The Chebyshev wavelets operational matrix of integration and product operation matrix

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Operational matrices of integration and product based on Chebyshev wavelets are presented. A general procedure for forming these matrices is given. These matrices play an important role in modelling of problems. Numerical examples are given to demonstrate applicability of these matrices.

Keywords: Chebyshev wavelets; operational matrix; product operation matrix

1. Introduction

Orthogonal functions and polynomial series have received considerable attention in dealing with various problems of dynamic systems. The main characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations, thus greatly simplifying the problem. The approach is based on converting the underlying differential equations into an integral equations through integration, approximating various signals involved in the equation by truncated orthogonal series and using the operational matrix $P$ of integration, to eliminate the integral operations. The matrix $P$ is given by

$$\int_0^t \Phi(s)ds \simeq P\Phi(t),$$

where $\Phi(t) = [\phi_0(t), \phi_1(t), \ldots, \phi_{n-1}(t)]^T$ and the matrix $P$ can be uniquely determined on the basis of the particular orthogonal functions. One of the operators that play an important role in modelling of equations is defined as follows

$$\Phi(t)^T F \simeq \tilde{F}\Phi(t),$$

where $\tilde{F}$ is an $n \times n$ matrix, called the product operational matrix.

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The elements $\phi_0(t), \phi_1(t), \ldots, \phi_{n-1}(t)$ are the basis functions, orthogonal on a certain interval $[a, b]$. Special attention has been given to applications of Walsh function [2], block-pulse function [4], Legendre series [7], Legendre polynomials [1] and Chebyshev polynomials [6].

All the above orthogonal functions, however, are supported on the whole interval $a \leq t \leq b$. This kind of global support is evidently a drawback for certain analysis work, especially systems involving abrupt variations or local function vanishing outside a short interval of time or space. Since the cancellation of many terms is required in order to obtain a reasonable accuracy, the wavelets analysis could be a possible tool for solving the difficulty in physics, communication and image processing [3]. In [5], Gu and Jiang, derived the Haar wavelets operational matrix of integration. Special attention has been given to applications of Legendre wavelets [10,13], Hybrid function [8,9,11,12].

In the present paper, we introduce the Chebyshev wavelets operational matrix of integration and product operation matrix. These matrices can be used to solve problems such as identification, analysis and optimal control, like that of the other orthogonal functions. The Chebyshev wavelets are first introduced, the operational matrices of integration and product are then derived. Illustrative examples are given to demonstrate the application of operational matrix of integration and product operation matrix for Chebyshev wavelets.

2. Properties of Chebyshev wavelets

2.1 Wavelets and Chebyshev wavelets

In recent years, wavelets have found their way into many different fields of science and engineering. Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter $a$ and the translation parameter $b$ vary continuously, we have the following family of continuous wavelets [5]:

$$\psi_{a,b}(t) = |a|^{-1/2} \psi \left( \frac{t-b}{a} \right), \quad a, b \in \mathbb{R}, \quad a \neq 0.$$  

Chebyshev wavelets $\psi_{nm}(t) = \psi(k, m, t)$ have three arguments; $k = 1, 2, 3, \ldots$, $n = 1, 2, 3, \ldots, 2^k$, $m$ is the order for Chebyshev polynomials and $t$ is the normalized time. They are defined on the interval $[0, 1)$ by:

$$\psi_{nm}(t) = \begin{cases} \alpha_m \frac{2^{k/2}}{\sqrt{\pi}} T_m(2^{k+1}t - 2n + 1), & \frac{n-1}{2^k} \leq t < \frac{n}{2^k}, \\ 0 & \text{otherwise} \end{cases}$$  

where:

$$\alpha_m = \begin{cases} \sqrt{2} & m = 0, \\ 2 & m = 1, 2, \ldots \end{cases}$$

Here $T_m(t)$ are the well-known Chebyshev polynomials of order $m$, which are orthogonal with respect to the weight function $w(t) = 1/\sqrt{1-t^2}$ and satisfy the following recursive formula:

$$T_0(t) = 1, \quad T_1(t) = t, \quad T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t), \quad m = 1, 2, 3, \ldots.$$  

The set of Chebyshev wavelets are an orthogonal set with respect to the weight function $w_n(t) = w(2^{k+1}t - 2n + 1)$. 


2.2 Function approximation

A function \( f(t) \) defined over \([0, 1)\) may be expanded as:

\[
f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} f_{nm} \psi_{nm}(t),
\]

where

\[
f_{nm} = (f(t), \psi_{nm}(t)).
\]

In Equation (3), \((\cdot, \cdot)\) denotes the inner product with weight function \(w_n(t)\).

If the infinite series in Equation (2) is truncated, then Equation (2) can be written as:

\[
f(t) \simeq \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} f_{nm} \psi_{nm}(t) = F^T \Psi(t),
\]

where \(F\) and \(\Psi(t)\) are \(2^k M \times 1\) matrices given by:

\[
F = \begin{bmatrix} f_{10}, f_{11}, \ldots, f_{1,M-1}, f_{20}, \ldots, f_{2,M-1}, \ldots, f_{2^k,0}, \ldots, f_{2^k,M-1} \end{bmatrix}^T
\]

\[
\Psi(t) = \begin{bmatrix} \psi_{10}(t), \psi_{11}(t), \ldots, \psi_{1,M-1}(t), \psi_{20}(t), \ldots, \psi_{2,M-1}(t), \ldots, \psi_{2^k,0}(t), \ldots, \psi_{2^k,M-1}(t) \end{bmatrix}^T
\]

2.3 Chebyshev wavelets operational matrix of integration

In this section, the operational matrix of integration \(P\) will be derived. First, in general, we have:

\[
\int_0^t \Psi(s) \, ds \simeq P \Psi(t),
\]

where \(\Psi(t)\) is given in Equation (6) and \(P\) is a \((2^k M) \times (2^k M)\) matrix given by:

\[
P = \begin{bmatrix} C & S & S & \ldots & S \\ O & C & S & \ldots & S \\ O & O & C & \ldots & S \\ \vdots & \vdots & \vdots & \ddots & S \\ O & O & O & \ldots & C \end{bmatrix},
\]

where \(S\) and \(C\) are \(M \times M\) matrices given by:

\[
S = \frac{\sqrt{2}}{2^k} \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ -\frac{1}{3} & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ -\frac{1}{15} & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{M(M-2)} & 0 & 0 & \ldots & 0 \end{bmatrix}
\]

\[
C = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ -\frac{1}{3} & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ -\frac{1}{15} & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{M(M-2)} & 0 & 0 & \ldots & 0 \end{bmatrix}
\]
and

\[
C = \frac{1}{2^k} \begin{bmatrix}
\frac{1}{2} & 1 & 0 & \cdots & 0 & 0 & 0 \\
\frac{1}{2\sqrt{2}} & 0 & 0 & \cdots & 0 & 0 & 0 \\
-\frac{1}{8\sqrt{2}} & 0 & 1 & 0 & \cdots & 0 & 0 \\
-\frac{1}{6\sqrt{2}} & -\frac{1}{4} & 0 & 1 & \cdots & 0 & 0 \\
& \vdots & & \vdots & \ddots & \vdots & \vdots \\
-\frac{1}{2\sqrt{2}(M-1)(M-3)} & 0 & 0 & \cdots & -\frac{1}{4(M-3)} & 0 & -\frac{1}{4(M-1)} \\
-\frac{1}{2\sqrt{2}M(M-2)} & 0 & 0 & \cdots & 0 & -\frac{1}{4(M-2)} & 0 \\
\end{bmatrix}.
\]

### 2.4 Chebyshev wavelets product operation matrix

The following property of the product of two Chebyshev wavelet function vectors will also be used:

\[
\Psi(t)^t F \simeq \tilde{F}\Psi(t).
\] (8)

Here \( F \) is given in Equation (5), \( \Psi(t) \) can be obtained similarly to Equation (6) and \( \tilde{F} \) is a \((2^k M) \times (2^k M)\) matrix. To illustrate the calculation procedures we choose \( M = 3 \) and \( k = 1 \). Using \( \Psi(t) \) we obtain

\[
\Psi(t)^t(t) = \begin{bmatrix}
\psi_{10} \psi_{10} & \psi_{10} \psi_{11} & \psi_{10} \psi_{12} & \psi_{10} \psi_{20} & \psi_{10} \psi_{21} & \psi_{10} \psi_{22} \\
\psi_{11} \psi_{10} & \psi_{11} \psi_{11} & \psi_{11} \psi_{12} & \psi_{11} \psi_{20} & \psi_{11} \psi_{21} & \psi_{11} \psi_{22} \\
\psi_{12} \psi_{10} & \psi_{12} \psi_{11} & \psi_{12} \psi_{12} & \psi_{12} \psi_{20} & \psi_{12} \psi_{21} & \psi_{12} \psi_{22} \\
\psi_{20} \psi_{10} & \psi_{20} \psi_{11} & \psi_{20} \psi_{12} & \psi_{20} \psi_{20} & \psi_{20} \psi_{21} & \psi_{20} \psi_{22} \\
\psi_{21} \psi_{10} & \psi_{21} \psi_{11} & \psi_{21} \psi_{12} & \psi_{21} \psi_{20} & \psi_{21} \psi_{21} & \psi_{21} \psi_{22} \\
\psi_{22} \psi_{10} & \psi_{22} \psi_{11} & \psi_{22} \psi_{12} & \psi_{22} \psi_{20} & \psi_{22} \psi_{21} & \psi_{22} \psi_{22} \\
\end{bmatrix}.
\] (9)

In Equation (9) we have

\[
\psi_{ij} \psi_{kl} = 0 \quad \text{if} \quad i \neq k.
\]

Also

\[
\psi_{10} \psi_{ij} = \frac{2}{\sqrt{\pi}} \psi_{ij} \\
\psi_{11} \psi_{1i} = \frac{2}{\sqrt{\pi}} \psi_{10} + \sqrt{\frac{2}{\pi}} \psi_{12}
\]

If we retain only the elements of \( \Psi(t) \), then we have

\[
\Psi^t = \frac{1}{\sqrt{\pi}} \begin{bmatrix}
2\psi_{10} & 2\psi_{11} & 2\psi_{12} & 0 & 0 & 0 \\
2\psi_{11} & 2\psi_{10} + \sqrt{2}\psi_{12} & \sqrt{2}\psi_{11} & 0 & 0 & 0 \\
2\psi_{12} & \sqrt{2}\psi_{11} & 2\psi_{10} & 0 & 0 & 0 \\
0 & 0 & 0 & 2\psi_{20} & 2\psi_{21} & 2\psi_{22} \\
0 & 0 & 0 & 2\psi_{21} & 2\psi_{20} + \sqrt{2}\psi_{22} & \sqrt{2}\psi_{21} \\
0 & 0 & 0 & 2\psi_{22} & \sqrt{2}\psi_{21} & 2\psi_{20} \\
\end{bmatrix}.
\]
Therefore the $6 \times 6$ matrix $\tilde{F}$ in Equation (8) can be written as

$$
\tilde{F} = \begin{bmatrix}
B_1 & 0 \\
0 & B_2
\end{bmatrix},
$$

(10)

where $B_i, i = 1, 2$, are $3 \times 3$ matrices given by

$$
B_i = \frac{1}{\sqrt{\pi}} \begin{bmatrix}
2f_{10} & 2f_{11} & 2f_{12} \\
2f_{11} & 2f_{10} + \sqrt{2}f_{12} & \sqrt{2}f_{11} \\
2f_{12} & \sqrt{2}f_{11} & 2f_{10}
\end{bmatrix}.
$$

(11)

In a general case, $\tilde{F}$ is a $(2^kM) \times (2^kM)$ matrix and has the form

$$
\tilde{F} = \begin{bmatrix}
B_1 & 0 & \cdots & 0 \\
0 & B_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_{2^k}
\end{bmatrix},
$$

(12)

where $B_i, i = 1, 2, \ldots, 2^k$, are similar to those in Equation (11).

3. Illustrative examples

**Example 1** Consider

$$
y'(t) + 4y(t) = x(t),
$$

$$
y(0) = 0,
$$

(13)

where

$$
x(t) = \begin{cases}
4t^2 + 2t & 0 \leq t \leq \frac{1}{2}, \\
4t + 2 & \frac{1}{2} < t \leq 1,
\end{cases}
$$

and the exact solution of Equation (13) is

$$
y(t) = \begin{cases}
t^2 & 0 \leq t \leq \frac{1}{2}, \\
t - \frac{1}{4} & \frac{1}{2} < t \leq 1.
\end{cases}
$$

Here we solve the same problem using Chebyshev wavelets, with $M = 3$ and $k = 1$. First we assume the unknown function $y'(t)$ is given by

$$
y'(t) = Y'\Psi(t)
$$

(14)

where

$$
Y' = [y'_{10}, y'_{11}, y'_{12}, y'_{20}, y'_{21}, y'_{22}]
$$

(15)

and

$$
\Psi(t) = [\psi_{10}, \psi_{11}, \psi_{12}, \psi_{20}, \psi_{21}, \psi_{22}]
$$

(16)
The elements of $\Psi(t)$ are given in Equation (6). Also we have

$$y(t) = \int_0^t y'(s) \, ds + y(0)$$

$$= \int_0^t \Psi'(s) \, ds$$

$$= \Psi'(t)$$

and

$$x(t) = X^T \Psi(t) = \begin{bmatrix} \frac{7\sqrt{\pi}}{16} & \frac{\sqrt{2\pi}}{4} & \frac{3\sqrt{2\pi}}{32} & \frac{\sqrt{2\pi}}{2^4} & 0 \end{bmatrix} \Psi(t)$$

(17)

Which when substituted into Equation (13), we have

$$Y^T \Psi(t) + 4Y^T P \Psi(t) = X^T \Psi(t) \implies Y^T (I + 4P) = X^T \implies Y^T = \begin{bmatrix} \frac{\sqrt{\pi}}{4} & \frac{\sqrt{2\pi}}{8} & 0 \end{bmatrix}.$$

Therefore

$$y(t) = Y^T P \Psi(t) = \begin{bmatrix} \frac{3\sqrt{\pi}}{64} & \frac{\sqrt{2\pi}}{32} & \frac{\sqrt{2\pi}}{128} & \frac{\sqrt{2\pi}}{4} & 0 \end{bmatrix} \Psi(t)$$

$$= \begin{cases} t^2, & 0 \leq t \leq \frac{1}{2}, \\ t - \frac{1}{4}, & \frac{1}{2} < t \leq 1. \end{cases}$$

Hence we obtain the same $y(t)$ as the exact solution.

**Example 2**  Bessel differential equation of order zero. Consider the following Bessel differential equation of order zero

$$ty''(t) + y'(t) + ty(t) = 0,$$

$$y(0) = 1, \quad y'(0) = 0.$$

(18)

A solution known as the Bessel function of the first kind of order zero denoted by $J_0(t)$ is

$$J_0(t) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(i!)^2} \left( \frac{t}{2} \right)^{2i}.$$

Here, we solve the same problem using Chebyshev wavelets with $M = 3$ and $k = 1$. First we assume that the unknown function $y''(t)$ is given by

$$y''(t) = Y^T \Psi(t).$$

(19)

Therefore

$$y'(t) = \int_0^t y''(s) \, ds + y'(0) = Y^T P \Psi(t)$$

(20)

and

$$y(t) = \int_0^t y'(s) \, ds + y(0) = \left( Y^T P^2 + D^T \right) \Psi(t),$$

(21)
Table 1. Estimated and exact values of $J_0(t)$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>Chebyshev wavelets</th>
<th>Solution of $J_0(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.9974</td>
<td>0.9975</td>
</tr>
<tr>
<td>0.2</td>
<td>0.9900</td>
<td>0.9900</td>
</tr>
<tr>
<td>0.3</td>
<td>0.9776</td>
<td>0.9776</td>
</tr>
<tr>
<td>0.4</td>
<td>0.9605</td>
<td>0.9604</td>
</tr>
<tr>
<td>0.5</td>
<td>0.9384</td>
<td>0.9385</td>
</tr>
<tr>
<td>0.6</td>
<td>0.9118</td>
<td>0.9120</td>
</tr>
<tr>
<td>0.7</td>
<td>0.8811</td>
<td>0.8812</td>
</tr>
<tr>
<td>0.8</td>
<td>0.8464</td>
<td>0.8463</td>
</tr>
<tr>
<td>0.9</td>
<td>0.8077</td>
<td>0.8075</td>
</tr>
<tr>
<td>1.0</td>
<td>0.7650</td>
<td>0.7652</td>
</tr>
</tbody>
</table>

where $D$ is given by

$$D = \begin{bmatrix} \sqrt{\pi} & 0 & \sqrt{\pi} & 0 & 0 \end{bmatrix}^T. \quad (22)$$

We can also express $t$ as

$$t = \begin{bmatrix} \sqrt{\pi} & \sqrt{2\pi} & 0 & \frac{3\sqrt{\pi}}{8} & \frac{\sqrt{2\pi}}{16} & 0 \end{bmatrix}^T \Psi(t) = E^T \Psi(t). \quad (23)$$

Substituting Equations (19)–(23) in Equation (18) we obtain

$$E^T \Psi(t) \Psi^T(t) Y + Y^T P \Psi(t) + E^T \Psi(t) \Psi^T(t) P^2 Y + E^T \Psi(t) \Psi^T(t) D = 0$$

From Equation (8) we get

$$\Psi(t) \tilde{E} Y + \Psi^T(t) P^T Y + \Psi^T(t) \tilde{E} P^2 Y + \Psi^T(t) \tilde{E} D = 0$$

or

$$\left( \tilde{E} + P^T + \tilde{E} P^2 \right) Y = -\tilde{E} D \quad (24)$$

where $\tilde{E}$ can be calculated similarly to Equation (10). Equation (24) is a set of algebraic equation which can be solved for $Y$. In Table 1, a comparison is made between the approximate values using the present approach together with the solution of $J_0(t)$.

References


