove optimal bounds for inductive independence, implying a number of algorithmic applications. An algorithm is provided that achieves — due to existing lower bounds — capacity that is asymptotically best possible using oblivious power assignments. Improved approximation algorithms are provided for a number of problems for oblivious power and for power control, including distributed scheduling, connectivity, secondary spectrum auctions, and dynamic packet scheduling.

1. Introduction

One of the strongest weapons for increasing the capacity of a wireless network is power control. Higher power increases the bandwidth of a single transmission link, while causing more interference to other simultaneously transmitting links. Given this tension, intelligent power control is crucial in increasing the spatial reuse of the available bandwidth. Thus it is not surprising that most contemporary wireless protocols use some form of power control. More recently, this phenomenon has also been studied theoretically; it was shown in a series of works that power control may improve the capacity of a wireless network in an exponential [37, 18] or even unbounded [8] way.

Unrestricted power control is, however, a double-edged sword. In order to achieve the theoretically best results, one must solve complex optimization problems, where transmission power of one node potentially depends on the transmission powers of all other nodes [29]. In real wireless networks, where communication demands change over time, this may not be an option. In practical protocols, the transmission power should be independent of other concurrent transmissions, which leaves it to only depend on the distance between transmitter and receiver. This is known as oblivious power control.

Many questions immediately rise in the wake of the previous assertion: What is the price of restricting power control to oblivious powers? Which of the infinitely many oblivious power schemes are good choices? Once an oblivious power scheme is chosen, what algorithmic results can be achieved?

In this work, we look at these questions in the context of the physical or SINR model of interference, a realistic model gaining increasing attention (see Section 1.2 for historical background and motivation and Section 2 for precise definitions). In this setting, our work answers a number of these questions optimally, completing an extensive line of work in the algorithmic study of the SINR model.

The specific problem at the center of our work is capacity maximization: Given a set of transmission links (each a transmitter-receiver pair), find the largest subset of links that can transmit simultaneously.

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Before the present work, the state-of-the-art was as follows. The mean power assignment, where a link of length \( \ell \) is assigned power (proportional to) \( \ell^{\alpha/2} \) (\( \alpha \) being a small physical constant), had emerged as the “star” among oblivious power assignments. It was shown that using mean power, one can approximate capacity maximization with respect to arbitrary power control within a factor of \( O(\log n \cdot \log \log \Delta) \) [18] and \( O(\log n + \log \log \Delta) \) [20], where \( \Delta \) is the ratio between the maximum and minimum transmission distance and \( n \) is the number of links in the system. This showed that the somewhat earlier lower bound of \( \Omega(n) \) [8] applied only when \( \Delta \) was doubly exponential. In terms of \( \Delta \), it was shown that one must pay a \( \Omega(\log \log \Delta) \) factor [18]. The best upper bounds were, as mentioned, either dependent on the size of the input [18, 20] and as such unbounded (in relation to \( \Delta \)), or exponentially worse (\( \log \Delta \)) [13, 1].

1.1. Our Contributions. In this paper, we study all power assignments of the form \( \ell^{p \alpha} \) for all fixed \( 0 < p < 1 \) (setting \( p = \frac{1}{2} \) gives us mean power). Our first result shows that the lower bound of \( \Omega(\log \log \Delta) \) is tight. That is, we give a simple algorithm that uses any oblivious power scheme from the above class, achieving solution quality within a \( O(\log \log \Delta) \)-factor of the optimum with unrestricted power control. For small to moderate values of \( \Delta \), e.g., when \( \Delta \) is at most polynomial in \( n \) (which presumably includes most real-world settings), our bound is an exponential improvement over all previous bounds, including the \( O(\log \Delta) \)-bound of [1] (see also [13]).

This result extends the “star status” from mean power to a large class of assignments. This class has been studied implicitly before in a wide array of work [32, 20, 26, 21] on “length-monotone, sub-linear” power assignments, but its relation to arbitrary power was not understood.

Our second main contribution is to improve a number of algorithmic results that use these power assignments. We shave a logarithmic approximation factor off a variety of problems, including distributed scheduling [32], secondary spectrum auctions [26], wireless connectivity [37, 23, 22], and dynamic packet scheduling [31, 2]. Using the capacity relation between oblivious and arbitrary power (our first result), we strengthen the bounds for these problems in the power control setting as well.

Though we have presented our work above in terms of algorithmic implications, what we actually prove are two structural results, from which this host of algorithmic applications follow essentially immediately. These results are important in their own right, e.g., implying tight bounds on certain efficiently computable measures of interference.

To provide an intuitive understanding of our results, it is useful to recall the graph theoretic notion of inductive independence [44]. A graph \( G \) has inductive independence number \( d \) if there is an ordering of the vertices \( v_1, v_2, \ldots, v_n \) such that each \( v_i \) has at most \( d \) edges to any independent set \( I \subseteq \{v_{i+1}, v_{i+2} \ldots v_n\} \). The inductive independence property is found in many graph classes (e.g., intersection graphs of convex planar objects are 3-inductive independent [44]), and it has powerful algorithmic implications [44, 19, 26]. For example, a simple \( d \)-approximation algorithm for the maximum independent set problem in such a graph is as follows: Process the vertices in the prescribed order, adding each vertex to the solution if it has no edges to nodes already in the solution. By the inductive independence property, the addition of a single vertex disqualifies at most \( d \) vertices of the optimal solution from being added in the future, which implies the claimed approximation factor.

In this paper, we deal with an interference measure that is a natural analog of inductive independence, applied to certain weighted graphs that model the SINR interference scenario. In this context, links are vertices, and the edge weights represent the extent of interference between links. The relevant ordering of the links is the ascending order by length, and “independent sets” are represented by feasible sets of links (links that can transmit simultaneously).

When feasibility is with respect to arbitrary power assignment, we show that the measure is bounded by \( O(\log \log \Delta) \) (Theorem 3), implying our first capacity result (and its applications). Technically, this is done by carefully extending the analysis of [18]. When feasibility is with respect to oblivious power from the above mentioned class, the measure can be bounded by a
constant (Theorem 4), implying the second set of algorithmic results. This involves a potentially novel contradiction technique (at least in the context of SINR analysis).

Our results hold for general metric spaces and all constants $\alpha > 0$. Apart from the specific applications pinpointed here, we expect any number of future algorithmic questions in the SINR model to directly benefit from these bounds.

1.2. Related Work. Gupta and Kumar [17] were among the first to give analytical results for wireless scheduling in the physical (SINR) model. Those early results analyzed special settings using e.g. certain node distributions, traffic patterns, transport layers etc. In reality, however, networks often differ from these specialized models and no algorithms were provided to optimize the capacity. On the other hand, graph-based models yielded algorithms like [33, 40] but such models do not capture the nature of wireless communication well, as demonstrated in [16, 35, 38]. Six years ago, Moscibroda and Wattenhofer [37] started combining the best of both worlds, studying algorithms for scheduling in arbitrary networks. Since then, the problems studied in this setting has reflected the diversity of the application areas underlying it – topology control [10, 39, 28], sensor networks [36], combined scheduling and routing [5], ultra-wideband [27], analog network coding [15].

In spite of this diversity, certain canonical problems have emerged, the study of which has resulted in improvements for other problems as well. The capacity problem is one such problem. After it was quickly shown to be NP-complete [13], a constant factor approximation algorithm for uniform power was achieved in [11, 24], eventually extended to essentially all interesting oblivious power schemes [20]. In [29, 30], a constant approximation to the capacity problem for arbitrary powers was obtained. As already mentioned before, the relation between capacity using oblivious power and capacity using arbitrary power was first studied in [18].

Linear power has turned out to be the easiest among fixed power assignments, being the only one with constant factor approximation for scheduling [9, 43] and a constant-bounded interference measure [9]. Whereas there are instances for which linear and uniform power are arbitrarily bad in comparison with mean power [37], a maximum feasible subset under mean power is known to be always within a constant factor of subsets feasible under linear or uniform power [42]. Recently it was shown in [6] that algorithms for capacity-maximization in the SINR model can be transferred to a model that takes Rayleigh-fading into account, losing only a $O(\log^* n)$ factor in the approximation ratio. This overview is far from being complete, surveys can be found in e.g. [14].

Technically, the idea of looking at the interaction between a feasible set and a link is known. The works of Halldorsson [18] and Kesselheim and Vöcking [32] are particularly relevant – the first in the context of oblivious-arbitrary comparison, and the second in the context of oblivious power. Our results improve the bounds in those papers to the best possible.

1.3. Outline of the Paper. Section 2 lays down the basic setting, including a formal description of the SINR model. In Section 3, we introduce the interference measure and our two structural results. We follow this in Section 4 by illustrating two applications of these results, one for each of the main theorems. Section 5 contains the proofs of the structural results, and Section 6 contains a medley of further applications.

2. Model and Definitions

Given is a set $L = \{l_1, l_2, \ldots, l_n\}$ of links, where each link $l_v$ represents a unit-size communication request from a transmitter $s_v$ to a receiver $r_v$, both of which are points in an arbitrary metric space. The distance between two points $x$ and $y$ is denoted $d(x, y)$. We write $d_{vw} = d(s_v, r_w)$ for short, and denote by $\ell_v$ the length of link $l_v$. Let $\Delta = \Delta(L)$ denote the ratio between the maximum and minimum length of a link in $L$. 
constants. We let \( a > 0 \) be the so-called path-loss constant, and \( N \) is a universal constant denoting the ambient noise, \( \beta \) denotes the minimum SINR (signal-to-interference-noise-ratio) required for a message to be successfully received, \( \alpha > 0 \) is the largest PC-feasible subset).

Let \( P_v \) denote the power assigned to link \( l_v \), or, in other words, \( s_v \) transmits with power \( P_v \). We focus on power assignments \( \mathcal{P}_p \), where \( P_v = \ell_v^{\alpha} \). This includes all the specific assignments of major interest: uniform (\( \mathcal{P}_0 \)), mean (\( \mathcal{P}_{1/2} \)), and linear power (\( \mathcal{P}_1 \)).

We say that \( S \) is \( \mathcal{P} \)-feasible, if Equation 1 is satisfied for each link in \( S \) when using power assignment \( \mathcal{P} \). We say that \( S \) is power control feasible (PC-feasible for short) if there exists a power assignment \( \mathcal{P} \) for which \( S \) is \( \mathcal{P} \)-feasible. We frequently write simply feasible when we refer to PC-feasible.

Let PC-Capacity denote the problem of finding a maximum cardinality subset of the links in \( L \) that is PC-feasible (that is we maximize the capacity of the channel used). Let \( OPT^\mathcal{P}(L) \) denote the optimal capacity (i.e., size of the largest \( \mathcal{P} \)-feasible subset) of a linkset \( L \) under power assignment \( \mathcal{P} \), and \( \overline{OPT}(L) \) denote the optimal capacity under any power assignment ((i.e., size of the largest PC-feasible subset)).

**Affectance.** We use the notion of affectance, introduced in [11] and refined in [24] and [32]. The affectance \( a_{w}^{\mathcal{P}}(v) \) of link \( l_v \) caused by another link \( l_w \), with a given power assignment \( \mathcal{P} \), is the interference of \( l_w \) on \( l_v \) relative to the power received, or

\[
a_{w}^{\mathcal{P}}(v) = \min \left( 1, c_v \frac{P_w/d_{wv}^{\alpha}}{P_v/d_{wv}^{\alpha}} \right) = \min \left( 1, c_v \frac{P_w}{P_v} \cdot \left( \frac{\ell_v}{d_{wv}} \right)^{\alpha} \right),
\]

where the factor \( c_v = \beta/(1 - \beta N \ell_v^{\alpha}/P_v) \) depends only properties of the link \( l_v \), and on universal constants. We let \( a_{w}^{\mathcal{P}}(v) \) denote \( a_{w}^{\mathcal{P}}(w) \). We shall frequently drop the power assignment reference \( \mathcal{P} \), which means then that we assume \( \mathcal{P}_p \). As it is convention, we define \( a_v(v) := 0 \) since \( v \) does not interfere with itself. For sets \( S \) and \( T \) of links and a link \( l_v \), let \( a_v(S) = \sum_{w \in S} a_{w}^{\mathcal{P}}(v) \), \( a_S(v) = \sum_{w \in S} a_{w}^{\mathcal{P}}(v) \), and \( a_S(T) = \sum_{w \in S} a_{w}^{\mathcal{P}}(T) \). Using this notation, Equation 1 can be rewritten as \( a_{S}^{\mathcal{P}}(v) \leq 1 \) (except for the near-trivial case of \( S \) containing only two links).

We introduce two more affectance notations. Let \( b_{v}(w) = b_{w}(v) = a_{v}(w) + a_{w}(v) \) be the symmetric version of affectance. Let \( \hat{a}_{v}(w) \) (\( \hat{b}_{v}(w) \)) be the length-ordered version, defined to be \( a_{v}(w) \) (\( b_{v}(w) \)) if \( \ell_v \leq \ell_w \) and 0 otherwise, respectively. (This assumes that link-lengths form a total order.) These are extended in similar ways to affectances to and from sets as defined for \( a_{v}(w) \). Notice that \( a_S(S) = \hat{b}_S(S) = b_S(S)/2 \).

**Non-weak links** A link is said to be non-weak if \( c_v \leq 2\beta \). This is equivalent to \( \frac{P_v}{\ell_v^{\alpha}} \geq 2\beta N \). Intuitively, this means that the link uses at least slightly more power than the absolute minimum needed to overcome ambient noise (the constant 2 can be replaced with any fixed constant larger than 1). Our theorems often assume links to be non-weak. This reasonable and often-used assumption [32, 1, 7, 12] can be achieved, if necessary, by scaling the powers.

**Length classes** A length class is any set \( R \) of links with \( \Delta(R) \leq 2 \) (i.e., link lengths vary by a factor no more than 2). Clearly, any link set \( L \) can be partitioned into \( \log \Delta(L) \) length classes. We also refer to this as nearly-equilength class.

**Independence** We refer to links \( l_v \) and \( l_w \) as \( q \)-independent if they satisfy \( d_{vw} \cdot d_{wv} \geq q^2 \cdot \ell_w \ell_v \). A set of mutually \( q \)-independent links is said to be \( q \)-independent.
Independence is a pairwise property, and thus weaker than feasibility. The condition is equivalent to \( a_P^v(u) \cdot a_P^v(u) \leq \frac{c_v}{q_v} \), independent of the power assignment \( P \). A feasible set is necessarily \( \beta^{1/\alpha} \)-independent [18], but there is no good relationship in the opposite direction.

In this paper we provide an independence-strengthening result with better tradeoffs than the so-called “signal-strengthening” result of [24]. The proof is in Appendix A.

**Lemma 1.** Any feasible set of links can be partitioned into \( 2q^\alpha/\beta + 1 \) or fewer \( q \)-independent sets.

### 3. Structural Properties

We start by defining the interference measure at the center of this work.

**Definition 2.** Let \( L \) be a set of links and \( \mathcal{P}, \mathcal{Q} \) be two power assignments. Then

\[
I_{\mathcal{Q}}^\mathcal{P}(L) \equiv \max_{S \in F_{\mathcal{Q}}(L)} \max_{l \in L} \hat{b}_P^v(S),
\]

where \( F_{\mathcal{Q}}(L) \) is the collection of subsets of \( L \) that are \( \mathcal{Q} \)-feasible.

When \( \mathcal{P}_p \) is used as one (or both) of the assignments, we use \( p \) instead of \( \mathcal{P}_p \) in the sub(super)-scripts – thus \( I_{\mathcal{Q}}^p(L) \) instead of \( I_{\mathcal{P}_p}^\mathcal{Q}(L) \).

As mentioned in the introduction, this definition is analogous to the inductive independence number of a graph. In our setting, the weighted graph is formed on the links, with the weight of the (undirected) edge between links \( l_u \) and \( l_v \) being \( b_u(v) = b_v(u) \) (computed according to power assignment \( \mathcal{P} \)). The ordering is the ascending order of length. Then, \( I_{\mathcal{Q}}^\mathcal{P}(L) \) is an upper bound on how much weight a link can have into a \( \mathcal{Q} \)-feasible set containing longer links, just as the inductive independence number is an upper bound on how many edges a vertex can have to an independent set consisting of higher-ranked vertices.

When using different power assignments \( I_{\mathcal{Q}}^\mathcal{P}(L) \) gives us a handle on how power assignments compare to each other. We primarily use it in the setting where \( \mathcal{P} = \mathcal{P}_p \), for some \( p \in (0, 1] \), and \( \mathcal{Q} \) is (an) optimal arbitrary power assignment (that maximizes the capacity with respect to \( L \)), allowing us to relate oblivious power to arbitrary power.

Here we give two structural results that characterize the utility of oblivious power assignments. Both of these are best possible and answer long standing open questions. The first characterizes the *price of oblivious power*, i.e., the quality of solutions using oblivious power assignment relative to those achievable by unrestricted power assignments. The second is characterization of the function when both \( \mathcal{P} \) and \( \mathcal{Q} \) are the same assignment (specifically, \( \mathcal{P}_p \) for some \( p \in (0, 1] \)).

**Theorem 3.** For any set \( L \) of non-weak links, any \( 0 < p < 1 \), and any power assignment \( \mathcal{Q} \),

\[
I_{\mathcal{Q}}^p(L) = O(\log \log \Delta).
\]

**Theorem 4.** Fix a power assignment \( \mathcal{P}_p \) for any \( 0 < p \leq 1 \). Then any set \( L \) of non-weak links is \( O(1) \)-inductively independent under \( \mathcal{P}_p \), i.e.,

\[
I_{\mathcal{P}_p}^\mathcal{P}(L) = O(1).
\]

Both theorems will be proven in Section 5. The first theorem improves upon the \( O(\log \log \Delta + \log n) \) bound that is stated implicitly in [20] (and extends it to many more power assignments). The second improves upon the \( O(\log n) \) bound proven in [32]. Both of these new theorems are optimal (up to constant factors).

### 4. Applications

Before embarking upon the somewhat technical proofs of these two theorems, we shall highlight two applications of these theorems, one for each. Further implications are provided in Section 6.
4.1. Capacity Approximation. Using the characterization described above, it is possible to derive a simple single-pass algorithm for maximizing capacity. This is, in fact, the same algorithm as used in [20] to maximize fixed power capacity within a constant factor. It is a type of a greedy algorithm that falls under the notion of “fixed priority”, as defined by Borodin et al. [4]. Recall the $d$-approximation to the max-independent set problem described in the introduction. We added vertices to the solution set in order, and vertices with edges to the solution set so far were disqualified. Our algorithm below is the natural weighted version of it – each vertex is assigned a budget, and is disqualified from being in the solution if the weight of the edges to it from the solution so far exceeds the budget.

Algorithm 1 $\text{Gr}(\text{Set } L = \{l_1, l_2, \ldots, l_n\} \text{ of links in increasing order of length})$

1: $S_0 \leftarrow \emptyset$
2: for $i = 1$ to $n$ do
3: $S_i \leftarrow S_{i-1}$
4: if $\hat{b}^p_{S_{i-1}}(l_i) < 1/2$ then
5: $S_i \leftarrow S_i \cup \{l_i\}$
6: end if
7: end for
8: $X = \{l_v \in S_n : a^p_{S_n}(v) \leq 1\}$

Theorem 5. Let $L$ be a set of links. For any $\mathcal{P}_p$ for which $L$ is non-weak, $\text{Gr}$ chooses a $\mathcal{P}_p^*$-feasible set $X$ such that $|X| \geq \frac{|R|}{2(2I^p_Q(L)+1)}$ for any power assignment $Q$ and any set $R \subseteq F_Q(L)$.

Proof. The structure of the proof is inspired by that of, e.g., [29]. Let $S := S_n$ and $X$ be the sets computed by Algorithm $\text{Gr}$ on input $L$. First we show that the size of $R$ is not much larger than the size of $S$, second we relate the size of $X$ to $S$ and conclude the statement. Consider any power assignment $Q$ and feasible set $R$ as specified by the statement of the theorem. Let $R'$ be $R' := R \setminus S$.

By definition of $I^p_Q(L)$, we know that $\hat{b}^p_i(R) \leq I^p_Q(L)$, for each $l_i \in S$. Thus,

$$\hat{b}^p_S(R) \leq I^p_Q(L) \cdot |S|,$$

Now,Algorithm $\text{Gr}$ chose none of the links in $R'$. Using the acceptance criteria of line 4 and the definition of $b^p$ yields that $\hat{b}^p_{S_j}(j) \geq \hat{b}^p_{S_{j-1}}(j) \geq 1/2$, for each $l_j \in R'$, implying that

$$\hat{b}^p_S(R') \geq |R'|/2.$$

Combining Equation 2 and Equation 3,

$$|R'| \leq 2 \cdot \hat{b}^p_S(R') \leq 2 \cdot \hat{b}^p_S(R) \leq 2I^p_Q(L) \cdot |S|.$$

Thus,

$$|R| \leq |R'| + |S| \leq (2I^p_Q(L) + 1)|S|.$$
Also, the definition of $Gr$ ensures that the average affectance of links in $S$ is small (at most half). To see this, observe that

$$\sum_{l_v \in S} a_S(v) = \sum_{l_i \in S} \sum_{l_j \in S} a_j(i)$$

$$= \sum_{l_i \in S} \sum_{l_j \in S, j < i} (a_j(i) + a_i(j))$$

$$= \sum_{l_i \in S} \sum_{l_j \in S, j < i} \hat{b}_j(i)$$

$$= \sum_{l_i \in S} \hat{b}_{S_{i-1}}(i)$$

$$\leq \frac{1}{2} |S|,$$

which implies that the average in-affectance, $a_S(v)$ is $\frac{1}{|S|} a_S(S) \leq \frac{1}{2}$. Explanation of numbered (in)equalities in the above computation are as follows:

1. By rearrangement. Here $j < i$ refers to the indices of the links as sorted by Algorithm $Gr$. We also use that by the definition of affectance, $\sum_{l_i \in S} a_i(i) = 0$.
2. By the way Algorithm $Gr$ iterates over the links, $j < i$ implies that $\ell_j \leq \ell_i$. Thus $\hat{b}_j(i) = a_j(i) + a_i(j)$, by definition of $\hat{b}$.
3. Since $S_{i-1} = \{l_j : l_j \in S, j < i\}$ as specified by Algorithm $Gr$.
4. By the acceptance criteria of Line 4 of the algorithm.

At least half the links have at most double the average affectance, or

$$|X| = |\{l_v \in S | a_S(v) \leq 1\}| \geq \frac{1}{2} |S| .$$

Combining Equation 4 and Equation 5 yields the statement of the theorem. 

**Theorem 6.** For any $\mathcal{P}_p$, there is a $O(\log \log \Delta)$-approximation algorithm for $PC$-Capacity that uses $\mathcal{P}_p$.

**Proof.** We consider $OPT$, a maximum capacity solution with arbitrary power, and a power assignment $Q$ that makes $OPT$ feasible. We can apply Theorem 5 to note that $Gr$ produces a solution that is at most a factor $O(1 + \mathcal{I}_Q^p(L))$ off the optimal solution. This observation proves the statement using Theorem 3. 

When there is a maximum power level and most links are weak, we can still attain the same approximation ratio, as done in [20], by solving the problem separately for the weak links using maximum power.

### 4.2. Distributed Scheduling

A fundamental problem in wireless algorithms is to schedule a given set of links in a minimum number of slots. For $\mathcal{P}_p$ ($0 \leq p \leq 1$), $O(\log n)$-approximate centralized algorithms are known [20]. In [32], the first distributed algorithm was given, with a $O(\log^2 n)$-approximation ratio. Since it is distributed, the algorithm includes an acknowledgment mechanism (via packets sent from receivers to transmitters) to enable links to know when they have succeeded (and subsequently stop running the algorithm). Assuming “free” acknowledgments, [21] improved the bound to $O(\log n)$ (using the same algorithm), but [32] remained the best result when acknowledgments have to be implemented explicitly.

Here we show that,

**Theorem 7.** There is a randomized distributed $O(\log n)$-approximate algorithm for $\mathcal{P}_p$-Scheduling which implements explicit acknowledgments, for any $0 \leq p \leq 1$.

For $p = 0$ and $p = 1$, this was shown in [21], thus we only need to focus on $p \in (0, 1)$.

To explain this result, we introduce another complexity measure.
**Definition 8.** [32] The maximum average affectance \( A^p(L) \) of a link set \( L \) is \( A^p(L) := \max_{R \subseteq L} a_R^p(R) / |R| \).

It is easily verified that \( A^p(L) = O(\max_Q P_Q^p(L) \cdot \chi(L)) \), where \( \chi(L) \) denotes the minimum number of slots in a feasible schedule of \( L \) (using arbitrary power). Similarly \( A^p(L) = O(I_p^p(L) \cdot \chi_p(L)) \) where \( \chi_p(L) \) denotes the minimum number of slots in a \( P_p \)-feasible schedule of \( L \).

**Corollary 9.** For any linkset \( L \), \( A^p(L) = O(\log \log \Delta \cdot \chi(L)) \) and \( A^p(L) = O(\chi_p(L)) \).

It was shown in [32] that the distributed scheduling algorithm completes in \( O(A^p(L) \log n) \) rounds. Thus, the second bound in Corollary 9 immediately gives us the \( O(\log n) \) approximation. The approximation bound in [32] was worse because it only showed that \( A^p(L) = O(\chi_p(L) \log n) \).

For comparison with arbitrary power, we can similarly use Corollary 9 to achieve a \( O(\log n \cdot \log \log \Delta) \) approximation including acknowledgements, improving on the \( O(\log n \cdot \log n + \log \log \Delta) \)-factor implied by [32] and [20]. Let \( \text{PC-Scheduling} \) be the power-control version of the problem.

**Corollary 10.** There is a randomized distributed algorithm for \( \text{PC-Scheduling} \) that is \( O(\log \log \Delta \cdot \log n) \)-approximate with respect to arbitrary power control optima. It can use any \( P_p \) power assignment, \( 0 < p < 1 \).

5. **Proofs of the Structural Results**

5.1. **Proof of Theorem 3.** We need two lemmas (Lemma 11 and 12) to bound affectances of a link to and from a set of links. The first handles the long links in \( S \) with relatively high affectance. It originates in [18] (Lemma 4.4), but is generalized here in two ways: to any power assignment \( P_p \), and to sets with the weaker property of 2-independence. The proof of this Lemma is given in Appendix B.

Denote \( \hat{p} = \frac{1}{\min(1-p,p)} \) for the rest of this section.

**Lemma 11.** Let \( p \) be a constant, \( 0 < p < 1 \), \( \tau \) be a parameter, \( \tau \geq 1 \), and \( \Lambda = (4(2\beta \tau)^{1/\alpha})^{\hat{p}} \). Let \( l_v \) be a link and let \( Q \) be a 2-independent set of non-weak links in an arbitrary metric space, where each link \( l_w \in Q \) satisfies \( \max(a_{w}^p(w),a_{w}^p(v)) \geq 1/\tau \) and \( \ell_w \geq \Lambda \cdot \ell_v \). Then, \( |Q| = O(\log \log \Delta) \).

Lemma 11 bounds the number of longer links that affect a given link by a significant amount. For affectances below that threshold, we bound their contributions for each length class separately.

We first need the following geometric argument. Intuitively, we want to convert statements involving the link \( l_v \) into statements about appropriate links within the 2-independent set \( S \).

**Proposition 5.1.** Let \( l_v \) be a link. Let \( S \) be a 2-independent set of nearly-equilength links and \( l_u \) be the link in \( S \) with \( d_{uw} \) minimum. Then, \( \max(d_{uw}, d_{uw}) \leq 6d_{uw} \), for any link \( l_w \) in \( S \).

**Proof.** Let \( D = d_{uw} \) and note that by definition \( d_{uw} \leq D \). By the triangular inequality and the definition of \( l_u \),

\[
\begin{align*}
d_{uw} & \leq d(s_w, r_v) + d(r_v, s_u) + d(s_u, r_u) \\
& = d_{uw} + d_{uv} + \ell_u \leq 2D + \ell_u.
\end{align*}
\]

Similarly,

\[
\begin{align*}
d_{uw} & \leq d_{uw} + d_{uw} + \ell_w \leq 2D + \ell_w.
\end{align*}
\]

Applying 2-independence, on one hand, and multiplying Equation 6 and Equation 7, on the other hand, we have that

\[
4\ell_u \ell_w \leq d_{uw} \cdot d_{uw} < (2D + \ell_u) \cdot (2D + \ell_w).
\]

It is then easily verified that \( D \geq \min(\ell_u, \ell_w)/2 \geq \max(\ell_u, \ell_w)/4 \), using that the links are nearly-equilength. The claim then follows from Equation 6 and Equation 7.

This leads to the second lemma.
Lemma 12. Let \( q \) be a positive real value and \( l_v \) be a link. Let \( S \) be a 2-independent and feasible set of non-weak links belonging to a single length-class of minimum length at least \( q^{\beta/\alpha} \cdot \ell_v \). Then, \( b^p_v(S) \leq (\max_{w \in S} b^p_v(w)) + O(1/q) \).

Proof. Consider the link \( l_w \) in \( S \) with \( d_{uw} \) minimum. By Proposition 5.1, \( \max(d_{wu}, d_{uw}) \leq 6d_{uw} \), for any link \( l_w \) in \( S \).

Since \( \ell_v \leq \ell_u \), it holds that \( c_v \leq c_u \). Then, we have that

\[
a^p_w(v) = c_v \left( \frac{\ell_v^{1-p} \ell_w^p}{d_{uw}} \right)^\alpha 
\leq c_u \left( \frac{(\ell_u/q^{\beta/\alpha})^{1-p} \ell_w^p}{d_{wu}/6} \right)^\alpha 
= \frac{6^\alpha}{q^{\beta(1-p)}} a^p_w(u) \leq \frac{6^\alpha}{q} a^p_w(u) .
\]

Also, using that the links in \( S \) are non-weak, \( c_w \leq 2c_u \), we obtain that

\[
a^p_v(w) = c_w \left( \frac{\ell_v^p \ell_w^{1-p}}{d_{vw}} \right)^\alpha 
\leq 2c_u \left( \frac{(\ell_u/q^{\beta/\alpha})^p \ell_w^{1-p}}{d_{wu}/6} \right)^\alpha 
= 2c_u \frac{6^\alpha}{q^{\beta p}} \left( \frac{2 \cdot e^{p(1-p)}}{d_{wu}} \right)^\alpha 
\leq 2 \frac{2^{\alpha+1} \cdot 6^\alpha}{q} a^p_v(u) ,
\]

where we use in the second-to-last inequality that the links are nearly-equilength. This extends to \( a^p_v(S) = a^p_S(v) \) such that

\[
b^p_v(S) - b^p_v(u) = a^p_S \setminus \{u\}(v) + a^p_v(S \setminus \{u\}) 
\leq (1 + 2^{\alpha+1}) \frac{6^\alpha}{q} a^p_S(u) 
\leq (1 + 2^{\alpha+1}) \frac{6^\alpha}{q} ,
\]

where the last inequality uses the feasibility of \( S \).

We are now ready to prove the core result, Theorem 3.

Proof. [of Theorem 3] Choose any \( l_v \in L \) and any feasible subset \( S \subseteq L \). We show that \( b^p_v(S) = O(\log \log \Delta) \). By definition of \( b \), we can assume that all links in \( S \) are larger than \( l_v \), since \( b \) is defined in such a way that all shorter links do not contribute to its value. With this assumption, \( b^p_v(S) = b^p_v(S) \). We use the independence-strengthening lemma (Lemma 1) to partition \( S \) into at most \( 2^{\alpha+1}/\beta \) + 1 different 2-independent feasible sets. Let \( S' \) be one of these sets.

Let \( D := \log \Delta(L) \). We say that a link \( l_w \) in \( S \) is short if \( \ell_v \leq \ell_w < D^{\beta/\alpha} \cdot \ell_v \) and long if \( \ell_w \geq D^{\beta/\alpha} \cdot \ell_v \). We partition \( S' \) into three sets:

\( S_1 \): Long links \( l_w \) with \( b_v(w) \geq 1/D \),
\( S_2 \): Long links \( l_w \) with \( b_v(w) < 1/D \), and
\( S_3 \): Short links.

First, we bound the affectance \( b_v(S_i) \) of each set \( S_i \) separately. By Lemma 11, \(|S_1| = O(\log \log \Delta(S_1))\) and thus \( b_v(S_1) \leq 2|S_1| = O(\log \log \Delta(S)) = O(\log \log \Delta(L)) \). Next we observe that due
to the choice of $D$, the set $S_2$ can be partitioned into $D$ or less length classes. Each such class $X$ satisfies the hypothesis of Lemma 12 with $q := D$ (recall that $S_1$ is a 2-independent subset of $S'$). This implies that $b_v(X) = O(1/D)$ and $b_v(S_2) = O(1)$. The set $S_3$ can be partitioned into $\log D \leq \frac{\log \Delta}{\alpha}$ length classes. For each group $X$, we apply Lemma 12 with $q = 1$, giving that $b_v(X) = O(1)$, for a total of $b_v(S_3) = O(\log \log \Delta)$. Thus, $b_v(S') = b_v(S_1) + b_v(S_2) + b_v(S_3) = O(\log \log \Delta)$, and $b_v(S) \leq (\frac{\log \Delta}{\alpha} + 1)b_v(S') = O(\log \log \Delta)$. \hfill \Box

5.2. Proof of Theorem 4. The following lemma is the crucial element.

**Lemma 13.** Let $L$ be a $P_r$-feasible set of non-weak links and $l_v$ be a link (not necessarily in $L$). Then, $\hat{a}_v(L) = O(1)$.

**Proof.** Let $L(n)$ be the set of all $P_r$-feasible sets of non-weak links of size $n$. Define $g(n)$ (a function of $n$) to be the “optimum upper bound” on $\hat{a}_v$, that is, $g(n) := \sup_{L \in L(n)} \sup_{\ell_v} \hat{a}_v(L)$. Such a function exists, since $\hat{a}_v(L) \leq n$ for any set $L$ of size $n$ and any $l_v$. We claim that $g(n)$ is indeed $O(1)$, which implies the lemma. For contradiction, assume $g(n) = \omega(1)$.

Since $g(n) = \omega(1)$, we can choose a large enough $n_0$ such that all of the following hold:

1. There exists $L \in L(n_0)$ and $l_v$ such that:
   
   $$(8) \quad \hat{a}_v(L) \geq \frac{1}{2} g(n_0).$$

   Observe that independently of $n_0$ such a $L$ and $l_v$ always exist by the definition of $g$.

2. Define $f(n) = \frac{1}{2} 2^\frac{n}{\alpha \beta}$. Then,

   $$(9) \quad f(n_0) \geq (16 \cdot 3^\alpha \beta)^{1/(\alpha \beta)}.$$ 

   Here $c_3$ is a fixed constant to be specified later.

3. $g(n_0) \geq 16 \cdot (4^\alpha + 1)$

We prove our lemma by deriving a contradiction to Equation 8. To prove this, we partition the link set $L$ into $L_1$ and $L_2$ where $L_1 := \{l_w : \ell_w \leq f(n_0) \cdot \ell_v\}$ and set $L_2 := L \setminus L_1$.

**Claim 5.2.** $\hat{a}_v(L_1) < \frac{1}{4} g(n_0)$.

**Proof.** By definition of $\hat{a}$, we can ignore links in $L_1$ smaller than $l_v$. Since the maximum length in $L_1$ is less than or equal to $f(n_0) \cdot \ell_v$, the remaining links in $L_1$ can be divided into log $f(n_0)$ length classes. Consider any such length class $C$. By Lemma 1, $C$ can be partitioned in to $\frac{2^{\alpha + 1}}{\beta}$ + 1 sets that are feasible and 2-independent. For any such set $C'$, we can invoke Lemma 12 to show that $a_v(C') = O(1)$ and thus $a_v(C) = O(\frac{2^{\alpha + 1}}{\beta} + 1)$. By setting $c_3$ to be this constant, we get,

$$a_v(L_1) \leq c_3 \log f(n_0) \leq c_3 \left( \frac{1}{4c_3} g(n_0) - 1 \right)$$

$$< \frac{1}{4} g(n_0),$$

where we used the definition of $f(n)$ in Equality 1. \hfill \Box

**Claim 5.3.** $\hat{a}_v(L_2) \leq \frac{1}{4} g(n_0)$.

**Proof.** Consider $l_w \in L_2$ such that $d(s_v, s_w)$ is minimized. Denote this quantity by $D$. Let $L_3$ be the set of links in $L_2$ with receivers in $B(s_v, D/2)$ (the ball of radius $D/2$ around $s_v$), and set $L_4 := L_2 \setminus L_3$.

Let us first handle affectances to $L_3$ using the following (proof in Appendix B):

**Proposition 5.4.** $|L_3| \leq 2 \cdot 4^\alpha + 1.$
Now using this proposition,

$$a_v(L_3 \cup \{l_w\}) \leq |L_3| + 1 \leq 2 \cdot (4^\alpha + 1) \leq \frac{1}{8} g(n_0) \, .$$

The last inequality follows from Equation 10.

We now consider \(L_4 \setminus \{l_w\}\). Consider any \(l_u \in L_4 \setminus \{l_w\}\). Using that \(r_u\) is at least \(D/2\) away from \(s_v\) (due to being in \(L_4\)) and the fact that we chose \(D := d(s_v, s_w)\), the triangle inequality yields \(d(s_v, r_u) \geq \frac{1}{3} d(s_w, r_u)\). Thus,

$$a_v(L_4 \setminus \{l_w\}) \leq \sum_{l_u \in L_4 \setminus \{l_w\}} c_u \cdot \frac{P_v}{d(s_v, r_u)\alpha} \frac{\ell_u^\alpha}{P_u} \leq 3^\alpha 2^\beta \sum_u P_w \frac{P_v}{d(s_w, r_u)\alpha} \frac{\ell_u^\alpha}{P_u} = 3^\alpha 2^\beta \frac{P_v}{P_w} a_w(L_4) \, .$$

(11)

The first equality holds because \(l_w\) and \(l_u\) belong to the same feasible set, thus \(a_w(u) = c_u \frac{P_w}{P_u} \frac{\ell_u^\alpha}{P_u} \). The second inequality used the assumption \(c_u \leq 2^\beta\) for non-weak links.

Since the power function \(P_p\) is non-decreasing and \(\ell_u \geq f(n_0)\cdot \ell_v\) due to the choice of \(L_2 \supseteq L_4\), \(P_w \geq P_p(f(n_0) \cdot \ell_v) = f(n_0)\alpha^\alpha P_v\). Thus, \(\frac{P_v}{P_w} \leq \frac{1}{f(n_0)^\alpha} \leq \frac{1}{16 \cdot 3^\alpha \beta} \) using Equation 9. Combining this inside with Inequality 11 and using that \(a_w(L_4) \leq g(n_0)\) due to the definition of \(g(n)\), we conclude that

$$a_v(L_4 \setminus \{l_w\}) \leq 3^\alpha 2^\beta \frac{1}{16 \cdot 3^\alpha \beta} g(n_0) \leq \frac{1}{8} g(n_0) \, .$$

Therefore the assumption \(g(n) = \omega(1)\) is wrong. This completes the proof of Claim 5.3. □

Combining Claims 5.2 and 5.3, we get that \(a_v(L) < \frac{1}{2} g(n_0)\), contradicting Equation 8. This completes the proof of Lemma 13. □

We can now complete the proof of Theorem 4.

Proof. Consider any \(S \in F_p(L)\) and any \(l_v \in L\). We show that \(\hat{b}_v^p(S) = O(1)\), proving the theorem: By definition, \(\hat{b}_v^p(S) \leq \sum_{l_w \geq l_v} a_v(w) + \sum_{l_u \geq l_v} a_w(v)\). For the first term, we obtain, using the definition of \(\hat{b}_v^p(S)\) and Lemma 13, that

$$\sum_{l_w \geq l_v} a_v(w) = \hat{a}_v(L) = O(1) \, .$$

The second sum \(\sum_{l_u \geq l_v} a_w(v)\) is known to be \(O(1)\) by [32, Lemma 7]. The proof is completed. □

We remark that the bound in neither theorem holds when there are weak links.

6. Further Applications

Both of our structural results have a number of further applications, improving the approximation ratio for many fundamental and important problems in wireless algorithms. All our improvements come from noticing that many existing approximation algorithms have bounds that are implicitly based on \(P_w^p(L)\) or \(P_p^v(L)\) (or both). Plugging in our improved bounds for these thus gives the (poly)-logarithmic improvements for a variety of applications. Here we often omit proofs of our claims, as they are all of the same flavor.
Connectivity. Wireless connectivity — the problem of efficiently connecting a set of wireless nodes in an interference aware manner — is one of the central problems in wireless network research [23]. Such a structure may underlie a multi-hop wireless network, or provide the underlying backbone for synchronized operation of an adhoc network. In a wireless sensor network, the structure can function as an information aggregation mechanism.

Recent results have shown that any set of wireless nodes can be strongly connected in $O(\log n \cdot (\log n + \log \log \Delta))$ slots using mean power in both centralized [23] and distributed [22] algorithms. These results are directly improved by Theorem 6:

**Theorem 14.** Any set of links can be strongly connected in $O(\log n \cdot \log \log \Delta)$ slots using power assignment $P_p$. This can be computed by either a poly-time centralized algorithm or a $O(poly(\log n) \log \Delta)$-time distributed algorithm.

Results for variations of connectivity such as minimum-latency aggregation scheduling and applications of connectivity such as maximizing the aggregation rate in a sensor network benefit from similar improvements. We refer the reader to [23] for a discussion of these problems and their numerous applications.

Spectrum Sharing Auctions. In light of recent regulatory changes by the Federal Communications Commission (FCC) opening up the possibility of dynamic white space networks (see, for example, [3]), the problem of dynamic allocation of channels to bidders (these are the wireless devices) via an auction has become highly important [45, 46].

The combinatorial auction problem in the SINR model is as follows: Given $k$ identical channels and $n$ users (links), with each user having a valuation for each of the $2^k$ possible subset of channels, find an allocation of the users to channels so that each channel is assigned a feasible set and the social welfare is maximized.

For the SINR model, recent work [26, 25] has established a number of results depending on different valuation functions. Since these results are based on the inductive independence number, Theorem 4 improves virtually all of them by a $\log n$ factor. For instance, an algorithm was given in [26] for general valuations that achieves a $O(\sqrt{k} \log n \cdot P_p(L)) = O(\sqrt{k} \log^2 n)$-approximation. We achieve an improved result by simply plugging in Theorem 4.

**Corollary 15.** Consider the combinatorial auction problem in the SINR setting, for any fixed power assignment $P_p$ with $0 < p \leq 1$. There exist algorithms that achieve a $O(\sqrt{k} \log n)$-factor for general valuations [26], a $O(\log n + \log k)$ approximation for symmetric valuations and a $O(\log n)$ approximation for Rank-matroid valuations [25].

Dynamic Packet Scheduling. Dynamic packet scheduling to achieve network stability is one of the fundamental problems in (wireless) network queueing theory [41]. In spite of its long history, this fundamental problem has been considered only recently in the SINR model (see [34, 31, 2]).

The problem calls for an algorithm that can keep queue sizes bounded in a wireless network under stochastic arrivals of packets at transmitters. A measure called efficiency between 0 and 1 is used to capture how well a given algorithm does compared to a hypothetical best algorithm. We refer the reader to the aforementioned papers for exact definitions and motivations related to this problem.

The state-of-the-art results for this problem have been achieved very recently and simultaneously in [2] and [31]. In spite of differences in the algorithm and assumptions made, both are based on the scheduling algorithm of [32] and achieve a similar result. Recall that the maximum average affectance is $A^p(L) = \max_{R \subseteq L} \frac{\alpha^p_p(R)}{|R|}$ and $\chi^p(L)$ is the minimum number of slots in a $P_p$-feasible schedule of $L$. Let $\phi(L) = \frac{A_p(L)}{\chi^p(L)}$.

The result in [31, 2] can be succinctly expressed as follows.

**Theorem 16.** [31, 2] There exists a distributed algorithm that achieves $\Omega \left( \frac{1}{\log n(1+\phi(L))} \right)$-efficiency for any link set $L$. 


Since the best bound on $\phi(L)$ known was $O(\log n)$ \cite{32}, both papers claimed $\Omega\left(\frac{1}{\log n}\right)$-efficiency. Results in this paper show that $\phi(L) = O(1)$ (see second part of Corollary 9), we get the following improved result:

**Corollary 17.** There exists a distributed algorithm that achieves $\Omega\left(\frac{1}{\log n}\right)$-efficiency for any power assignment $P_p$ ($0 < p \leq 1$).

Since Corollary 9 also shows that $\phi(L) = O\left(\frac{\chi(L)}{\log n}\right) = O(\log n \cdot \log \log \Delta)$, we also get the following improved bound for power control:

**Corollary 18.** There is a distributed algorithm with $\Omega\left(\frac{1}{\log n \cdot \log \log \Delta}\right)$-efficiency, with respect to power control optima.

**References**


Definition 19. We say that links $l_v$ and $l_w$ are $t$-close under power assignment $\mathcal{P}$ if,
\[
\max(a^\mathcal{P}_v(w), a^\mathcal{P}_w(v)) \geq t.
\]

For the rest of this section, denote $\hat{p} := \frac{1}{\min(1-p,p)}$.

Lemma 11 Let $p$ be a constant, $0 < p < 1$, $\tau$ be a parameter, $\tau \geq 1$, and $\Lambda = (4(2\beta\tau)^{1/\alpha})^{\hat{p}}$. Let $l_v$ be a link and let $Q$ be a 2-independent set of non-weak links in an arbitrary metric space, that are both $\frac{1}{\tau}$-close to $l_v$ under power assignment $\mathcal{P}_p$ and at least a $\Lambda$-factor longer than $l_v$. Then, $|Q| = O(\log \log \Delta)$.

Proof. The set $Q$ consists of two types of links: those that affect $l_v$ by at least $\frac{1}{\tau}$ under power assignment $\mathcal{P}_p$, and those that are affected by $l_v$ by that amount. We consider first the links of the former type.

Consider a pair $l_w,l_w'$ in $Q$ that affect $l_v$ by at least $1/\tau$ under $\mathcal{P}_p$, and suppose without loss of generality that $\ell_w \geq \ell_w'$. Let $l_1$ be the shortest link in $Q$. The affectance of $l_w$ on $l_v$ implies that
\[
\frac{c_v}{d_{wv}} \left(\frac{\ell_w \ell_v}{\ell_{w'}}\right)^{\alpha} \geq \frac{1}{\tau},
\]
which can be transformed to $d_{wv} \leq \ell_{wv}^p\ell_{v}^{1-p}(c_v\tau)^{1/\alpha}$, and similarly, $d_{w'v} \leq \ell_{w'}^p\ell_{v}^{1-p}(c_v\tau)^{1/\alpha}$. Recall that since $l_v$ is non-weak, $c_v \leq 2\beta$. By the triangular inequality, we have that

$$d_{w'w} \leq d(s_{w'}, r_v) + d(r_v, s_w) + d(s_w, r_w) = d_{w'v} + d_{wv} + \lambda_w$$

$$\leq 2\ell_{w}^{p}\ell_{v}^{1-p}(c_v\tau)^{1/\alpha} + \lambda_w$$

$$\leq 2\ell_{w}^{p}\ell_{v}^{1-p}(2\beta \tau)^{1/\alpha} + \lambda_w$$

$$\leq \ell_{w}^{p}\ell_{v}^{1-p} + \lambda_w \leq 2\ell_{w} ,$$

using that $\Delta \ell_{w} \leq \ell_1 \leq \ell_{w}$. Similarly,

(12) $$d_{w'w} \leq \ell_{w'w} + \frac{1}{2} \ell_{w}^{p}\ell_{v}^{1-p} .$$

Applying 2-independence, on one hand, and multiplying Equation 12 and 12, on the other, we obtain that

(13) $$4\ell_{w}\ell_{w'} \leq d_{w'w} \cdot d_{w'w} \leq 2\ell_{w'}\ell_{w} + \ell_{w}^{p}\ell_{v}^{1-p} \cdot \ell_{w} ,$$

Cancelling a $2\ell_{w}$-factor, simplifying and rearranging, we have that

(14) $$\ell_{w}^{p} \geq \frac{2\ell_{w'}^{p}}{\ell_{v}^{1-p}} .$$

Label the links in $Q$ as $l_1, l_2, \ldots, l_{|Q|}$ in increasing order of length, and define $\lambda_i = l_i/l_1$. By dividing both sides of Equation 14 by $\ell_{1}^{p}$, we get that

$$\lambda_i^{p} \geq \lambda_i ,$$

Then, $\lambda_2 \geq 2^{1/p}$ and by induction $\lambda_i \geq 2^{1/p} i$. Note that $\Delta(Q) = l_{|Q|}/l_1 = \lambda_{|Q|} \geq 2^{(1/p)i}$, so $\Delta(Q) = 1 \leq \log_{1/p} \log \Delta$, and the claim follows.

The other case of links $l_w$ with $a_w(w) \geq 1/\tau$ is symmetric, with the roles of $p$ and $1 - p$ switched, leading to a bound of $1 + \log_{1/(1-p)} \log \Delta$. \hfill \Box

We shall in particular apply the lemma with $\tau = \log \Delta$.

Proof of Proposition 5.4.

Proof. By Lemma 1, $L_3$ can be divided into $2 \cdot 4^a + 1$ sets, each of which is 4-independent. For contradiction, if $|L_3| > 2 \cdot 4^a + 1$, then at least one of these sets must be of size at least 2. Thus, there would be two different links $l_x$ and $l_y$ that are members of $L_3$ and are 4-independent.

However, since $l_x, l_y \in L_3$, we can argue that

$$d(x, y) \leq \ell_{x} + d(r_x, r_y) \leq \ell_{x} + D \leq \ell_{x} + 2\ell_{x} \leq 3\ell_{x} ,$$

Explanation of numbered inequalities:

1. By triangle inequality.

2. Observing that both $r_x$ and $r_y$ are in $B(s_v, D/2)$ (due to the definition of $L_3$) and using triangle inequality.

3. Since $\ell_{x} = d(s_x, r_x) \geq D/2$ as $r_x \in B(r_v, D/2)$ (since $l_x \in L_3$) and $d(s_x, r_v) \geq D$ (by definition of $D$)

We can similarly show that $d(y, x) \leq 3\ell_{y}$. Then $d(x, y) \cdot d(y, x) \leq 9\ell_{x}\ell_{y}$, contradicting 4-independence. \hfill \Box