Research Article

On Controllability and Observability of Fuzzy Dynamical Matrix Lyapunov Systems

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We provide a way to combine matrix Lyapunov systems with fuzzy rules to form a new fuzzy system called fuzzy dynamical matrix Lyapunov system, which can be regarded as a new approach to intelligent control. First, we study the controllability property of the fuzzy dynamical matrix Lyapunov system and provide a sufficient condition for its controllability with the use of fuzzy rule base. The significance of our result is that given a deterministic system and a fuzzy state with rule base, we can determine the rule base for the control. Further, we discuss the concept of observability and give a sufficient condition for the system to be observable. The advantage of our result is that we can determine the rule base for the initial value without solving the system.

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1. INTRODUCTION

The importance of control theory in applied mathematics and its occurrence in several problems such as mechanics, electromagnetic theory, thermodynamics, and artificial satellites are well known. In general, fuzzy systems are mainly classified into three categories, namely pure fuzzy systems, T-S fuzzy systems, and fuzzy logic systems, using fuzzifiers and defuzzifiers. In this paper, we use fuzzy matrix Lyapunov system to describe fuzzy logic system. The purpose of this paper is to provide sufficient conditions for controllability and observability of first-order fuzzy matrix Lyapunov system modeled by

\[
\begin{align*}
X'(t) &= A(t)X(t) + X(t)B(t) + F(t)U(t), \\
X(0) &= X_0, \quad t > 0, \\
Y(t) &= C(t)X(t) + D(t)U(t),
\end{align*}
\]

where \(U(t)\) is an \(n \times n\) fuzzy input matrix called fuzzy control and \(Y(t)\) is an \(n \times n\) fuzzy output matrix. Here \(A(t), B(t), F(t), C(t),\) and \(D(t)\) are matrices of order \(n \times n\), whose elements are continuous functions of \(t\) on \(J = [0, T] \subset \mathbb{R}\) \((T > 0)\).

The problem of controllability and observability for a system of ordinary differential equations was studied by Barnett and Cameron [1] and for matrix Lyapunov systems by Murty et al.[2]. Fuzzy control usually decomposes a complex system into several subsystems according to the human expert’s understanding of the system and uses a simple control law to emulate the human control strategy. There exist two major types of fuzzy controllers, namely Mamdani fuzzy controllers and Takagi-Sugeno (TS) fuzzy controllers. They mainly differ in the consequence of fuzzy rules: the former uses fuzzy sets whereas the latter employs (linear) functions. Takagi and Sugeno [3, 4] propose a type of fuzzy model in which the consequent part of the rules is defined not by the membership function but by a crisp analytical function. More and more interest appears to shift towards TS fuzzy controllers in recent years, as evidenced by the increasing number of papers in this direction and due to their applications in real world problems (e.g., [5–12]).

Recently, the controllability and observability criteria for fuzzy dynamical control systems were discussed by Ding and Kandel [13, 14]. In this paper, by converting the fuzzy matrix Lyapunov system into a Kronecker product system we obtain sufficient conditions for controllability and observability of the system (1) satisfying (2).

The paper is well organized as follows. In Section 2, we present some basic definitions and results relating to fuzzy sets [13] and Kronecker product of matrices. Further, we obtain a unique solution of the system (1), when \(U(t)\) is a
crisp continuous matrix. In Section 3, we generate a fuzzy dynamical Lyapunov system, and also obtain its solution set. In Section 4, we present a sufficient condition for the controllability of the system and illustrate the results by suitable examples. In Section 5, we obtain a sufficient condition for the observability of the fuzzy dynamical Lyapunov system, and the theorem is highlighted by a suitable example. Finally, in Section 6, we present some conclusions and future works.

This paper extends some of the results of Ding and Kandel [13, 14] developed for system of fuzzy di...
A fuzzy set-valued mapping $F : J \rightarrow E^n$ is called fuzzy integrably bounded if $F_0(t)$ is integrably bounded.

**Definition 2.** Let $F : J \rightarrow E^n$ be a fuzzy integrably bounded mapping. The fuzzy integral of $F$ over $J$ denoted by $\int^t_J F(t)dt$ is defined level-set-wise by

$$\left[ \int^t_J F(t)dt \right]^\alpha = (A) \int^t_{f_\alpha} F(u)dt, \quad 0 < \alpha \leq 1. \quad (12)$$

Let $F : J \times E^n \rightarrow E^n$, and consider the fuzzy differential equation

$$u' = F(t, u), \quad u(0) = u_0. \quad (13)$$

**Definition 3.** A mapping $u : J \rightarrow E^n$ is a fuzzy weak solution to (13) if it is continuous and satisfies the integral equation

$$u(t) = u_0 + \int^t_0 F(s, u(s))ds, \quad \forall t \in J. \quad (14)$$

If $F$ is continuous, then this weak solution also satisfies (13) and we call it fuzzy strong solution to (13).

Now, we present some properties and rules for Kronecker products and basic results related to matrix Lyapunov systems.

**Definition 4** (see [2]). Let $A \in C^{m \times n}$ and $B \in C^{p \times q}$. Then the Kronecker product of $A$ and $B$ written $A \otimes B$ is defined to be the partitioned matrix

$$A \otimes B = \begin{bmatrix}
    a_{11}B & a_{12}B & \cdots & a_{1n}B \\
    a_{21}B & a_{22}B & \cdots & a_{2n}B \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1}B & a_{m2}B & \cdots & a_{mn}B
\end{bmatrix} \quad (15)$$

which is an $mp \times nq$ matrix and is in $C^{mp \times nq}$.

**Definition 5** (see [2]). Let $A = [a_{ij}] \in C^{m \times n}$; one denotes

$$\hat{A} = \text{Vec}A = \begin{bmatrix}
    A_1 \\
    A_2 \\
    \vdots \\
    A_n
\end{bmatrix}, \quad \text{where} \quad A_j = \begin{bmatrix}
    a_{1j} \\
    a_{2j} \\
    \vdots \\
    a_{mj}
\end{bmatrix}, \quad (1 \leq j \leq n). \quad (16)$$

The Kronecker product has the following properties and rules [2].

1. $(A \otimes B)^* = A^* \otimes B^*$ ($A^*$ denotes transpose of $A$).
2. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.
3. The mixed product rule
   $$(A \otimes B)(C \otimes D) = (AC \otimes BD).$$

This rule holds, provided the dimension of the matrices is such that the various expressions exist.

4. $\|A \otimes B\| = \|A\| \|B\|$.
5. If $A(t)$ and $B(t)$ are matrices, then
   $$(A \otimes B)' = A' \otimes B + A \otimes B' \quad (\text{d/dt}).$$
6. $\text{Vec}(AYB) = (B^* \otimes A)\text{Vec}Y$.
7. If $A$ and $B$ are matrices both of order $n \times n$, then
   (i) $\text{Vec}(AX) = (In \otimes A)\text{Vec}X$,  
   (ii) $\text{Vec}(XA) = (A^* \otimes In)\text{Vec}X$.

Now, by applying the Vec operator to the matrix Lyapunov system (1) satisfying (2) and using the above properties, we have

$$\hat{X}'(t) = G(t)\hat{X}(t) + (In \otimes F(t))\hat{U}(t), \quad \hat{X}(0) = \hat{X}_0, \quad (17)$$

$$\hat{Y}(t) = (In \otimes C(t))\hat{X}(t) + (In \otimes D(t))\hat{U}(t), \quad (18)$$

where $G(t) = (B^* \otimes In) + (In \otimes A)$ is an $n^2 \times n^2$ matrix and $\hat{X} = \text{Vec}X(t)$, $\hat{U} = \text{Vec}U(t)$ are column matrices of order $n^2$.

The corresponding linear homogeneous system of (17) is

$$\hat{X}'(t) = G(t)\hat{X}(t), \quad \hat{X}(0) = \hat{X}_0. \quad (19)$$

**Lemma 1.** Let $\phi(t)$ and $\psi(t)$ be the fundamental matrices for the systems

$$X'(t) = A(t)X(t), \quad X(0) = I_n, \quad (20)$$

$$[X^*(t)]' = B^*(t)X^*(t), \quad X(0) = I_n, \quad (21)$$

respectively. Then the matrix $\psi(t) \otimes \phi(t)$ is a fundamental matrix of (19) and the solution of (19) is $\hat{X}(t) = (\psi(t) \otimes \phi(t))\hat{X}_0$.

**Proof.** Consider

$$(\psi(t) \otimes \phi(t))' = \big((\psi(t)' \otimes \phi(t)) + (\psi(t) \otimes \phi'(t))\big)$$

$$= (B^*(t)\psi(t) \otimes \phi(t)) + (\psi(t) \otimes A(t)\phi(t))$$

$$= (B^*(t) \otimes In)(\psi(t) \otimes \phi(t)) + (In \otimes A(t))(\psi(t) \otimes \phi(t))$$

$$= [B^*(t) \otimes In + In \otimes A(t)](\psi(t) \otimes \phi(t))$$

$$= G(t)(\psi(t) \otimes \phi(t)). \quad (22)$$

Also $\psi(0) \otimes \phi(0) = I_n \otimes I_p = I_{np}$.

Hence, $\psi(t) \otimes \phi(t)$ is a fundamental matrix of (19). Clearly, $\hat{X}(t) = (\psi(t) \otimes \phi(t))\hat{X}_0$ is a solution of (19). \hfill \Box

**Theorem 2.** Let $\phi(t)$ and $\psi(t)$ be the fundamental matrices for the systems (20) and (21). Then the unique solution of the initial value problem (17) is given by

$$\hat{X}(t) = (\psi(t) \otimes \phi(t))\hat{X}_0 + \int^t_0 \big((\psi(t-s) \otimes \phi(t-s))(In \otimes \hat{F}(s))\hat{U}(s)ds. \quad (23)$$
Proof. First we show that the solution of (17) is of the form 
\( \hat{X}(t) = (\psi(t) \otimes \phi(t))\hat{X}_0 + \hat{X}(t) \), where \( \hat{X}(t) \) is a particular solution of (17) and is given by

\[
\hat{X}(t) = \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes \hat{F}(s))\hat{U}(s)ds. \tag{24}
\]

Let \( u(t) \) be any other solution of (17), write \( w(t) = u(t) - \hat{X}(t) \), then \( w \) satisfies (19), hence \( w = (\psi(t) \otimes \phi(t))\hat{X}_0, u(t) = (\psi(t) \otimes \phi(t))\hat{X}_0 + \hat{X}(t) \).

Consider the vector \( \tilde{X}(t) = (\psi(t) \otimes \phi(t))\nu(t) \), where \( \nu(t) \) is an arbitrary vector to be determined so as to satisfy (17),

\[
\tilde{X}'(t) = (\psi(t) \otimes \phi(t))'\nu(t) + (\psi(t) \otimes \phi(t))\nu'(t)
\implies G(t)\tilde{X}(t) + (I_n \otimes \hat{F}(t))\hat{U}(t)
= G(t)(\psi(t) \otimes \phi(t))\nu(t) + (\psi(t) \otimes \phi(t))\nu'(t)
\implies (I_n \otimes \hat{F}(t))\hat{U}(t)
= (\psi(t) \otimes \phi(t))\nu(t)
\implies \nu(t)
= \int_0^t (\psi^{-1}(s) \otimes \phi^{-1}(s))(I_n \otimes \hat{F}(s))\hat{U}(s)ds.
\]

Hence, the desired expression follows immediately by noting the fact that \( \phi(t)\phi^{-1}(s) = \phi(t - s) \) and \( \psi(t)\psi^{-1}(s) = \psi(t - s) \).

\[\square\]

3. FORMATION OF FUZZY DYNAMICAL LYAPUNOV SYSTEMS

Let \( u_i(t) \in E^i, t \in J, i = 1, 2, \ldots, n^2 \), and define

\[
\hat{U}(t) = (u_1(t), u_2(t), \ldots, u_{n^2}(t))
= u_1(t) \times u_2(t) \times \cdots \times u_{n^2}(t)
= \{ (u^a_1(t), u^a_2(t), \ldots, u^a_{n^2}(t)) : \alpha \in [0, 1] \}
= \{ (\hat{u}_1(t), \hat{u}_2(t), \ldots, \hat{u}_{n^2}(t)) : \hat{u}_i(t) \in u^a_i(t), \alpha \in [0, 1] \},
\]

where \( u^a_i(t) \) is the \( \alpha \)-level set of \( u_i(t) \). From the above definition of \( \hat{U}(t) \) and Theorem 1, it can be easily seen that \( \hat{U}(t) \in E^{n^2} \).

Now by using the fuzzy control \( \hat{U}(t) \), we show that the following system

\[
\hat{X}'(t) = G(t)\hat{X}(t) + (I_n \otimes F(t))\hat{U}(t), \quad \hat{X}(0) = \hat{X}_0, \tag{27}
\]

\[
\hat{Y}(t) = (I_n \otimes C(t))\hat{X}(t) + (I_n \otimes D(t))\hat{U}(t) \tag{28}
\]
determines a fuzzy system.

Assume that \( \hat{U}(t) \) is continuous in \( E^{n^2} \). The set \( \hat{U}^a = u_1(t) \times u_2(t) \times \cdots \times u_{n^2}(t) \) is a convex and compact set in \( R^{n^2} \). For any positive number \( T \), consider the following differential inclusions:

\[
\hat{X}'(t) \in G(t)\hat{X}(t) + (I_n \otimes F(t))\hat{U}^a(t), \quad t \in [0, T],
\]

\[
\hat{X}(0) = \hat{X}_0. \tag{29}
\]

Let \( \hat{X}^a \) be the solution of (29) satisfying (30).

Claim (i). \( |\hat{X}(t)|^a \in P_{\delta}(R^{n^2}), \) for every \( 0 \leq \alpha \leq 1, t \in [0, T] \).

First, we prove that \( \hat{X}^a \) is nonempty, compact, and convex in \( C([0, T], R^{n^2}) \). Since \( \hat{U}^a(t) \) has measurable selection, we have that \( \hat{X}^a \) is nonempty.

Let \( K = \max_{t \in [0, T]} \| \phi(t) \|, L = \max_{t \in [0, T]} \| \psi(t) \|, M = \max_{t \in [0, T]} \| F(t) \|, \).

If for any \( \hat{X} \in \hat{X}^a \), then there is a selection \( u(t) \in \hat{U}^a(t) \) such that

\[
\hat{X}(t) = (\psi(t) \otimes \phi(t))\hat{X}_0
+ \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s))u(s)ds. \tag{31}
\]

Then

\[
||\hat{X}(t)|| \leq ||(\psi(t) \otimes \phi(t))\hat{X}_0||
+ \int_0^t ||(\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s))u(s)||ds
= ||\psi(t)||||\phi(t)||||\hat{X}_0||
+ \int_0^t ||\psi(t-s)||||\phi(t-s)||||F(s)||||u(s)||ds
\leq KL||\hat{X}_0|| + KLNMT. \tag{32}
\]

Thus \( \hat{X}^a \) is bounded.

For any \( t_1, t_2 \in [0, T] \),

\[
\hat{X}(t_1) - \hat{X}(t_2)
= (\psi(t_1) \otimes \phi(t_1))\hat{X}_0
+ \int_0^{t_1} (\psi(t_1-s) \otimes \phi(t_1-s))(I_n \otimes F(s))u(s)ds
- (\psi(t_2) \otimes \phi(t_2))\hat{X}_0
- \int_0^{t_2} (\psi(t_2-s) \otimes \phi(t_2-s))(I_n \otimes F(s))u(s)ds.
\]

(33)
Therefore
\[ \| \hat{X}(t_1) - \hat{X}(t_2) \| \leq \| (\psi(t_1) \otimes \phi(t_1) - (\psi(t_2) \otimes \phi(t_2)) \| \| \hat{X}_0 \| \\
+ \int_{t_1}^{t_2} \| (\psi(t_1 - s) \otimes \phi(t_1 - s)) (I_n \otimes F(s)) u(s) \| ds \\
+ \int_0^{t_1} \| (\psi(t_1 - s) \otimes \phi(t_1 - s) - (\psi(t_2 - s) \otimes \phi(t_2 - s)) \| ds. \] (34)

Since \( \phi(t) \) and \( \psi(t) \) are both uniformly continuous on \( [0, T] \), \( \hat{X} \) is equicontinuous. Thus, \( \hat{X}^a \) is relatively compact. If \( \hat{X}^a \) is closed, then it is compact.

Let \( \hat{X}_k \in \hat{X}^a \) and \( \hat{X}_k \to \hat{X} \). For each \( \hat{X}_k \), there is a \( u_k \in \hat{U}^a(t) \) such that
\[ \hat{X}_k(t) = (\psi(t) \otimes \phi(t)) \hat{X}_0 \\
+ \int_0^t (\psi(t - s) \otimes \phi(t - s)) (I_n \otimes F(s)) u_k(s) ds. \] (35)

Since \( u_k \in \hat{U}^a(t) \) is closed, then there exists a subsequence \( \{u_k \}_k \) converging weakly to \( u \in \hat{U}^a(t) \). From Mazur's theorem [20], there exists a sequence of numbers \( \lambda_j > 0 \), \( \sum \lambda_j = 1 \) such that \( \sum \lambda_j u_k \) converges strongly to \( u \).

Thus, from (35) we have
\[ \sum \lambda_j \hat{X}_k(t) = \sum \lambda_j (\psi(t) \otimes \phi(t)) \hat{X}_0 \\
+ \int_0^t (\psi(t - s) \otimes \phi(t - s)) (I_n \otimes F(s)) \sum \lambda_j u_k(s) ds. \] (36)

From Fatou's lemma, taking the limit as \( j \to \infty \) on both sides of (36), we have
\[ \hat{X}(t) = (\psi(t) \otimes \phi(t)) \hat{X}_0 \\
+ \int_0^t (\psi(t - s) \otimes \phi(t - s)) (I_n \otimes F(s)) u(s) ds. \] (37)

Thus, \( \hat{X}(t) \in \hat{X}^a \), and hence \( \hat{X}^a \) is closed.

Let \( \hat{X}_1, \hat{X}_2 \in \hat{X}^a \), then there exist \( u_1, u_2 \in \hat{U}^a(t) \) such that
\[ \hat{X}_1(t) = G(t) \hat{X}_1(t) + (I_n \otimes F(t)) u_1(t), \]
\[ \hat{X}_2(t) = G(t) \hat{X}_2(t) + (I_n \otimes F(t)) u_2(t). \] (38)

Let \( \hat{X} = \lambda \hat{X}_1(t) + (1 - \lambda) \hat{X}_2(t), 0 \leq \lambda \leq 1 \), then
\[ \hat{X}' = \lambda \hat{X}'_1(t) + (1 - \lambda) \hat{X}'_2(t) \]
\[ = \lambda (G(t) \hat{X}_1(t) + (I_n \otimes F(t)) u_1(t)) \\
+ (1 - \lambda) (G(t) \hat{X}_2(t) + (I_n \otimes F(t)) u_2(t)) \] (39)
\[ = G(t) [\lambda \hat{X}_1(t) + (1 - \lambda) \hat{X}_2(t)] \]
\[ + (I_n \otimes F(t)) [\lambda u_1(t) + (1 - \lambda) u_2(t)]. \]

Since \( \hat{U}^a(t) \) is convex, \( \lambda u_1(t) + (1 - \lambda) u_2(t) \in \hat{U}^a(t) \), we have
\[ \hat{X}'(t) = G(t) \hat{X}(t) + (I_n \otimes F(t)) \hat{U}^a(t), \] (40)

that is \( \hat{X} \in \hat{X}^a \). Thus \( \hat{X}^a \) is convex. Therefore, \( \hat{X}^a \) is nonempty, compact, and convex in \( C([0, T], R^n) \). Thus, from Arzela-Ascoli theorem, we know that \( [\hat{X}(t)]^a \) is compact in \( R^n \) for every \( t \in [0, T] \). Also it is obvious that \( [\hat{X}(t)]^a \) is convex in \( R^n \). Thus, we have \( [\hat{X}(t)]^a \in P_e(R^n) \), for every \( t \in [0, T] \). Hence the claim.

Claim (ii). \( [\hat{X}(t)]^a \subset [\hat{X}(t)]^{a_1} \), for all \( 0 \leq a_1 \leq a_2 \leq 1 \).
Let \( 0 \leq a_1 \leq a_2 \leq 1 \). Since \( \hat{U}^{a_2}(t) \subset \hat{U}^{a_1}(t) \), we have
\[ \hat{U}^{a_2}(t) = u_{a_2}^1(t) \times u_{a_2}^2(t) \times \cdots \times u_{a_2}^n(t) \]
\[ \subset u_{a_1}^1(t) \times u_{a_1}^2(t) \times \cdots \times u_{a_1}^n(t) \] (41)
\[ = \hat{U}^{a_1}(t). \]

Thus, we have the selection inclusion \( S_{U^{a_2}(t)} \subset S_{U^{a_1}(t)} \) and the following inclusion:
\[ \hat{X}'(t) \subset G(t) \hat{X} + (I_n \otimes F(t)) \hat{U}^{a_1}(t) \] (42)

Consider the differential inclusions
\[ \hat{X}(t) \subset G(t) \hat{X} + (I_n \otimes F(t)) \hat{U}^{a_1}(t), \quad t \in [0, T], \] (43)
\[ \hat{X}(t) \subset G(t) \hat{X} + (I_n \otimes F(t)) \hat{U}^{a_2}(t), \quad t \in [0, T]. \] (44)

Let \( \hat{X}^{a_2} \) and \( \hat{X}^{a_1} \) be the solution sets of (43) and (44), respectively. Clearly, the solution of (43) satisfies the following inclusion:
\[ \hat{X}(t) \subset (\psi(t) \otimes \phi(t)) \hat{X}_0 \\
+ \int_0^t (\psi(t - s) \otimes \phi(t - s)) (I_n \otimes F(s)) S_{U^{a_1}(s)} ds \] (45)

Thus \( \hat{X}^{a_2} \subset \hat{X}^{a_1} \), and hence \( \hat{X}^{a_2}(t) \subset \hat{X}^{a_1}(t) \). Hence the claim.
Claim (iii). If \{\alpha_k\} is a nondecreasing sequence converging to \(\alpha > 0\), then \(\hat{X}_x(t) = \bigcap_{k \geq 1} \hat{X}^{\alpha_k}(t)\).

Let \(\hat{U}^{\alpha_k}(t) = u^{\alpha_k}_1 \times u^{\alpha_k}_2 \times \cdots \times u^{\alpha_k}_{n_k}\), \(\hat{U}^a(t) = u^a_1 \times u^a_2 \times \cdots \times u^a_{n_k}\), and consider the inclusions
\[
\hat{X}'(t) \in G(t)\hat{X} + (I_n \otimes F(t)) \hat{U}^{\alpha_k}(t),
\]
\[
\hat{X}'(t) \in G(t)\hat{X} + (I_n \otimes F(t)) \hat{U}^a(t).
\]

Let \(\hat{X}^{\alpha_k}\) and \(\hat{X}^a\) be the solution sets of (46) and (47), respectively. Since \(u_i(t)\) is a fuzzy set and from Theorem 1, we have
\[
u_i^a = \bigcap_{k \geq 1} u_i^{\alpha_k}.
\]

Consider
\[
\hat{U}^a(t) = u^a_1 \times u^a_2 \times \cdots \times u^a_{n_k},
\]
\[
= \bigcap_{k \geq 1} u^{\alpha_k}_1 \times \bigcap_{k \geq 1} u^{\alpha_k}_2 \times \cdots \times \bigcap_{k \geq 1} u^{\alpha_k}_{n_k}
\]
\[
= \bigcap_{k \geq 1} (u^{\alpha_k}_1 \times u^{\alpha_k}_2 \times \cdots \times u^{\alpha_k}_{n_k})
\]
\[
= \bigcap_{k \geq 1} \hat{U}^{\alpha_k}(t)
\]
and then \(S_{\hat{U}^a(t)}^1 = S_{\bigcap_{k \geq 1} \hat{U}^{\alpha_k}(t)}^1\). Therefore
\[
\hat{X}'(t) \in G(t)\hat{X} + (I_n \otimes F(t)) \hat{U}^{\alpha_k}(t)
\]
\[
= G(t)\hat{X} + (I_n \otimes F(t)) \bigcap_{k \geq 1} \hat{U}^{\alpha_k}(t)
\]
\[
\in G(t)\hat{X} + (I_n \otimes F(t)) \hat{U}^a(t), \quad k = 1, 2, \ldots.
\]

Thus, we have \(\hat{X}^a \subset \bigcap_{k \geq 1} \hat{X}^{\alpha_k}\), which implies that
\[
\hat{X}^a \subset \bigcap_{k \geq 1} \hat{X}^{\alpha_k}.
\]

Let \(\hat{X}\) be the solution set to the inclusion
\[
\hat{X}'(t) \in G(t)\hat{X} + (I_n \otimes F(t)) \hat{U}^a(t), \quad k \geq 1.
\]
Then,
\[
\hat{X}(t) \in (\psi(t) \otimes \phi(t))\hat{X}_0
\]
\[
+ \int_0^t (\psi(t-s) \otimes \phi(t-s)) (I_n \otimes F(s)) S_{\hat{U}^a(s)}^1 ds.
\]

It follows that
\[
\hat{X}(t) \in (\psi(t) \otimes \phi(t))\hat{X}_0
\]
\[
+ \int_0^t (\psi(t-s) \otimes \phi(t-s)) (I_n \otimes F(s)) S_{\hat{U}^a(s)}^1 ds.
\]

This implies that \(\hat{X} \in \hat{X}^a\). Therefore,
\[
\bigcap_{k \geq 1} \hat{X}^{\alpha_k} \subset \hat{X}^a.
\]

From (51) and (55), we have
\[
\hat{X}^a = \bigcap_{k \geq 1} \hat{X}^{\alpha_k},
\]
and hence,
\[
\hat{X}^a(t) = \bigcap_{k \geq 1} \hat{X}^{\alpha_k}(t).
\]

From Claims 3–3 and applying Theorem 1, there exists \(\hat{X}(t) \in E^{n^2}\) on \([0, T]\) such that \(\hat{X}^a(t)\) is a solution to the differential inclusions (29) and (30). Hence, the system (27), (28) is a fuzzy dynamical Lyapunov system, and it can be expressed as
\[
\hat{X}'(t) = G(t)\hat{X}(t) + (I_n \otimes F(t)) \hat{U}(t), \quad \hat{X}(0) = \{\hat{X}_0\},
\]
\[
\hat{Y}(t) = (I_n \otimes C(t))\hat{X}(t) + (I_n \otimes D(t)) \hat{U}(t).
\]

The solution set of the fuzzy dynamical system (58), (59) is given by
\[
\hat{X}(t) \in (\psi(t) \otimes \phi(t))\hat{X}_0
\]
\[
+ \int_0^t (\psi(t-s) \otimes \phi(t-s)) (I_n \otimes F(s)) S_{\hat{U}^a(s)}^1 ds.
\]

Remark 1. Consider a special case. If the input is in the form
\[
\hat{U}(t) = \hat{u}(t) \times \hat{u}_2(t) \times \cdots \times \hat{u}_i(t) \times \cdots \times \hat{u}_{n^2}(t),
\]
where \(\hat{u}_i(t) \in R^1, \ k \neq i,\) are crisp numbers, then the \(i\)th component of the solution set of (27) is a fuzzy set in \(R^1\).

Proof. The proof follows along similar lines as in the above discussion. \(\square\)
4. CONTROLLABILITY OF FUZZY DYNAMICAL LYAPUNOV SYSTEMS

In this section, we discuss the concept of controllability of the fuzzy system (58) satisfying (59).

Definition 6. The fuzzy system (58), (59) is said to be completely controllable if for any initial state $\hat{x}(0) = \hat{x}_0$ and any given final state $\hat{x}_f$ there exists a finite time $t_1 > 0$ and a control $\hat{u}(t)$, $0 \leq t \leq t_1$, such that $\hat{x}(t_1) = \hat{x}_f$.

Lemma 2. If $F$ is a fuzzy set, then $\int_0^T F \, dt = TF$.

Proof. Let $[F]^a$ be the $\alpha$-level set of $F$. Since

$$\int_0^T F \, dt \, dt = T[F]^a, \tag{62}$$

From the definition of fuzzy set, we have $\int_0^T F \, dt = TF$. □

Lemma 3. Let $P, Q$ be two fuzzy sets and let $h(t)$ be a nonzero continuous function on $[0, T]$, satisfying

$$\int_0^T h(t) P \, dt = \int_0^T h(t) Q \, dt \quad \text{(63)}$$

then $P = Q$.

Proof. For each $\alpha$-level, we have

$$\int_0^T h(t)[P]^a \, dt = \left[ \int_0^T h(t) P \, dt \right]^a = \left[ \int_0^T h(t) Q \, dt \right]^a = \int_0^T h(t)[Q]^a \, dt. \tag{64}$$

Suppose that $P \neq Q$, then for some $\alpha \in [0, 1]$, we have $[P]^a \neq [Q]^a$. Without loss of generality, we assume that $P, Q \in E^1$. Let $P_{max} = [P_{min}(\alpha), P_{max}(\alpha)]$ and $Q_{max} = [Q_{min}(\alpha), Q_{max}(\alpha)]$. Then, we have either (i) $P_{min}(\alpha) \neq Q_{min}(\alpha)$ or (ii) $P_{max}(\alpha) \neq Q_{max}(\alpha)$ holds.

If (i) holds, then

$$\int_0^T h(t) P_{min}(\alpha) \, dt \neq \int_0^T h(t) Q_{min}(\alpha) \, dt. \tag{65}$$

If (ii) holds, then

$$\int_0^T h(t) P_{max}(\alpha) \, dt \neq \int_0^T h(t) Q_{max}(\alpha) \, dt. \tag{66}$$

Thus, in both cases (i) and (ii), we have

$$\int_0^T h(t)[P_{min}(\alpha), P_{max}(\alpha)] \, dt \neq \int_0^T h(t)[Q_{min}(\alpha), Q_{max}(\alpha)] \, dt. \tag{67}$$

This implies that

$$\int_0^T h(t) P^a \, dt \neq \int_0^T h(t) Q^a \, dt, \tag{68}$$

which is a contradiction to (64). Hence $P = Q$. □

Definition 7 (see [20]). Let $u, v \in E^1, k \in R^1$, and let $[u]^a$ be the $\alpha$-level set of $u$. One defines the sum of $u$ and $v$ by

$$[u + v]^a = [u]^a + [v]^a = \{a + b : a \in [u]^a, b \in [v]^a\}, \tag{69}$$

the difference between $u$ and $v$ by

$$[u - v]^a = [u]^a - [v]^a = \{a - b : a \in [u]^a, b \in [v]^a\}, \tag{70}$$

and the scalar product by

$$[ku]^a = k[u]^a = \{ka : a \in [u]^a\}. \tag{71}$$

Definition 8 (see [20]). Let $x, y \in E^{n^2}$ and $x = x_1 \times x_2 \times \cdots \times x_{n^2}$, $y = y_1 \times y_2 \times \cdots \times y_{n^2}, x_i, y_i \in E^1, i = 1, 2, \ldots, n^2$. If $y = z + x$, then $z = y - x$ which is defined by

$$[z]^a = [y - x]^a = [y]^a - [x]^a = \begin{bmatrix} [y_1]^a - [x_1]^a \\ \vdots \\ [y_{n^2}]^a - [x_{n^2}]^a \end{bmatrix}. \tag{72}$$

If $y = w - x$, then $w = y + x$ which is defined by

$$[w]^a = [y + x]^a = [y]^a + [x]^a = \begin{bmatrix} [y_1]^a + [x_1]^a \\ \vdots \\ [y_{n^2}]^a + [x_{n^2}]^a \end{bmatrix}. \tag{73}$$

Definition 9 ([20]). Let

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n^2} \\ c_{21} & c_{22} & \cdots & c_{2n^2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n^21} & c_{n^22} & \cdots & c_{n^2n^2} \end{bmatrix} \tag{74}$$

be an $n^2 \times n^2$ matrix, $p = p_1 \times p_2 \times \cdots \times p_{n^2}$, let $p_i \in E^1, i = 1, 2, \ldots, n^2$, be a fuzzy set in $E^{n^2}$, and let $[p_i]^a$ be $\alpha$-level sets of $p_i$. Define the product $Cp$ of $C$ and $p$ as

$$[Cp]^a = C[p]^a = \begin{bmatrix} c_{11}[p_1]^a + \cdots + c_{1n^2}[p_{n^2}]^a \\ c_{21}[p_1]^a + \cdots + c_{2n^2}[p_{n^2}]^a \\ \vdots & \vdots & \ddots & \vdots \\ c_{n^21}[p_1]^a + \cdots + c_{n^2n^2}[p_{n^2}]^a \end{bmatrix}. \tag{75}$$

All these definitions yield the following lemma.

Lemma 4. $Cp$ is a fuzzy set in $E^{n^2}$.

Proof. The proof is similar to proof of Lemma 3.1 [13]. □
Theorem 3. The fuzzy system (58),(59) is completely controllable if the \( n \times n \) symmetric controllability matrix
\[
W(0, T) = \int_0^T (\psi(t - T) \otimes \phi(T - t))(I_n \otimes F(t))
\]
\[
\times (I_n \otimes F(t))^* (\psi(T - t) \otimes \phi(T - t))^* dt
\]
is nonsingular. Furthermore, the fuzzy control \( \hat{U}(t) \) which transfers the state of the system from \( \hat{X}(0) = \hat{X}_0 \) to a fuzzy state \( \hat{X}(T) = \hat{X}_f \) is in \( x_{f_1}, x_{f_2}, \ldots, x_{f_{n^2}} \) can be determined by the following fuzzy rule base:
\[
R : \text{IF } \hat{x}_1 \text{ is in } x_{f_1}, \ldots, \hat{x}_{n^2} \text{ is in } x_{f_{n^2}},
\]
\[
\text{THEN } \hat{u}_1 \text{ is in } u_1, \ldots, \hat{u}_{n^2} \text{ is in } u_{n^2},
\]

where
\[
(\hat{u}_1(t), \hat{u}_2(t), \ldots, \hat{u}_i(t), \ldots, \hat{u}_{n^2}(t))
\]
\[
= \frac{1}{T} \frac{1}{T} (I_n \otimes F(t))^{-1} (\psi(T - t) \otimes \phi(T - t))^{-1}
\]
\[
\times (\hat{x}_1(T), \hat{x}_2(T), \ldots, \hat{x}_{f_1}, \ldots, \hat{x}_{f_{n^2}}(T))
\]
\[
- (I_n \otimes F(t))^* (\psi(T - t) \otimes \phi(T - t))^*
\]
\[
\times W^{-1}(0, T)(\psi(T) \otimes \phi(T))\hat{X}_0, \quad i = 1, 2, \ldots, n^2.
\]

Proof. Suppose that the symmetric controllability matrix \( W(0, T) \) is nonsingular. Therefore \( W^{-1}(0, T) \) exists. Multiplying \( W^{-1}(0, T)(\psi(T) \otimes \phi(T))\hat{X}_0 \) on both sides of (76), we have
\[
(\psi(T) \otimes \phi(T))\hat{X}_0 = \int_0^T (\psi(T - t) \otimes \phi(T - t))(I_n \otimes F(t))
\]
\[
\times (I_n \otimes F(t))^* (\psi(T - t) \otimes \phi(T - t))^*
\]
\[
\times W^{-1}(0, T)(\psi(T) \otimes \phi(T))\hat{X}_0 dt.
\]

Now our problem is to find the control \( \hat{U}(t) \) such that \( \hat{X}(T) = \hat{X}_f \)
\[
= (\psi(T) \otimes \phi(T))\hat{X}_0
\]
\[
+ \int_0^T (\psi(T - t) \otimes \phi(T - t))(I_n \otimes F(t))\hat{U}(t)dt.
\]

Since \( \hat{X} \) is fuzzy and from Lemma 4, \( \hat{U}(t) \) must be fuzzy, otherwise the fuzzy left side of (80) cannot be equal to the crisp right side. By Lemma 2, \( \hat{X}_f \) can be written as
\[
\hat{X}_f = \frac{1}{T} \int_0^T \hat{X}_f dt
\]
\[
= \frac{1}{T} \int_0^T (\psi(T - t) \otimes \phi(T - t))(I_n \otimes F(t))
\]
\[
\times (I_n \otimes F(t))^* (\psi(T - t) \otimes \phi(T - t))^* \hat{X}_f dt.
\]

From (80) and (81), we have
\[
\frac{1}{T} \int_0^T (\psi(T - t) \otimes \phi(T - t))(I_n \otimes F(t))
\]
\[
\times (I_n \otimes F(t))^* (\psi(T - t) \otimes \phi(T - t))^* \hat{X}_f dt
\]
\[
= (\psi(T) \otimes \phi(T))\hat{X}_0
\]
\[
+ \int_0^T (\psi(T - t) \otimes \phi(T - t))(I_n \otimes F(t))\hat{U}(t)dt.
\]

Combining (79) and (82), we have
\[
\frac{1}{T} \int_0^T (\psi(T - t) \otimes \phi(T - t))(I_n \otimes F(t))
\]
\[
\times (I_n \otimes F(t))^* (\psi(T - t) \otimes \phi(T - t))^* \hat{X}_f dt
\]
\[
= \int_0^T (\psi(T - t) \otimes \phi(T - t))(I_n \otimes F(t))
\]
\[
\times (I_n \otimes F(t))^* (\psi(T - t) \otimes \phi(T - t))^* \hat{X}_f dt
\]
\[
+ \int_0^T (\psi(T - t) \otimes \phi(T - t))(I_n \otimes F(t))\hat{U}(t)dt.
\]

It follows that
\[
\int_0^T (\psi(T - t) \otimes \phi(T - t))(I_n \otimes F(t))\hat{U}(t)dt
\]
\[
= \int_0^T (\psi(T - t) \otimes \phi(T - t))(I_n \otimes F(t))
\]
\[
\times \left\{ \frac{1}{T} (I_n \otimes F(t))^{-1} (\psi(T - t) \otimes \phi(T - t))^{-1} \hat{X}_f
\]
\[
- (I_n \otimes F(t))^* (\psi(T - t) \otimes \phi(T - t))^* \hat{X}_fight\} dt.
\]
By using Lemma 3, we get
\[
\hat{U}(t) = \frac{1}{T} (I_n \otimes F(t))^{-1} (\psi(T - t) \otimes \phi(T - t))^{-1} \hat{X}_f \\
- (I_n \otimes F(t))^* (\psi(T - t) \otimes \phi(T - t))^* \\
\times W^{-1}(0, T)(\psi(T) \otimes \phi(T))\hat{X}_0.
\] (85)

Now we have two special cases for (85). First, let \( \hat{X}(T) = \hat{X}_f = (\hat{x}_1(T), \hat{x}_2(T), \ldots, \hat{x}_n(T)) \) be a crisp point, then we will get a corresponding control \( \hat{U}(t) = (\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n) \), satisfying (85).

Second, let \( \hat{X}(T) = (\hat{x}_1(T), \hat{x}_2(T), \ldots, x_f(T)) \), then the corresponding control \( \hat{U}(t) \) will take the form \( \hat{U}(t) = (\hat{u}_1, \hat{u}_2, \ldots, u_i, \ldots, \hat{u}_n) \) in which the ith component of \( \hat{U}(t) \) is a fuzzy set in \( E^i \). Obviously, \( \hat{u}_i(t) \) is in \( u_i(t) \), the grade of the membership can be determined by \( \mu_{x\otimes}(\hat{x}_i(T)) \), the grade of the membership of \( \hat{x}_i(T) \) in \( x_f \). Thus, based on the above discussion, we have a fuzzy rule base for the control \( \hat{U} \), and is given by (77) and (78).

Remark 2. The nonsingularity of the symmetric controllability matrix \( W(0, T) \) in Theorem 3 is only a sufficient condition but not necessary because the fuzzy rule base cannot guarantee the nonsingularity of the controllability matrix.

Example 1. Consider the fuzzy dynamical matrix Lyapunov system (1) satisfying (2) with
\[
A(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\

F(t) = \begin{bmatrix} e^t \\ 0 \end{bmatrix}, \quad C(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\

D(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad T = \frac{\pi}{2}, \quad X_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\] (86)

Let \( \hat{X}_f = (x_{f1}, x_{f2}, x_{f3}, x_{f4}) \in E^4 \), where
\[
[\hat{X}_f]^a = \begin{bmatrix} [x_{f1}]^a \\ [x_{f2}]^a \\ [x_{f3}]^a \\ [x_{f4}]^a \end{bmatrix} = \begin{bmatrix} [0.1(\alpha - 1), 0.1(1 - \alpha)] \\ [\alpha - 1, 1 - \alpha] \\ [0.1(\alpha - 1), 0.1(1 - \alpha)] \\ [\alpha - 1, 1 - \alpha] \end{bmatrix}.
\] (87)

We select the points \( \hat{x}_1 = 0.5, \hat{x}_2 = 0.25, \hat{x}_3 = 0.05, \) and \( \hat{x}_4 = 0.025 \) which are in \( x_{f1}, x_{f2}, x_{f3}, \) and \( x_{f4} \) with grades of 0.5, 0.75, 0.5, and 0.75, respectively. The fundamental matrices of (20), (21) are
\[
\phi(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}, \quad \psi(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}.
\] (88)

Now the fundamental matrix of (19) is
\[
\psi(t) \otimes \phi(t) = \begin{bmatrix} e^{\cos t} & -e^t \sin t & 0 & 0 \\ e^t \sin t & e^t \cos t & 0 & 0 \\ 0 & 0 & e^t \cos t & -e^t \sin t \\ 0 & 0 & e^t \sin t & e^t \cos t \end{bmatrix}.
\] (89)

Consider
\[
(\psi(t) \otimes \phi(t)) (I_n \otimes F(t)) (I_n \otimes F(t))^* (\psi(t) \otimes \phi(t))^* \times e^t
\]
\[
= e^t \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \theta & -\sin \theta \\ 0 & 0 & \sin \theta & \cos \theta \end{bmatrix} e^t
\]
\[
= \begin{bmatrix} e^\pi & 0 & 0 & 0 \\ 0 & e^\pi & 0 & 0 \\ 0 & 0 & e^\pi & 0 \\ 0 & 0 & 0 & e^\pi \end{bmatrix}.
\] (90)

where \( \theta = \pi/2 - t. \) Therefore,
\[
W(0, \frac{\pi}{2}) = \int_0^{\pi/2} \begin{bmatrix} e^\pi & 0 & 0 & 0 \\ 0 & e^\pi & 0 & 0 \\ 0 & 0 & e^\pi & 0 \\ 0 & 0 & 0 & e^\pi \end{bmatrix} dt = \frac{\pi}{2} e^{\pi/2}.
\] (91)

Clearly, it is nonsingular.
Thus, from Theorem 3, the input \( \hat{U} \) can be chosen by the following \( \alpha \)-level sets:

\[
\hat{U}^\alpha(t) = \frac{2e^{-t}}{\pi} e^{-\theta} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\cos \theta & \sin \theta & 0 & 0 \\
-\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & \cos \theta & \sin \theta \\
0 & 0 & -\sin \theta & \cos \theta
\end{bmatrix}
\]

and the corresponding control function to the point \((0.5, 0.25, 0.05, 0.025)^T\) is

\[
\hat{U}(t) = \begin{bmatrix}
\tilde{u}_1 \\
\tilde{u}_2 \\
\tilde{u}_3 \\
\tilde{u}_4
\end{bmatrix} = \frac{2e^{-\pi/2}}{\pi} \begin{bmatrix}
0.5(\sin t + \cos t) \\
0.75(\sin t - \cos t) \\
0.5(\sin t + \cos t) \\
0.75(\sin t - \cos t)
\end{bmatrix} e^{-t}.
\]

Example 2. Consider the fuzzy dynamical matrix Lyapunov system (1) satisfying (2) with

\[
A(t) = \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}, \quad B(t) = \begin{bmatrix}
t + 1 & 0 \\
0 & t + 1
\end{bmatrix},
\]

\[
F(t) = \begin{bmatrix}
1 & 0 \\
0 & (2 - t)e^{t-1}
\end{bmatrix}, \quad C(t) = \begin{bmatrix}
e^{-t} & 0 \\
e^t & 0
\end{bmatrix},
\]

\[
D(t) = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}, \quad T = 1, \quad X_0 = \begin{bmatrix}
1 \\
0
\end{bmatrix}.
\]

Let \( \hat{X}_f = (x_{f_1}, x_{f_2}, x_{f_3}, x_{f_4}) \in E^4 \), where

\[
[\hat{X}_f]^a = \begin{bmatrix}
[x_{f_1}]^a \\
[x_{f_2}]^a \\
[x_{f_3}]^a \\
[x_{f_4}]^a
\end{bmatrix} = \begin{bmatrix}
0.5\alpha + 0.5, 1 \\
0.8\alpha + 0.2, 1 \\
[0.2(\alpha - 1), 0.2(1 - \alpha)]
\end{bmatrix}.
\]

We select the points \( \hat{x}_1 = 0.75, \hat{x}_2 = 0.8, \hat{x}_3 = 0.75, \) and \( \hat{x}_4 = 0.1 \) which are in \( x_{f_1}, x_{f_2}, x_{f_3}, \) and \( x_{f_4} \) with grades of the membership being 0.5, 0.75, 0.25, and 0.5, respectively. The fundamental matrices of (20), (21) are

\[
\phi(t) = \begin{bmatrix}
\frac{t + 1}{t + 1} & 0 \\
0 & \frac{t + 1}{t + 1}
\end{bmatrix}, \quad \psi(t) = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & e^{t+1}
\end{bmatrix}.
\]

Now the fundamental matrix of (19) is

\[
(\psi(t) \otimes \phi(t)) = \begin{bmatrix}
(t + 1)^2 & 0 & t(t + 1) & 0 \\
0 & 1 & 0 & \frac{t}{t + 1}
\end{bmatrix}
\]

\[
0 & 0 & e^{t+1} & 0 \\
0 & 0 & 0 & \frac{e^{t+1}}{t + 1}
\end{bmatrix}.
\]

(93)
It is easily seen that

\[(\psi(1-t) \otimes \phi(1-t))(I_n \otimes F(t))(I_n \otimes F(t))^* \]

\[\times (\psi(1-t) \otimes \phi(1-t))^* \]

\[= \begin{bmatrix} 2t^2 - 6t + 5 & 0 & (1-t)e^{1-t} & 0 \\ 0 & (2t^2 - 6t + 5)e^{2(t-1)} & (1-t)e^{t-1} \\ (1-t)e^{1-t} & 0 & e^{2(1-t)} & 0 \\ 0 & (1-t)e^{t-1} & 0 & 1 \end{bmatrix}. \]

(99)

Therefore,

\[W\left(0, \frac{\pi}{2}\right) \]

\[= \int_0^1 \begin{bmatrix} 2t^2 - 6t + 5 & 0 & (1-t)e^{1-t} & 0 \\ 0 & (2t^2 - 6t + 5)e^{2(t-1)} & (1-t)e^{t-1} \\ (1-t)e^{1-t} & 0 & e^{2(1-t)} & 0 \\ 0 & (1-t)e^{t-1} & 0 & 1 \end{bmatrix} dt \]

\[= \begin{bmatrix} 8 & 0 & 1 & 0 \\ 0 & \frac{3}{2} & 1 & 2e^{-1} \end{bmatrix}. \]

(100)

Clearly, it is nonsingular.

Thus, from Theorem 3, the input \(\hat{U}\) can be chosen by the following \(\alpha\)-level sets, given by

\[\hat{U}^\alpha(t) = \begin{bmatrix} \frac{1}{2-t} & 0 & (t-1) & e^{1-t} & 0 \\ 0 & \frac{e^{1-t}}{2-t} & 0 & t-1 & 2^{-t} \\ 0 & 0 & e^{1-t} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]

\[\begin{bmatrix} [0.5\alpha + 0.5, 1] \\ [0.8\alpha + 0.2, 1] \\ 0 \\ 0 \end{bmatrix} \] and the corresponding control function to the point \((0.75, 0.8, 0.75, 0.1)\) is

\[\hat{\hat{U}}(t) = \begin{bmatrix} \hat{\hat{u}}_1 \\ \hat{\hat{u}}_2 \\ \hat{\hat{u}}_3 \\ \hat{\hat{u}}_4 \end{bmatrix} \]

\[= \begin{bmatrix} \frac{2 + 3(t-1)e^{t-1}}{4(2-t)} \\ \frac{3e^{1-t} + 2(t-1)}{4(2-t)} \\ e^{1-t} \\ 0 \end{bmatrix} \]

(102)

5. OBSERVABILITY OF FUZZY DYNAMICAL LYAPUNOV SYSTEMS

In this section, we discuss the concept of observability of the fuzzy system (58), (59).

Definition 10. The fuzzy system (58), (59) is said to be completely observable over the interval \([0, T]\) if the knowledge of rule base of input \(\hat{U}\) and output \(\hat{Y}\) over \([0, T]\) suffices to determine a rule base of initial state \(\hat{X}_0\).

Let \(u_i^\ell, y_i^\ell, i = 1, 2, \ldots, n^2, \ell = 1, 2, \ldots, m\), be fuzzy sets in \(E^1\). We assume that the rule base for the input and output is

\[R^\ell: \text{IF } \hat{u}_i^\ell(t) \text{ is in } u_i^\ell(t), \ldots, \hat{u}_n^\ell(t) \text{ is in } u_n^\ell(t), \]

THEN \(\hat{y}_1(t) \text{ is in } y_1^\ell(t), \ldots, \hat{y}_m(t) \text{ is in } y_m^\ell(t)\),

(103)

\[\ell = 1, 2, \ldots, m,\]

and the relation between input and output is

\[\hat{Y}(t) = (I_n \otimes C(t))\hat{X}(t) + (I_n \otimes D(t))\hat{\hat{U}}(t). \]

(104)

Theorem 4. Assume that the fuzzy rule base (103) holds, then the system (58), (59) is completely observable over the interval \([0, T]\) if \((I_n \otimes C(T))(\psi(T) \otimes \phi(T))\) is nonsingular. Furthermore, if \(\hat{X}_0 = (\hat{x}_0^1, \hat{x}_0^2, \ldots, \hat{x}_0^n)\), then one has the following rule base for the initial value \(\hat{X}_0\):

\[R^\ell: \text{IF } \hat{u}_i^\ell(T) \text{ is in } u_i^\ell(T), \ldots, \hat{u}_n^\ell(T) \text{ is in } u_n^\ell(T), \]

IF \(\hat{y}_1(T) \text{ is in } y_1^\ell(T), \ldots, \hat{y}_m(T) \text{ is in } y_m^\ell(T), \)

THEN \(\hat{x}_0^\ell(t) \text{ is in } x_0^\ell(1), \ldots, x_0^\ell(t) \text{ is in } x_0^\ell(n)\),

\[\ell = 1, 2, \ldots, m,\]

(105)
where

\[ x_0^\ell(i) = \left( (I_n \otimes C(T)) \left( \psi(T) \otimes \phi(T) \right) \right)^{-1} \]

\[ \times \left\{ V_1^\ell(T) - (I_n \otimes D(T)) \hat{U}(T) - (I_n \otimes C(T)) \right\} \]

\[ \times \int_0^T \left( \psi(T-s) \otimes \phi(T-s) \right) (I_n \otimes F(s)) H_1^\ell(s) ds \right\}, \]

(106)

\[ \hat{X}_0 = \left[ (I_n \otimes C(T)) \left( \psi(T) \otimes \phi(T) \right) \right]^{-1} \]

\[ \times \left\{ \hat{Y}(T) - (I_n \otimes D(T)) \hat{U}(T) - (I_n \otimes C(T)) \right\} \]

\[ \times \int_0^T \left( \psi(T-s) \otimes \phi(T-s) \right) (I_n \otimes F(s)) \hat{U}(s) ds \right\}, \]

(107)

\[ H_1^\ell(t) = \hat{u}_1(t) \times \cdots \times u_i^\ell(t) \times \cdots \times \hat{u}_n^\ell(t), \]

\[ V_1^\ell(t) = \hat{y}_1(t) \times \cdots \times y_i^\ell(t) \times \cdots \times \hat{y}_m^\ell(t), \]

\[ i = 1, 2, \ldots, n^2, \quad \ell = 1, 2, \ldots, m. \]

**Proof.** Without loss of generality, we prove this theorem by considering \( \ell = 1. \) Let

\[ \hat{u}(t) = (\hat{u}_1(t), \hat{u}_2(t), \ldots, \hat{u}_n(t)), \]

\[ \hat{y}(t) = (\hat{y}_1(t), \hat{y}_2(t), \ldots, \hat{y}_m(t)). \]

Let \( \mu_{u_i^\ell(t)}(\hat{u}(t)) \) be the grade of the membership of \( \hat{u}(t) \) in \( u_i^\ell(t) \), and let \( \mu_{y_i^\ell(t)}(\hat{y}(t)) \) be the grade of the membership of \( \hat{y}(t) \) in \( y_i^\ell(t) \). Since \( (I_n \otimes C(T)) \left( \psi(T) \otimes \phi(T) \right) \) is nonsingular and from (60), we have

\[ \hat{X}_0 = \left[ (I_n \otimes C(T)) \left( \psi(T) \otimes \phi(T) \right) \right]^{-1} \]

\[ \times \left\{ \hat{Y}(T) - (I_n \otimes D(T)) \hat{U}(T) - (I_n \otimes C(T)) \right\} \]

\[ \times \int_0^T \left( \psi(T-s) \otimes \phi(T-s) \right) (I_n \otimes F(s)) \hat{U}(s) ds \right\}. \]

(108)

Now, the initial value \( \hat{X}_0 \) is no more a crisp value, but should be a fuzzy set. In order to determine each component of \( \hat{X}_0 \), let us assume

\[ H_1^\ell(t) = \hat{u}_1(t) \times \cdots \times u_i^\ell(t) \times \cdots \times \hat{u}_n(t), \]

\[ V_1^\ell(t) = \hat{y}_1(t) \times \cdots \times y_i^\ell(t) \times \cdots \times \hat{y}_m(t), \]

(109)

\[ i = 1, 2, \ldots, n^2. \]

From Remark 1, we know that the set \( \left( \psi(T) \otimes \phi(T) \right) \hat{X}_0 \)

\[ \times \int_0^T (\psi(T-s) \otimes \phi(T-s)) (I_n \otimes F(s)) H_1^\ell(s) ds \]

is a fuzzy set in \( E_1^\ell \). From Lemma 4, we know that the product

\[ (I_n \otimes C(T)) \int_0^T (\psi(T-s) \otimes \phi(T-s)) (I_n \otimes F(s)) H_1^\ell(s) ds \]

is a fuzzy set in \( E_{n^2}^\ell \). Hence, \( \hat{X}_0 \) is a fuzzy set in \( E_{n^2}^\ell \), and the 1st component of it denoted by \( x_0^\ell(i) \) is a fuzzy set in \( E_1^\ell \). The grade of the membership of \( x_0^\ell(i) \) in \( x_0^\ell(i) \) is defined by

\[ \mu_{\hat{X}_0^\ell(i)}(\hat{X}_0) = \min \left\{ \mu_{u_i^\ell(t)}(\hat{u}(t)), \mu_{y_i^\ell(t)}(\hat{y}(t)) \right\}. \]

Now, we are in a position to determine the rule base for the initial value and it is given by (105), (106), (107), and (108).

In general, it is difficult to compute \( x_0^\ell(i) \), but to solve the real problems, we choose the following approximation. Now, we take the point \( (\bar{x}_0^\ell(i), \hat{X}_0^\ell(i)) \) and the zero-level set \( [x_0^\ell(i)]^0 \)

to determine a triangle as the new fuzzy set \( \hat{X}_0^\ell(i) \).

We can use the center average defuzzifier

\[ x_0^\ell(i) = \frac{\sum_{j=1}^{m} (\bar{x}_0^\ell(i)) \mu_{\hat{X}_0^\ell(j)}(\hat{X}_0^\ell(j))}{\sum_{j=1}^{m} \mu_{\hat{X}_0^\ell(j)}(\hat{X}_0^\ell(j))} \]

(110)

to determine the initial value \( \hat{X}_0^\ell = (\bar{x}_0^\ell, \hat{X}_0^\ell, \ldots, \hat{X}_0^\ell) \). To obtain more accurate value for the initial state, more rule bases may be provided.
Example 3. Consider the fuzzy matrix Lyapunov system
\[
X'(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} X(t) + X(t) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} e^t \\ 0 \end{bmatrix} U, \\
0 \leq t \leq \frac{\pi}{2},
\]
\[
Y(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} X(t).
\]

The \(\alpha\)-level sets of fuzzy input \(\hat{U}(t)\) and fuzzy output \(\hat{Y}(t)\) by rule base 1 and rule base 2 are given as follows.

Rule 1:
\[
\hat{U}^{(1)} = \begin{bmatrix} [0, -0.75(\alpha - 1)] \\ [0.75(\alpha - 1) + 1, 1] \\ [-0.5(\alpha - 1)] \\ [0.5(\alpha - 1) + 1, 1] \end{bmatrix},
\]
\[
\hat{Y}^{(1)} = \begin{bmatrix} [0, -2(\alpha + 1)] \\ [0.5\alpha + 2.5, 3] \\ [-1.5(\alpha - 1)] \\ [0.5(\alpha - 1) + 3.3] \end{bmatrix}.
\]

Rule 2:
\[
\hat{U}^{(2)} = \begin{bmatrix} [0, -0.8(\alpha - 1)] \\ [0.8\alpha + 0.2, 1] \\ [-0.5(\alpha - 1)] \\ [0.5\alpha + 0.5, 1] \end{bmatrix},
\]
\[
\hat{Y}^{(2)} = \begin{bmatrix} [0, -1.5(\alpha - 1)] \\ [\alpha + 1, 2] \\ [-2.5(\alpha - 1)] \\ [2\alpha + 1, 3] \end{bmatrix}.
\]

From rule base 1, we select
\[
\hat{u}^1 = (\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4) = (0.5, 0.85, 0.4, 0.75),
\]
the grades of the membership of \(\hat{u}_1, \hat{u}_2, \hat{u}_3, \text{ and } \hat{u}_4\) are 1/3, 0.8, 0.2, and 1/2, respectively. Also
\[
\hat{y}^1 = (\hat{y}_1, \hat{y}_2, \hat{y}_3, \hat{y}_4) = (1, 2.8, 0.5, 2.9),
\]
the grades of the membership of \(\hat{y}_1, \hat{y}_2, \hat{y}_3, \text{ and } \hat{y}_4\) are 1/2, 0.6, 2/3, and 0.8, respectively.

From rule base 2, we select
\[
\hat{u}^2 = (\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4) = (0.5, 0.8, 0.25, 0.75),
\]
the grades of the membership of \(\hat{u}_1, \hat{u}_2, \hat{u}_3, \text{ and } \hat{u}_4\) are 3/8, 3/4, 1/2, and 1/2, respectively. Also
\[
\hat{y}^2 = (\hat{y}_1, \hat{y}_2, \hat{y}_3, \hat{y}_4) = (1, 1.75, 2, 1.5),
\]
\[
\begin{bmatrix}
\cos \left( \frac{\pi}{2} - s \right) - \sin \left( \frac{\pi}{2} - s \right) & 0 & 0 \\
\sin \left( \frac{\pi}{2} - s \right) \cos \left( \frac{\pi}{2} - s \right) & 0 & 0 \\
0 & 0 & \cos \left( \frac{\pi}{2} - s \right) - \sin \left( \frac{\pi}{2} - s \right) \\
0 & 0 & \sin \left( \frac{\pi}{2} - s \right) \cos \left( \frac{\pi}{2} - s \right)
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0.5 \\
0.4 \\
0.75 \\
ds
\end{bmatrix}
\]

\[
\begin{bmatrix}
-1.292, -0.542 - 0.75\alpha \\
-1.124, -0.27 - 0.854\alpha \\
-1.046 \\
-0.9532
\end{bmatrix}
\]

\[\mu_{x_0^3}(x_0^3) = \min \{0.8, 0.6\} = 0.6,\]

\[x_0^3(3) = e^{-\pi/2}
\]

when \( \alpha = 0 \), we get the biggest interval \([-1.124, -0.27]\) and \(x_0^3 = -0.9324\) is located in this interval. We choose its membership grade in \(x_0^3(2)\) as

\[\mu_{x_0^3}(x_0^3) = \min \left\{ \frac{1}{3}, \frac{1}{2} \right\} = \frac{1}{3} = 0.333,\]

\[x_0^3(2) = e^{-\pi/2}
\]
\[
\begin{bmatrix}
-1.142 \\
-0.9324 \\
[-1.25 + 0.5\alpha, -0.438 - 0.312\alpha] \\
[-1.3532, -0.8532 - 0.5\alpha]
\end{bmatrix},
\]

(129)

when \( \alpha = 0 \), we get the biggest interval \([-1.25, -0.438]\) and \( \bar{x}_0^3 = -1.046 \) is located in this interval. We choose its membership grade in \( x_0^3(3) \) as
\[
\mu_{x_0^3(3)}(\bar{x}_0^3) = \min \left\{ \frac{2}{3}, 0.2 \right\} = 0.2,
\]

\[
x_0^3(4) = e^{-\pi/2}
\]

\[
\left[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
2.8 & 1 & 0 & 0 \\
0.5 & 0 & 0 & 1 \\
[0.5(\alpha - 1) + 3, 3] & 0 & 0 & 1 \\
\end{array}
\right]
\]

\[
\begin{bmatrix}
\cos \left( \frac{\pi}{2} - s \right) & -\sin \left( \frac{\pi}{2} - s \right) & 0 & 0 \\
\sin \left( \frac{\pi}{2} - s \right) & \cos \left( \frac{\pi}{2} - s \right) & 0 & 0 \\
0 & 0 & \cos \left( \frac{\pi}{2} - s \right) & -\sin \left( \frac{\pi}{2} - s \right) \\
0 & 0 & \sin \left( \frac{\pi}{2} - s \right) & \cos \left( \frac{\pi}{2} - s \right)
\end{bmatrix}
\]

\[
\left[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{array}
\right]
\]

\[
\begin{bmatrix}
0.5 \\
0.85 \\
0.4 \\
[0.5(\alpha - 1) + 1, 1]
\end{bmatrix}
\]

\[
= \int_0^{e^{\pi/2}} e^{\pi/2 - s} ds
\]

(130)

Similarly for rule base 2, by the use of formula (106), (107), we obtain the values of \( \bar{x}_0^3, x_0^3(i) \), \( i = 1, 2, 3, 4 \) and given as follows:

\[
\begin{bmatrix}
-1.092 \\
-0.664 \\
-0.584 \\
-0.812
\end{bmatrix},
\]

\[
\begin{bmatrix}
-1.6 + 0.8\alpha, -0.488 - 0.312\alpha \\
-1.164, -0.364 - 0.8\alpha \\
-1.292, -0.492 - 0.8\alpha \\
-0.916, 0.092 - 1.008\alpha
\end{bmatrix},
\]

(132)

Also the grades of the membership of \( \bar{x}_0^3 = -1.092, \bar{x}_0^2 = -0.664, \bar{x}_0^3 = -0.584, \bar{x}_0^4 = -0.812 \) in \( x_0^3(1), x_0^3(2), x_0^3(3), x_0^3(4) \) are 0.333, 0.75, 0.2, 0.25, respectively. We can use the center average defuzzifier to determine \( \bar{X}_0 = (\bar{x}_0^3, \bar{x}_0^2, \bar{x}_0^3, x_0^3, x_0^4) \), where

\[
\bar{x}_0^3 = \frac{\sum_{i=1}^{4} \mu_{x_0^3(i)}(\bar{x}_0^3)}{\sum_{i=1}^{4} \mu_{x_0^3(i)}(\bar{x}_0^3)}
\]

\[
\begin{bmatrix}
-1.142 \\
-0.9324 \\
[-1.296, -0.796 - 0.5\alpha] \\
[-1.224, -0.62 - 0.604\alpha]
\end{bmatrix},
\]

(131)

when \( \alpha = 0 \), we get the biggest interval \([-1.224, -0.62]\) and \( \bar{x}_0^4 = -0.9532 \) is located in this interval. We choose its membership grade in \( x_0^4(4) \) as
\[
\mu_{x_0^4(4)}(\bar{x}_0^4) = \min \left\{ \frac{1}{2}, 0.8 \right\} = 0.5.
\]

(133)
6. CONCLUSIONS

In this paper, we have investigated a way to incorporate matrix Lyapunov systems with a set of fuzzy rules. Here, a deterministic matrix Lyapunov system with fuzzy inputs and fuzzy outputs can generate a fuzzy dynamical matrix Lyapunov system (FDMLS). Based on this result, we can study both controllability and observability properties of the FDMLS. First, we have provided a sufficient condition for the controllability of the FDLS, that is, for a given fuzzy state with a fuzzy rule base, we can determine a control which transfers the initial state to the given state in a finite time. The advantage of our approach is that all levels are represented by mathematical formulas. Example 1 shows how to determine the control by our formula. Next, we have studied the observability property which concerns the following problem, that is, given the input and output rule bases we can determine a rule base for the initial state with a formula. Example 3 illustrates the significance of our method by which we can determine the rule base for initial value without solving the FDLS. Our future research works will concentrate on the applications of these systems (FDMLS) to real world problems.

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