Original article

An unbounded stabilization problem of bilinear systems

A. Boutoulout*, R. El Ayadi, M. Ouzahra

TSI Group – MACS Laboratory Moulay Ismail University, Faculty of Sciences, Box 11201 Zitoune, Meknès, Morocco

Department of Mathematics and Informatics, ENS, University of Sidi Mohamed Ben Abdellah, Fès, Morocco

Received 29 October 2011; received in revised form 18 August 2012; accepted 4 April 2013

Abstract

In this paper, we propose a family of feedback controls that guarantee the strong stabilization of unbounded parabolic bilinear systems, where the operator of control is supposed unbounded in the sense that it is bounded from the state space into some extension. An explicit decay estimate is established. An illustrating example is given.

Keywords: Unbounded bilinear system; Feedback control; Strongstabilization; Decay estimate; Admissibility

1. Introduction

In this paper we consider infinite-dimensional bilinear control systems of the form:

\[
\frac{dz(t)}{dt} = Az(t) + v(t)Bz(t), \quad z(0) = z_0 \in H,
\]

where the state space \( H \) is a real Hilbert with inner product \( \langle \cdot, \cdot \rangle \) and corresponding norm \( \| \cdot \| \), \( A \) generates a strongly continuous semigroup of contractions \( (S(t))_{t \geq 0} \) on \( H \) and \( B \) is an unbounded linear operator from \( H \) to a Banach extension \( X \) of \( H \) with a continuous injection \( H \hookrightarrow X \), so that \( \| B_y \|_X \leq M \| y \|, \quad \forall y \in H \), for some \( M > 0 \). This is the case, when for example the control is exercised over the boundary or at a point of the geometrical domain of the system.

The scalar function \( v(\cdot) \) denotes the control.

We shall suppose that \( A \) admits an extension, still denoted by \( A \), which generates a strongly continuous semigroup on \( X \), also denoted by \( (S(t))_{t \geq 0} \), so we have

\[
\| S(t)B_y \|_X \leq M e^{\omega t} \| B \|_{L(H,X)} \| y \|, \quad \forall y \in H,
\]

for some constants \( M > 0 \), \( \omega \geq 0 \).

Bilinear controls are essential in modeling reaction-diffusion convection processes controlled by means of so-called catalysts that can accelerate or decelerate the reaction at hand, which is the case for various chemical or biological chain reactions [22] (see for instance [23,26,30]). In many control problems where (bio)chemical reactions and transport phenomena occur, control actions may give rise to the unboundedness aspect of the operator of control in the obtained bilinear model. This is the case for example if the control take place at the boundaries of the system’s evolution

* Corresponding author. Tel.: +212 0663831669.

E-mail addresses: Boutouloutali@yahoo.fr (A. Boutoulout), rachid_el_ayadi@yahoo.fr (R. El Ayadi), m.ouzahra@yahoo.fr (M. Ouzahra).

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http://dx.doi.org/10.1016/j.matcom.2013.04.026

domain. This may also occur in the case of controls with time delay. The authors in [5,7,9] study the “unboundedness” of the control operators based on the notion of “admissible” operators and appropriate regularity assumptions (see also [4,9,16,19]).

The operator \( B \) is said to be \((p, q)\)-admissible for \( 1 < p, q < +\infty \) such that \( \frac{1}{p} + \frac{1}{q} \leq 1 \), if for any \( t > 0 \), the integral \( \int_0^t v(s) S(t − s)B y(s) \, ds \) lies in \( H \) and depends continuously on \( v \in L^p(0, +\infty; \mathbb{R}) \) and \( y \in L^q(0, +\infty; H) \). In the other word the following bilinear operator \( \Phi_t : L^p(0, +\infty; \mathbb{R}) \times L^q(0, +\infty; H) \rightarrow H \) defined by \( \Phi_t(y, v) = \int_0^t v(s) S(t − s)B y(s) \, ds \) is continuous, and let \( C_I \) denotes its norm. In this context, if \( B \) is \((p, q)\)-admissible then the open loop system (1) admits a unique mild solution \( z \in \mathcal{C}[0, +\infty[, H] \) (see [5,9]).

In [7], the author treats the well-posedness and stability of the closed-loop system:

\[
\dot{z}(t) = A z(t) + f(< Bz(t), z(t)>) B z(t), \quad z(0) = z_0, \tag{3}
\]

under the following assumptions:

(a) \( f : \mathbb{R} \rightarrow \mathbb{R} \) is nonnegative, nondecreasing and continuous,

(b) \( A \) generates a strongly continuous semigroup of contractions on \( H \),

(c) \( D((I − A)^{1/2}) = D((I − A^*)^{1/2}) \) with equivalent norms,

(d) \( < (I − A)y, y > \geq C \| (I − A)^{1/2} y \|^2 \), \( \forall y \in D(A) \) (for some \( C > 0 \)),

(e) \( B \) is a positive self-adjoint and bounded from the subspace \( V = D((I − A)^{1/2}) \) of \( H \) to its dual space.

In [17], a quadratic control has been proposed to get the estimate \( \|z(t)\| = O(\frac{1}{\sqrt{t}}) \) which cannot be, in general, improved with quadratic controls.

In this paper, we study the polynomial stabilizability of the system (1) and we give an explicit estimate of the stabilized state.

The plan of the paper is as follows: In the next section, we present an appropriate decomposition of the system (1) via the spectral properties of \( A \), and we apply this approach to study the stabilization problem of (1). In the third section, we give an illustrating example.

2. Stabilization problem

2.1. Decomposition of the system and well-posedness

In what follows, we suppose that \( A \) is self-adjoint with compact resolvent, so there are only finitely many eigenvalues \( (\lambda_i)_{1 \leq i \leq N} \) such that \( \lambda_i \geq −\eta \) for some \( \eta > 0 \), each with finite dimensional eigenspace \( [21,33] \), and hence the space \( H \) can be decomposed according to

\[
H = H_\eta \oplus H_s, \tag{4}
\]

where \( H_\eta = \text{Vect}(\varphi_i, 1 \leq i \leq N) \), \( H_s = \text{Vect}(\varphi_i, i > N) \) and for all \( i \geq 1 \), \( \varphi_i \) is an eigenvector associated to the eigenvalue \( \lambda_i \) and we have

\[
\|S_s(t)\| \leq M_1 \exp(-\eta t), \quad \forall t \geq 0 \quad (\text{for some } M_1 > 0), \tag{5}
\]

where \( S_s(t) \) denotes the restriction of the semigroup \( S(t) \) in \( H_s \).

Let us consider the two systems:

\[
\frac{dz_u(t)}{dt} = A u z_u(t) + v(t) B u z_u(t), \quad z_u(0) = z_{u0} \in H_u, \tag{6}
\]

\[
\frac{dz_s(t)}{dt} = A s z_s(t) + v(t) B s z_s(t), \quad z_s(0) = z_{s0} \in H_s, \tag{7}
\]

where \( A_u \) and \( B_u \) are respectively the restrictions of \( A \) and \( B \) in \( H_u \), \( A_s \) and \( B_s \) are respectively the restrictions of \( A \) and \( B \) in \( H_s \), and \( z_u \) and \( z_s \) are the components of \( z \in H \) on \( H_u \) and \( H_s \) respectively. On the other hand, since the spectrum of \( A \) is discrete, then we can chose \( \eta \) such that \( \{ \lambda_i / \lambda_i > −\eta \} = \{0\} \) and hence \( A_u = 0 \). In other words, the kernel of \( A \) is \( H_u \). Thus the approach consists on splitting the system into the kernel of \( A \) and its supplementary \( H_s \).
Based on the decomposition (6)—(7) of (1), we will study the stabilizability of the system (1) using the following control

\[ v_\alpha(t) = - <z_\alpha(t), B_\alpha z_\alpha(t)>^\alpha, \quad \text{with } \alpha > 0. \]  

(8)

We now recall the following result

**Theorem 1.** [12] Let \( A \) be a non linear operator, with domain \( D(A) \) in a finite dimensional Hilbert space \( H \). If \( A \) is maximal monotone, then \( -A \) generates a semigroup of contractions \( e^{-tA} \) on \( D(A) \). Furthermore, for all \( z_0 \in D(A) \), the system

\[ \frac{dz}{dt} = -Az, \quad z(0) = z_0 \]

admits a unique solution \( z \in C([0, +\infty]; D(A)) \) given by \( z(t) = e^{-tA}z_0 \).

First, let us analyze the well-posedness of the closed-loop system:

\[ z_\alpha'(t) = A_{u\alpha}z_\alpha(t), \quad z_\alpha(0) = z_{\alpha0}, \]  

(9)

where \( A_{u\alpha} = - < y, B_\alpha y>^\alpha B_\alpha y \), for all \( y \in D(A_{u\alpha}) := \{x \in H_u/B_\alpha x \in H\} \).

**Proposition 1.** Let \( A \) generate a semigroup \( S(t) \) of contractions on \( H \) and let \( B \) be linear, unbounded and such that:

(i) \( <B_\alpha y, z> = <y, B_\alpha z> \), for all \( y, z \in D(A_{u\alpha}) \).

(ii) \( <B_\alpha z, z> \geq 0 \), for all \( y \in D(A_{u\alpha}) \).

Then for all \( z_{\alpha0} \in D(A_{u\alpha}) \), the system (9) admits a unique solution \( z_\alpha \in C([0, +\infty]; D(A_{u\alpha})) \). Furthermore, \( A_{u\alpha} \) generates a semigroup of contractions on \( D(A_{u\alpha}) \) given by:

\[ z_\alpha(t) = e^{tA_{u\alpha}}z_{\alpha0}, \quad \forall z_{\alpha0} \in D(A_{u\alpha}). \]

**Proof.** Let \( \phi : D(A_{u\alpha}) \rightarrow \mathbb{R} \) be defined by \( \phi(x) = \frac{1}{\alpha+1} <B_\alpha x, x>^{\alpha+1}. \)

Then, using (i), we obtain \( \phi'(x) = <B_\alpha x, x>^\alpha B_\alpha x \), where \( \phi' \) indicates the Gateaux derivative of \( \phi \). From (ii) it follows that \( \phi \) is convex and hence \( \phi' : D(A_{u\alpha}) \rightarrow H_u \) is a monotone operator (see [11]). Moreover \( D(A_{u\alpha}) \) is a closed subspace of \( H_u \), it follows that \( \phi' \) is maximal monotone and so is \( -A_{u\alpha} \). Hence \( A_{u\alpha} \) generates a nonlinear semigroup of contractions \((e^{tA_{u\alpha}})_{t \geq 0}\) on \( D(A_{u\alpha}) \). The conclusion follows from Theorem 1 and the fact that \( H_u \) is of finite dimensional.

**Remark 1.** Since \( H_u \) is of finite dimensional, then for all \( z_{\alpha0} \in D(A_{u\alpha}) \) and \( t > 0 \), we have \( z_\alpha(t) \in D(A_{u\alpha}) \) and \( z_\alpha \) admits a right derivative at \( t \) (see [12]).

2.2. Asymptotic behavior of solutions

In this part, we will analyze the asymptotic behavior of \( z_\alpha(t) \).

Before turning our attention to asymptotic behavior of the solutions, we recall some definitions of the stabilizability for the bilinear system (1).

**Definition 1.** We say that the origin is strongly stabilizable on a set \( Y \subset H \) if there exists a control \( v \) such that, for all initial states \( z_0 \) in \( Y \), the mild solution \( z(t) \) of (1) starting at \( z_0 \) satisfies

\[ \lim_{t \to +\infty} z(t) = 0 \]

(10)

The origin is said to be globally strongly stabilizable if it is strongly stabilizable on \( Y = H \).
Definition 2. We say that the origin is exponentially stabilizable on a set \( Y \subset H \) if there exists a control \( v \) such that, for all initial states \( z_0 \) in \( Y \), there exist \( M, \sigma > 0 \) (depending on \( z_0 \)) such that the mild solution \( z(t) \) of (1) starting at \( z_0 \) satisfies

\[
\|z(t)\| \leq Me^{-\sigma t}\|z_0\|. \quad \forall t \geq 0. \tag{11}
\]

The origin is said to be globally exponentially if it is exponentially stabilizable on \( Y = H \).

The next lemma provides a useful estimate.

Lemma 2. Let \( T > 0 \) and suppose that:

(i) \( A \) generates a semigroup \( S(t) \) of contractions on \( H \),
(ii) \( B \) is unbounded such that: \( \langle B_yz, z \rangle = \langle y, B_uz \rangle \) and \( \langle B_uz, z \rangle \geq 0 \), for all \( y, z \in \mathcal{D}(A_{u\alpha}) \).

Then the solution \( z_0(t) \) of the system (6) verifies:

1. for \( \alpha \geq 1 \),

\[
\langle B_uz_0(t), z_0(t) \rangle \geq K_{\|z_0\|} \lambda(t)^{\frac{1}{\alpha-1}}, \quad \forall t > 0, \tag{12}
\]

2. for \( \alpha \leq 1 \),

\[
\langle B_uz_0(t), z_0(t) \rangle \geq K_{\|z_0\|} \lambda(t)^{\frac{1}{\alpha-1}}, \quad \forall t > 0, \tag{13}
\]

where \( \lambda(t) = \int_t^{t+T} \langle B_uz_0(s), z_0(s) \rangle^\alpha ds + K_{\|z_0\|} \) is a non-negative constant depending on \( \|z_0\| \) and \( T \).

Proof. Let \( z_0(t) \in \mathcal{D}(A_{u\alpha}) \) and let \( z_0(t) \) be the unique solution of (9).

We have

\[
\frac{1}{2} \frac{d}{dt} \|z_0(t)\|^2 = \langle A_{u\alpha}z_0(t), z_0(t) \rangle \leq 0, \quad \forall t > 0.
\]

Using the fact that \( A_{u\alpha} \) is dissipative and that \( t \rightarrow z_0(t) \) is continuous, we deduce that

\[
\frac{1}{2} \left[ \|z_0(s)\|^2 - \|z_0(t)\|^2 \right] = \int_s^t \langle A_{u\alpha}z_0(\tau), z_0(\tau) \rangle \ d\tau \leq 0, \quad \forall t \geq s \geq 0. \tag{14}
\]

It follows that

\[
\|z_0(t)\| \leq \|z_0(s)\|, \quad \forall t \geq s \geq 0. \tag{15}
\]

In particular, we have

\[
\|z_0(t)\| \leq \|z_0(0)\|, \quad \forall t \geq 0. \tag{16}
\]

Moreover, we have

\[
\langle B_uz_0, z_0 \rangle = \langle B_u[z_0 - z_0], z_0 \rangle = \langle B_u z_0(s), z_0(s) \rangle + \langle B_u[z_0(s) - z_0], z_0 \rangle + \langle B_u z_0(s), z_0(s) \rangle. \tag{17}
\]

Then, using (16), we get

\[
|\langle B_uz_0, z_0 \rangle | \leq 2\delta R \|z_0\| \|z_0(s) - z_0\| + |\langle B_u z_0(s), z_0(s) \rangle |,
\]

where \( \delta = \|B_u\|_{L(D(A_{u\alpha}), H)} \).

Based on the variation of parameters formula and taking into account the fact that \( A_u = 0 \), we can write:

\[
z_0(s) - z_0 = \int_0^s B_u z_0(t) \ d\tau.
\]
Then

\[ \|z_u(s) - z_{u0}\| \leq \delta \int_0^T |v(\tau)| \|z_{u0}\| \ d\tau. \]

The Hölder’s inequality yields:

\[ \|z_u(s) - z_{u0}\| \leq \delta \|z_{u0}\| \left[ T^{1/(\alpha + 1)}[\lambda(0)]^{\alpha/(\alpha + 1)}, \ \forall s \in [0, T], \right. \]

Replacing \( z_{u0} \) by \( z_u(t) \) in (17) and (18) and using (16), we get

\[ | < B_u z_u(t), z_u(t) > | \leq d \left[ \int_0^T < B_u z_u(s + t), z_u(s + t) >^{\alpha + 1} ds \right]^{\alpha/(\alpha + 1)} + | < B_u z_u(s + t), z(s + t) > |. \]

with \( d = 25^2 \| z_0 \| ^2 T^{\alpha/(\alpha + 1)} \), which may be written as:

\[ | < B_u z_u(t), z_u(t) > | \leq d \left[ \lambda(t) \right]^{\alpha/(\alpha + 1)} + | < B_u z_u(s + t), z_u(s + t) > |. \]

Integrating this last inequality over the interval \([0, T]\), it follows that

\[ \int_0^T | < B_u z_u(t), z_u(t) > | ds \leq d \ T \left[ \lambda(t) \right]^{\alpha/(\alpha + 1)} + \int_0^T | < B_u z_u(s + t), z_u(s + t) > | ds. \]

Using Hölder’s inequality and the monotony of \( B_u \), we deduce that

\[ T \ | < B_u z_u(t), z_u(t) > | \leq d \ T \left[ \lambda(t) \right]^{\alpha/(\alpha + 1)} + T^{\alpha/(\alpha + 1)}[\lambda(t)]^{1/(\alpha + 1)}. \]

From (14), we get:

\[ - \int_0^t < A u z_u(\tau), z_u(\tau) > d\tau \leq \|z_{u0}\|^2, \ \forall t > 0. \]

Then using the monotony of \( B_u \) and the relation

\[ - \int_0^t < A u z_u(\tau), z_u(\tau) > d\tau = - \int_0^t < A u z_u(\tau), z_u(\tau) > d\tau + \int_0^t < B_u z_u(\tau), z_u(\tau) >^{\alpha + 1} d\tau, \]

we deduce that \( v_u \in L^{(\alpha+1)/\alpha}(0, +\infty) \). Then \( \lambda(t) \to 0 \) and hence for all \( \alpha \geq 1 \), we have \( \lambda(t)^{\alpha/(\alpha + 1)} = O(\lambda(t)^{1/(\alpha + 1)}), \) as \( t \to +\infty \). This implies (12). Similarly for all \( \alpha \leq 1 \), we have \( \lambda(t)^{1/(\alpha + 1)} = O(\lambda(t)^{\alpha/(\alpha + 1)}), \) as \( t \to +\infty \), which gives (13).

Now we are ready to state our first main result which concerns the stabilization of the subsystem (6).

First, let us recall the following lemma.

**Lemma 3.** [25] Let \( u \) denotes a positive increasing function such that \( u(0) = 0 \) and set \( v(s) = s - (I + u)^{-1}(s) \) where \( I \) denotes the identity function. Let \( (s_k)_{k \geq 0} \) be a sequence of positive numbers such that

\[ u(s_{k+1}) + s_{k+1} \leq s_k, \ \forall k \geq 0. \]

Then \( s_k \leq x(k) \), where \( x(t) \) is the solution of

\[ x'(t) + v(x(t)) = 0, \ x(0) = x_0. \] (19)

**Theorem 4.** Suppose that

(i) \( A \) generates a semigroup \( S(t) \) of contractions on \( H \),

(ii) \( B \) is unbounded such that \( < B_{u} y, \cdot > = < y, B_{u} z > \), \( \forall y, z \in D(A_{u0}) \), and

\[ < B_{u} y, y > \geq \mu \| y \|^2, \ \forall y \in D(A_{u0}) \mbox{ (for some } \mu > 0). \] (20)

Then the feedback law (8) strongly stabilizes (9) with the following decay estimates:

(a) \[ \| z_u(t) \| = O(t^{-1/2\alpha}), \ \mbox{as } t \to +\infty, \ \mbox{if } \alpha \geq 1, \] (21)

(b) \[ \| z_u(t) \| = O(t^{-\alpha/2}), \ \mbox{as } t \to +\infty, \ \mbox{if } 0 < \alpha \leq 1. \] (22)
Theorem 2. Let $T > 0$.

Firstcase: $\alpha \geq 1$

Let $z_{u0} \in \mathcal{D}(A_{ua})$. From (12) and (20) we deduce that

$$
\mu \| z_u(kT) \|^2 \leq C_1 [\lambda(kT)]^{1/(\alpha+1)}, \forall k \geq 0, \text{ (for some } C_1 > 0).
$$

Tacking $t = kT + T$ and $s = kT$ in (14), and using the fact that

$$
< B_{u}y, y >_{\alpha+1} \leq - < A_{ua}y, y > \leq 0, \forall y \in \mathcal{D}(A_{ua}),
$$

we get

$$
2\lambda(kT) \leq \| z_u(kT) \|^2 - \| z_u(kT + T) \|^2, \forall k \geq 0.
$$

It follows that:

$$
2 \left[ \frac{\mu}{C_1} \right]^{(\alpha+1)} \| z_u(kT) \|^{2\alpha+2} \leq \| z_u(kT) \|^2 - \| z_u(kT + T) \|^2, \forall k \geq 0.
$$

Letting $C_2 = 2[\frac{\mu}{C_1}]^{(\alpha+1)}$ and $s_k = \| z_u(kT) \|^2$, the last inequality can be written as

$$
C_2 s_k^{\alpha+1} + s_{k+1} \leq s_k, \forall k \geq 0.
$$

Using the fact that $s_{k+1} \leq s_k$, we obtain

$$
C_2 s_{k+1}^{\alpha+1} + s_{k+1} \leq s_k, \forall k \geq 0.
$$

Let $u(x) = \frac{1}{\alpha} x^{\alpha+1}$ and $v(s) = s - (I + u)^{-1}(s)$. Using the result of Lemma 3, we get $s_k \leq x(k), \forall k \geq 0$, where $x(t)$ is the solution of the Eq. (19).

The inequality $x(k) \geq s_k$ combined with the fact that $x(t)$ decreases, gives $x(t) \geq 0, \forall t \geq 0$. Furthermore, it is easy to see that $v(s)$ is an increasing function such that for all $s \geq 0$, we have $0 \leq v(s) \leq u(s)$. Then we obtain $-C_2 x(t)^{\alpha+1} \leq x(t) \leq 0$, which implies $x(t) = O(t^{-1/\alpha})$, as $t \to + \infty$. Since $s_k \leq x(k)$, we deduce that $s_k = O(kT + T)$ and using the fact that $\| z_u(t) \|$ decreases, we deduce (21).

Secondcase: $0 < \alpha \leq 1$

Let $z_{u0} \in \mathcal{D}(A_{ua})$. From (13) and (20) we deduce that

$$
\mu \| z_u(kT) \|^2 \leq C_3 [\lambda(kT)]^{\alpha/\alpha+1}, \forall k \geq 0, \text{ (for some } C_3 > 0).
$$

This inequality together with (14) gives

$$
2 \left[ \frac{\mu}{C_3} \right]^{(\alpha+1)/\alpha} \| z_u(kT) \|^{2\alpha+2}/\alpha \leq \| z_u(kT) \|^2 - \| z(kT + T) \|^2
$$

It follows from similar arguments as in the first case that $x(t) = O(t^{-\alpha})$, as $t \to + \infty$, which implies (22). \hfill \square

Remark 2.

1. From (21) we deduce that for all $\alpha \geq 1$, we have $v_{\alpha}(t) = O(t^{-1})$, as $t \to + \infty$.

2. From (22) we deduce that for all $0 < \alpha \leq 1$, we have $v_{\alpha}(t) = O(t^{-\alpha^2})$, as $t \to + \infty$.

Let us recall the following well-posedness result.

Theorem 5. [5] Consider the system (1) and suppose that $B$ is $(p, q)$-admissible with $p, q$ satisfying $1 < p < + \infty, 1 < q < + \infty$. Then for any $y_0 \in H, u \in L_p^p(0, + \infty)$, the system admits a unique mild solution $y \in C([0, + \infty[, H)$.

We also recall the following version of Gronwall’s inequality.

Lemma 6. [35] Let the functions $x, a, b$ and $k$ be continuous and nonnegative on $J = [\alpha, \beta]$, and let $1 \leq q \leq + \infty$ such that

$$
x(t) \leq a(t) + b(t) \left( \int_a^t k(s) x^q(s) \, ds \right)^{1/q}, \forall t \in J.
$$
Then
\[ x(t) \leq a(t) + b(t) \left( \int_{a}^{b} k(s) \ e(s) \ a^q(s) \ ds \right)^{1/q}, \quad \forall t \in J, \]
where \( e(t) = \exp \left( -\int_{a}^{b} k(s) \ b(s)^q \ ds \right). \)

The following second main result concerns the asymptotic behavior of the subsystem (7) controlled with (8), for \( \alpha \geq 1. \)

**Theorem 7.** Let assumptions of Theorem 4 hold and suppose that \( B_s \) is \((p, q)\)-admissible such that \( \frac{1}{p} + \frac{1}{q} < 1 \) and \( C_t = O(t^\beta) \), as \( t \to +\infty \), for some \( \beta \in \mathbb{R}. \)

If \( \beta \) is such that \( \beta + \frac{1}{p} + \frac{1}{q} - 1 < 0 \), then for all \( \alpha \geq 1 \) the control (8) exponentially stabilizes (7).

**Proof.** Remarking that for all \( \beta < 0; \ p^\beta = O(1), \) we can suppose that \( \beta \geq 0. \)

Since \( B_s \) is \((p, q)\)-admissible, then using Theorem 5 we deduce that the open loop system (7) has a unique global mild solution \( z_s \in C([0, +\infty[; H_s), \) which verifies the variation of constants formula:

\[ z_s(t + T) = S_s(T)z_s(t) + \int_{t}^{t+T} v_s(\tau) \ S_s(t + T - \tau) \ B_s \ z_s(\tau) d\tau, \quad \forall t, T \geq 0. \]

It follows that
\[ \|z_s(t + T)\| \leq \|S_s(T)z_s(t)\| + \| \int_{0}^{T} \tilde{v}_s(\tau) \ S_s(T - \tau) B_s \ \tilde{z}_s(\tau) d\tau \|, \]
with \( \tilde{v}_s(\tau) = v_s(\tau + t) \) and \( \tilde{z}_s(\tau) = z(\tau + t), \quad \forall \tau \in [0, T]. \)

Since \( B_s \) is \((p, q)\)-admissible, then there exists \( C_T \geq 0 \) such that
\[ \|z_s(t + T)\| \leq \|S_s(T)z_s(t)\| + C_T \ \|\tilde{v}_s(.)\|_{L^p(0,T)} \ \|\tilde{z}_s(.)\|_{L^q(0,T;H_s)}. \]

Using the spectrum growth assumption (5), we get
\[ \|z_s(t + T)\| \leq M_1 \ \|z_s(.)\| \exp(\eta T) + C_T \ \|v_a(.)\|_{L^p(t,t+T)} \ \|z_s(.)\|_{L^q(t+t+T;H_s)}. \]

This equation can be written as
\[ \|z_s(t + T)\| \leq M_1 \ \|z_s(.)\| \exp(\eta t) \exp(-\eta(T + t)) + C_{T+t-t} \ \|v_a(.)\|_{L^p(t,t+T)} \ \|z_s(.)\|_{L^q(t+T;H_s)} \left[ \int_{t}^{t+T} \|z_s(\tau)\|^q d\tau \right]^{1/q}. \]

Applying the above Lemma 6, we get
\[ \|z_s(t + T)\| \leq M_1 \ \|z_s(.)\| \exp(-\eta T) + M_1 \ C_T \ \|v_a(.)\|_{L^p(t,t+T)} \ g(t) \ \|z_s(.)\|, \]
where
\[ g(t) = \left( \int_{t}^{t+T} e(s) \ \exp(-qs(s - t)) \ ds \right)^{1/q}, \]
and
\[ e(s) = \exp \left( -\int_{s}^{T} (C_{T-s})^q \ \|v(.)\|^q_{L^p(t,s)} \ d\tau \right), \quad \forall \ s \geq t, \]
so \( e(T + t) = \exp \left( -\int_{t}^{t+T} (C_{T-t})^q \ \|v(.)\|^q_{L^p(t,t+T)} \ d\tau \right) = \exp \left( -\int_{0}^{T} (C_s)^q \ \|v(.)\|^q_{L^p(t+t+s)} \ ds \right). \]
It follows from \( v_{\alpha} = O(t^{-1}) \) and \( C_t = O(t^p) \) that there exists \( M_2 > 0 \) (which does not depend on \( T \)) such that
\[
\int_0^T (C_s)^q \| v_\alpha(\cdot) \|_{L^p(\tau, \tau+s)}^q \, ds \leq M_2 \int_0^T s^{\beta q} \left( t^{-p+1} - (t+s)^{-p+1} \right) /p \, ds.
\]
Thus
\[
\int_0^T (C_s)^q \| v(\cdot) \|_{L^p(\tau, \tau+s)}^q \, ds \leq M_2 \int_0^T s^{\beta q} \left( t^{-p+1} - (t+s)^{-p+1} \right) /p \, ds.
\]
Let \( N_0 = \max \left( 0, E \left( \frac{\ln(M_1)}{\eta} \right) \right) + 1 \), where \( E \left( \frac{\ln(M_1)}{\eta} \right) \) is the integer part of \( \frac{\ln(M_1)}{\eta} \).

For \( T \in [N_0, N_0 + 1] \) and for all \( t>N_0 + 1 \), we have
\[
\int_0^T (C_s)^q \| v(\cdot) \|_{L^p(\tau, \tau+s)}^q \, ds \leq M_2 \int_0^1 s^{\beta q} \left[ 1 - (1 + u)^{-p+1} \right] /p \, du.
\]
Moreover \( \int_0^1 s^{\beta q} \left[ 1 - (1 + u)^{-p+1} \right] /p \, du < +\infty \), then
\[
\sup_{N_0 \leq T \leq N_0 + 1} \int_0^T (C_s)^q \| v(\cdot) \|_{L^p(\tau, \tau+s)}^q \, ds = O(t^{(\beta+1)(1+p)/(1-q)}) - 1).
\]
Since \( \beta + \frac{1}{p} + \frac{1}{q} - 1 < 0 \), then
\[
\sup_{N_0 \leq T \leq N_0 + 1} \left( \frac{1}{1 - [1 - e(t + T)]^1/q} \right) = O(t), \quad as \ t \to \infty.
\]
Using the fact that \( (C_t)_{t \geq 0} \) increases, we get
\[
C_T \| v(\cdot) \|_{L^p(\tau, \tau+T)} g(t) \leq C_{N_0 + 1} \| v(\cdot) \|_{L^p(\tau, \infty)} \int_0^{+\infty} \exp(-q\eta u) du \sup_{N_0 \leq T \leq N_0 + 1} \left( \frac{1}{1 - [1 - e(t + T)]^1/q} \right),
\]
which gives
\[
\sup_{N_0 \leq T \leq N_0 + 1} \left( C_T g(t) \| v(\cdot) \|_{L^p(\tau, \tau+T)} \right) = O(t^{(1/p)-1}), \quad and \quad hence
\]
\[
\sup_{N_0 \leq T \leq N_0 + 1} \left( C_T g(t) \| v(\cdot) \|_{L^p(\tau, \tau+T)} \right) \to 0, \quad as \ t \to \infty.
\]
From the inequalities (24) and (27) we deduce that there exists \( t_1 > N_0 + 1 \) and \( 0 < Q < 1 \), which does not depend on \( T \), such that
\[
\| z_s(t + T) \| \leq Q \| z_s(t) \|, \quad \forall t \geq t_1, \quad \forall T \in [N_0, N_0 + 1].
\]
We can easily show that
\[
\| z_s(t + nT) \| \leq Q^n \| z_s(t_1) \|, \quad \forall n \in \mathbb{N}, \quad \forall T \in [N_0, N_0 + 1].
\]
For all \( t > t_1 + N_0(N_0 + 1) \), we have \( \frac{t-t_1}{N_0} > 1 \) and hence there exists an integer number \( n(t) \in \left[ \frac{t-t_1}{N_0}, \frac{t-t_1}{N_0 + 1} \right] \) and \( T \in [N_0, N_0 + 1] \) such that \( t = t_1 + n(t)T \).

Applying (28) we get
\[
\| z_s(t) \| \leq Q^{\alpha(t)} \| z_s(t_1) \| \leq Q^{\alpha(t-t_1)/N_0} \| z_s(t_1) \|.
\]
We conclude that the control (8) exponentially stabilizes the subsystem (7). \( \Box \)

The following theorem is the version of Theorem 7 corresponding to the case \( \alpha \leq 1 \).

**Theorem 8.** Let \( 1 < p, q < +\infty \) such that \( \frac{1}{p} + \frac{1}{q} < 1 \), let assumptions of Theorem 4 hold and suppose that \( B_s \) is \( (p, q) \)-admissible such that \( \frac{1}{p} + \frac{1}{q} < 1 \) and \( C_t = O(t^\beta) \), as \( t \to +\infty \), for some \( \beta \in \mathbb{R} \) such that \( \beta + \frac{1}{p} + \frac{1}{q} < 1 \). Then for all real \( \alpha \) such that \( \sup \left( \frac{1}{p+q}, \left[ \beta + \frac{1}{p} + \frac{1}{q} \right]/2 \right) < \alpha \leq 1 \), the control (8) exponentially stabilizes the system (7).

**Proof.** It is clear that we can suppose that \( \beta \geq 0 \).
Let $N_0 = \max \left(0, E \left( \frac{\ln(M_1)}{\eta} \right) \right) + 1$ and let $T \in [N_0, N_0 + 1]$. Using the same arguments as in the proof of Theorem 7, it follows from $v_a = O(t^{-\alpha^2})$ and $C_I = O(t^\beta)$ that there exists $M_2 > 0$ (which does not depend on $T$) such that

\[
\int_0^T (C_s)^q \|v_a(.)\|_{L^p(t,T+s)}^q \, ds \leq M_2 \int_0^T s^\beta q \left| t^{1-\alpha^2} - (t + s)^{1-\alpha^2} \right|^q / p \, ds.
\]

Then

\[
\int_0^T (C_s)^q \|v(.)\|_{L^p(t,T+s)}^q \, ds \leq M_2 \int_0^T t^\beta q \left[ 1 - (1 + u)^{1-\alpha^2} \right]^q / p \, du.
\]

Taking $t > N_0 + 1$ we obtain

\[
\int_0^T (C_s)^q \|v(.)\|_{L^p(t,T+s)}^q \, ds \leq M_2 \int_0^\infty t^\beta q \left[ 1 - (1 + u)^{1-\alpha^2} \right]^q / p \, du.
\]

This last inequality together with the fact that $\int_0^1 u^\beta q \left[ 1 - (1 + u)^{1-\alpha^2} \right]^q / p \, du < +\infty$, implies that

\[
\sup_{N_0 \leq T \leq N_0 + 1} \left( \int_0^T (C_s)^q \|v(.)\|_{L^p(t,s+t)}^q \, ds \right) = O(t^{\beta + 1_\rho + 1_\beta - \alpha^2}).
\]  
\[\text{(29)}\]

Moreover since $\beta + \frac{1}{p} + \frac{1}{q} - \alpha^2 < 0$, then \( \sup_{N_0 \leq T \leq N_0 + 1} \left( \frac{1}{1 - \alpha^2(t+T)} \right)^1 / q \right) = O(1) \) as \( t \to +\infty \). We deduce as in the proof of Theorem 7 that

\[
\sup_{N \leq T \leq N + 1} \left( C_T \|v(.)\|_{L^p(t,T+t)}g(t) \right) = O(t^{\frac{1}{p} - \alpha^2}), \text{ as } t \to +\infty \text{ which implies, since } \frac{1}{p} - \alpha^2 < 0, \text{ that}
\]

\[
\sup_{N_0 \leq T \leq N_0 + 1} \left( C_T \|v(.)\|_{L^p(t,t+T)}g(t) \right) \to 0, \text{ as } t \to +\infty.
\]  
\[\text{(30)}\]

The conclusion follows again from similar techniques as in the proof of Theorem 7. \( \square \)

**Remark 3.** If $C_I$ is bounded, then for all $1 < p, q < +\infty$ such that $\frac{1}{p} + \frac{1}{q} < 1$, the state $z(t)$ exponentially tends to 0.

The next theorem constitutes our third main result and concerns the asymptotic behavior of the whole state $z(t)$ of (1).

**Theorem 9.** Let assumptions of Theorem 4 hold and suppose that $B_*$ is $(p, q)$-admissible such that $\frac{1}{p} + \frac{1}{q} < 1$ and $C_I = O(t^\beta)$, as $t \to +\infty$, for some $\beta \in \mathbb{R}$. Then for all $z_0 = z_{a0} + z_{z0}$ such that $(z_{a0}, z_{z0}) \in D(A_{ua}) \times H_\sigma$, the system (1) controlled with (8) has a unique mild solution $z \in \mathcal{C}([0, +\infty[; H)$ and we have:

1. for $\alpha \geq 1$:

\[
\|z(t)\| = O(t^{-1/\alpha}), \text{ as } t \to +\infty.
\]  
\[\text{(31)}\]

2. for $\sup(\frac{1}{p/2}, [\beta + \frac{1}{p} + \frac{1}{q}] / 2) < \alpha \leq 1$:

\[
\|z(t)\| = O(t^{-\alpha/2}), \text{ as } t \to +\infty.
\]  
\[\text{(32)}\]

**Proof.** Let $z_0 = z_{a0} + z_{z0}$ such that $(z_{a0}, z_{z0}) \in D(A_{ua}) \times H_\sigma$. It is easy to check that $z(t) = z_a(t) + z_z(t)$ is a solution of (1). To establish the uniqueness of the solution $z$, let $z^{(1)}, z^{(2)} \in \mathcal{C}([0, +\infty[; H)$ be two mild solutions of (1) such that $B_{ua} z^{(1)}(t) \in H_\sigma$ and $B_{ua} z^{(2)}(t) \in H_\sigma$, $\forall t \geq 0$. Let $T > 0$ be fixed. For all $t \in [0, T]$, we have

\[
z^{(1)}(t) - z^{(2)}(t) = \int_0^t (u^{(1)}(\tau) - u^{(2)}(\tau)) \, B \left( z^{(1)}(\tau) - z^{(2)}(\tau) \right) d\tau + \int_0^t [u^{(1)}(\tau) - u^{(2)}(\tau)] \, S(t - \tau) \, B \, z^{(2)}(\tau) d\tau.
\]
where \( v^{(i)}(t) = -\langle B_u z^{(i)}_u(t), z^{(i)}_u(t) \rangle \) for \( i \in \{1, 2\} \). Then
\[
\|z^{(1)}(t) - z^{(2)}(t)\| \leq C_t \|v^{(1)}\|_{L^p(0,T)} \|z^{(1)} - z^{(2)}\|_{L^q(0,T; H)} + C_t \|v^{(1)} - v^{(2)}\|_{L^p(0,T)} \|z^{(2)}\|_{L^q(0,T; H)},
\]
and hence
\[
\|z^{(1)}(t) - z^{(2)}(t)\| \leq a(T) \|z^{(1)} - z^{(2)}\|_{L^q(0,T; H)} + b(T) \|v^{(1)} - v^{(2)}\|_{L^p(0,T)},
\]
where \( a(T) = C_T \|B\|_{L(D(A_{uo}), H)} \|z_u^{(1)}\|_{L^2_p(0,T)} \) and \( b(T) = C_T \|z^{(2)}\|_{L^q(0,T; H)}. \)

Moreover
\[
|v^{(1)}(\tau) - v^{(2)}(\tau)| \leq | < B_u z^{(1)}(\tau), z^{(1)}(\tau) - z^{(2)}(\tau) > | + | < B_u z^{(2)}(\tau), z^{(1)}(\tau) - z^{(2)}(\tau) > |
\]
Then
\[
|v^{(1)}(\tau) - v^{(2)}(\tau)| \leq \|B\|_{L(D(A_{uo}), H)} \left[ \|z^{(1)}(\tau)\| \|z^{(2)}(\tau)\| \right] \|z^{(1)}(\tau) - z^{(2)}(\tau)\|.
\]

Using the fact that \( z^{(1)}, z^{(2)} \in C([0, +\infty]; H) \), we deduce that
\[
\|v^{(1)}(\tau) - v^{(2)}(\tau)\|_{L^p(0,T)} \leq c(T) \|z^{(1)}(\tau) - z^{(2)}(\tau)\|_{L^p(0,T; H)},
\]
with \( c(T) = \|B\|_{L(D(A_{uo}), H)} \left[ \sup_{\tau \in [0,T]} \|z^{(1)}(\tau)\| \|z^{(2)}(\tau)\| \right]. \)

Injecting (34) in (33) we obtain
\[
\|z^{(1)}(t) - z^{(2)}(t)\| \leq a(T) \|z^{(1)} - z^{(2)}\|_{L^q(0,T; H)} + b(T) \ c(T) \ |z^{(1)} - z^{(2)}|_{L^p(0,T; H)}.
\]

Then
\[
\|z^{(1)}(t) - z^{(2)}(t)\| \leq M(T) \left[ \|z^{(1)} - z^{(2)}\|^q_{L^q(0,T; H)} + \|z^{(1)} - z^{(2)}\|^p_{L^p(0,T; H)} \right],
\]
where \( M(T) = \max(a(T), b(T)c(T)) \). In the other word, the scalar function \( \varphi(t) = \|z^{(1)}(t) - z^{(2)}(t)\| \) verifies
\[
\varphi(t) \leq M(T) \int_0^T \left[ \varphi^p(\tau) + \varphi^q(\tau) \right] d\tau. \tag{35}
\]

It follows from Gronwall’s inequality (see [2]), that \( \varphi(t) = 0 \) for all \( t \in [0, T] \) which implies the uniqueness of the solution of (1).

Finally, Since \( z_u(t) \) exponentially tends to zero, then the estimates (31) and (32) of \( z(t) \) follow from the ones of (21) and (22). \( \square \)

3. An example: heat equation

Bilinear controls are efficient in modeling of reaction-diffusion-convection processes and various chemical or biological chain reactions. For example, a nuclear fission results from the collision of neutrons with active uranium nuclei, and leads to the occurrence of new neutrons which, in turn, react with active nuclei in the same way and so on (see [1,8,24,27]). A simplified model is given by the following equation:
\[
\begin{align*}
\frac{\partial z(t, x)}{\partial t} &= \Delta z(t, x) + v(t)a(x)z(t, x), \quad \text{on } [0, +\infty[ \times \Omega \\
\frac{\partial z}{\partial n} &= 0, \quad \text{on } \Gamma \\
z(0, x) &= z_0, \quad \text{on } \Omega
\end{align*}
\] \tag{36}

where \( v(t) \) is the locally distributed control and \( a(x) \) characterizes the location and repartition of the control. Here the evolution domain \( \Omega \) is an open bounded domain in \( \mathbb{R}^d \) with \( C^\infty \) boundary \( \Gamma \).

To study this system, we take \( H=L^2(\Omega), Az=\Delta z, \) for all \( z \in D(A) = \{ z \in L^2(\Omega) \ / \ \frac{\partial z}{\partial n} = 0 \ \text{on } \Gamma \} \) and \( Bz=az \) and \( B_{zu}y_u = a(x) < y_u, \varphi > \varphi_1 \) with \( \varphi \in L^2(\Gamma) (\gamma \geq 2) \). We suppose that \( a \notin L^{\gamma+2}(\Omega) \), so that \( B \) is unbounded from \( L^2(\Omega) \) to \( L^2(\Omega) \), but it is bounded from \( L^2(\Omega) \) into the space \( X=L^{2/(2+\gamma)}(\Omega) \).
We can state the following result:

**Proposition 2.** Suppose that a satisfies \( a(x) \geq 0 \) on \( \Omega \) and \( \int_\Omega a(x)dx \neq 0 \).

Then the control \( v(t) = -\left( \int_\Omega a(x)dx \right)^{a} \left( \int_\Omega z_u(x, t) \ dx \right)^{2a} \) strongly stabilizes (36) with the estimate (31) for \( \alpha \geq 1 \), and the estimate (32) for \( \sup \left( \frac{1}{p^{1/2}}, \left[ \beta + \frac{1}{p} + \frac{1}{q} \right]^{1/2} \right) < \alpha \leq 1 \).

**Proof.** It is easy to check from \( a(x) \geq 0 \) on \( \Omega \) and \( \int_\Omega a(x)dx \neq 0 \) that the condition (20) hold. Now let \( 1 < p, q, r < +\infty \) such that \( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \) and let us estimate the norm \( C_t \) of the bilinear operator \( \Phi_t(u, z) = \int_0^t u(\tau)S(t - \tau)B_z(z(\tau))d\tau \).

From [13] there exists \( k_y \geq 0 \) such that for all \( 0 \leq \tau \leq t \), we have:

\[
S(t - \tau)B_z(z(\tau)) \in L^2(\Omega) \quad \text{and} \quad \|S(t - \tau)B_z(z(\tau))\|_{L^2(\Omega)} \leq k_y (t - \tau)^{d/2} (1/2 - 1/q^*), \|B_z(z(\tau))\|_{L^q(\Omega)},
\]

where \( q^* = \frac{2q}{2q + r} \). Then

\[
\|S(t - \tau)B_z(z(\tau))\|_{L^2(\Omega)} \leq k_y (t - \tau)^{-d/2} \|B_z(z(\tau))\|_{L^{q}(\Omega)},
\]

and hence

\[
\|\Phi_t(u, z)\| \leq k_y \int_0^t |u(\tau)| (t - \tau)^{-d/2} \|a(\cdot) z(\tau)\|_{L^{q}(\Omega)} d\tau.
\]

Using Schwarz’s inequality we obtain

\[
\|\Phi_t(u, z)\| \leq k_y \|a(\cdot)\|_{L^q(\Omega)} \int_0^t |u(\tau)| (t - \tau)^{-d/2} \|z(\tau)\|_{L^2(\Omega)} d\tau.
\]

It follows that

\[
\|\Phi_t(u, z)\| \leq k_y \|a(\cdot)\|_{L^q(\Omega)} \|u(\tau)\|_{L^p(0, t)} \|t - \cdot\|^{-d/2} \|L^q(0, t)\| \|z\|_{L^2(0, t)} \|L^2(\Omega)\|
\]

Moreover, it is easy to check that: \( \|t - \cdot\|^{-d/2} \|L^q(0, t)\| = \left( \frac{d}{1 - d/2 + d(q^*)} \right)^{d/2} \). It follows that \( B_z \) is \( (p, q) \)-admissible with norm \( C_t = O((d/2)q + (1/r)) \), and we have \( \frac{d}{2} + \frac{1}{r} + \frac{1}{p} + \frac{1}{q} - 1 = \frac{-d}{2q} < 0 \).

Then applying Theorem 9 we conclude that the control \( v(t) = -\left( \int_\Omega a(x)dx \right)^{a} \left( \int_\Omega z_u(x, t) \ dx \right)^{2a} \) strongly stabilizes (36) and guarantees the estimate (31) for \( \alpha \geq 1 \) and the estimate (32) for \( \sup \left( \frac{1}{p^{1/2}}, \left[ \beta + \frac{1}{p} + \frac{1}{q} \right]^{1/2} \right) < \alpha \leq 1 \).

**Remark 4.** We note that the existing result of [7] does not apply to study the stabilization of (36), since \( X = L^2(\Omega) \) is not a Hilbert space.

4. Conclusion

In this work, a family of quadratic controls has been proposed to study the strong stabilization of unbounded parabolic bilinear systems, an explicit estimation of the stabilized state is given. The paper leaves the open question of whether the obtained estimate can be established for other classes of unbounded bilinear systems, including hyperbolic case.

References


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A. Boutoulout et al. / Mathematics and Computers in Simulation xxx (2013) xxx–xxx