Further Results on Convergence of Cooperative Standard Cellular Neural Networks

Mauro Di Marco, Mauro Forti, Massimo Grazzini, Luca Pancioni
Department of Information Engineering, University of Siena
Via Roma 56, 53100 - Siena, Italy
Email: {dimarco,forti,grazzini,pancioni}@dii.unisi.it

Abstract—The paper considers a class of nonsymmetric cooperative standard cellular neural networks (SCNNs), which are defined by a cell-linking template, and are characterized by neuron activations modeled by a typical three-segment pwl function. The paper establishes conditions ensuring that the monotone solution semiflow associated to the considered class of SCNNs satisfies the LIMIT SET DICHOTOMY and is convergent toward equilibrium points. The conditions, which involve only static aspects of the equilibrium point configuration of the SCNNs, are easier to verify with respect to those in previous results in the literature. By means of a standard numerical program for locating the equilibrium points of pwl SCNNs, parameter ranges for which the conditions are verified, and the cooperative SCNNs are convergent, are established.

I. INTRODUCTION

A cooperative cellular neural network (CNN) is a CNN with nonnegative (excitatory) interconnections between distinct neurons. Such CNNs have received a great deal of attention in the literature and have been applied in the solution in real time of a variety of signal processing tasks, see, e.g., [1]–[8], and references therein.

Fundamental results on trajectory convergence have been established by Chua and Roska in [4] for a class of cooperative CNNs with an irreducible interconnection matrix defined via a cell-linking template. In the results in [4], a modified CNN model is considered where the original pwl neuron activation \( g(\rho) = (1/2)(|\rho + 1| - |\rho - 1|) \) of the standard (S) CNN model was replaced by a continuously differentiable strictly increasing (sigmoid) activation \( \sigma \). The use of the modified activation \( \sigma \) was necessary to prove convergence by directly applying the theory developed by Hirsch and Smith [9], [10] for addressing convergence of SCNNs. In particular, it has been shown that the semiflow associated to a cooperative SCNN is monotone but, due to the horizontal segments in the pwl activation \( g \), is not eventually strongly monotone (ESM). Moreover, by means of a counterexample it has been shown there are SCNNs with an irreducible interconnection matrix for which the LIMIT SET DICHOTOMY is violated. Analogous results hold for delayed SCNNs [12]. These results imply that in general it is not possible to use the tools in [9], [10], which are based on the LIMIT SET DICHOTOMY, to study convergence of SCNNs with pwl activations. On the other hand, in [13] it has been shown that there are relevant classes of cell-linking templates for which SCNNs with a pwl activation \( g \) satisfy the LIMIT SET DICHOTOMY, and are convergent, although the associated semiflow is not ESM.

The goal of this paper is to give further results on convergence of SCNNs with the cell-linking templates considered in [13], which are easier to apply in practical cases. The convergence results in [13] are based on an assumption concerning some global properties of the dynamics, as the absence of (attracting) heteroclinic/homoclinic connections. Such an assumption is expected to be satisfied generically, but it is difficult to check for specific SCNN parameters. In this paper we show that we can obtain convergence results analogous to those in [13] under an alternative assumption that is simpler to verify in practice. Indeed, the new assumption involves only static information on the equilibrium points and can be checked by any standard program for finding the equilibrium points of SCNNs with a pwl activation \( g \), at least for not too large SCNN dimensions. Finally, we numerically obtain parameter ranges for which the stated assumption is satisfied and the SCNNs are convergent.

II. PRELIMINARIES

We consider the space \( \mathbb{R}^n \) with the following vector-order relations [10]. Given \( x, y \in \mathbb{R}^n \), we let

\[
\begin{align*}
    x &\leq y \iff x_i \leq y_i, i = 1, \ldots, n \\
    x &< y \iff x \leq y, x \neq y \\
    x &\ll y \iff x_i < y_i, i = 1, \ldots, n.
\end{align*}
\]

Given two subsets \( U, V \) of \( \mathbb{R}^n \), \( U \leq V \) \( (U < V) \) means that \( x \leq y \) \( (x < y) \) for any \( x \in U \) and \( y \in V \).
As in [10], by a semiflow on $\mathbb{R}^n$ we mean a continuous map $\Phi : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $(t, x) \to \Phi_t(x)$, satisfying $\Phi_0(x) = x$ and $\Phi_t(\Phi_s(x)) = \Phi_{t+s}(x)$, for all $t, s \geq 0$ and $x \in \mathbb{R}^n$. We say that the semiflow $\Phi$ is monotone if for any $x \leq y$ we have $\Phi_t(x) \leq \Phi_t(y)$, $t \geq 0$. We say that $\Phi$ is eventually strongly monotone (ESM) if $\Phi$ is monotone and for any $x < y$ there exists $t > 0$ such that we have $\Phi_t(x) < \Phi_t(y)$, $t > \epsilon$. The omega-limit set, $\omega(x)$, of $\Phi_t(x)$ is the set of points $y \in \mathbb{R}^n$ such that there exists a sequence $t_k \to +\infty$ such that $\Phi_{t_k}(x) \to y$ as $k \to +\infty$. We say that $\xi \in \mathbb{R}^n$ is an equilibrium point (EP) of $\Phi$ if $\Phi_t(\xi) = \xi$, $t \geq 0$. A point $x \in \mathbb{R}^n$ is convergent (and we write $x \in \mathcal{C}$) if we have $\Phi_t(x) \to \xi$, as $t \to +\infty$, for some EP $\xi$. Equivalently, $\omega(x) = \{\xi\}$. The semiflow $\Phi$ on $\mathbb{R}^n$ is said to be: convergent if we have $\mathcal{C} = \mathbb{R}^n$; almost convergent if $\mathcal{C} = \mathbb{R}^n \setminus U$, where $U$ is a subset of $\mathbb{R}^n$ with Lebesgue measure zero.

III. A CLASS OF COOPERATIVE SCNNs

We consider a class of nonsymmetric, inputless SCNNs, which are defined by the cloning template $[r \ p \ s]$ and have periodic boundary conditions (a one-dimensional circular SCNN array). We suppose that the template parameters satisfy $r, s > 0$, i.e., the SCNNs are cooperative, and $p > 1$. In matrix-vector notation the SCNNs obey the system of differential equations

$$\dot{x} = -x + AG(x) \quad (1)$$

where $x = (x_1, \ldots, x_n)' \in \mathbb{R}^n$ is the vector of neuron state variables, $A$ is the neuron interconnection matrix, $G(x) = (g(x_1), \ldots, g(x_n))' : \mathbb{R}^n \to \mathbb{R}^n$ is a diagonal mapping such that

$$g(\rho) = \frac{1}{2}(|\rho + 1| - |\rho - 1|) \quad (2)$$

is a typical three-segment pwl neuron activation. We have

$$A = \begin{pmatrix}
  p & s & 0 & 0 & \cdots & r \\
  r & s & 0 & 0 & \cdots & 0 \\
  0 & r & s & \cdots & 0 \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  0 & \cdots & 0 & r & s \\
  s & \cdots & 0 & 0 & r & p
\end{pmatrix}.$$  

For any $x_0 \in \mathbb{R}^n$, we denote by $x(t; x_0)$, $t \geq 0$, the unique bounded solution of (1) with initial condition $x_0$ at $t = 0$. The solution semiflow associated to the SCNN (1) is given as $\Phi_t(x_0) = x(t; x_0)$ for all $t \geq 0$ and $x_0 \in \mathbb{R}^n$. It is shown in [11] that $\Phi$ is monotone, however due to the horizontal segments in the pwl activation $g$, $\Phi$ is not ESM.

By $E$ we denote the set of EPs of (1). In the paper we always suppose that the EPs of (1) are isolated. Moreover, if $\xi$ is an EP of (1), then we have $x_\xi \neq i \neq 1, \ldots, n$, i.e., the EP does not belong to the boundary of regions into which the state space $\mathbb{R}^n$ is subdivided by the pwl function $g$. This assumption is satisfied by generic SCNN parameters. We also remark that a sufficient condition for isolated EPs is that $A - E_n$, where $E_n$ is the identity matrix, and all the principal submatrices of $A - E_n$ are nonsingular. This can be easily checked via the explicitly known eigenvalues of circulant matrices. The hypothesis $|x_i| \neq 1$ for $i = 1, \ldots, n$ can be checked for given SCNN parameters by means of numerical programs for finding the EPs of (1) (see Sect. V).

Since $p > 1$, if $\xi \in E$ is a sink (i.e., an asymptotically stable EP), then $\xi$ belongs to a saturation region of (1). It can also be easily verified that if $\xi \in E$ belongs to a partial saturation region of (1), then $\xi$ is an unstable saddle-type EP, i.e., $\xi$ has eigenvalues with both negative and positive real parts. We also note that the origin is an EP of (1).

IV. MAIN RESULTS

A recent paper [13] has addressed convergence of the SCNNs (1) under the following assumption: (A) If $x \notin \mathcal{C}$ is a non-convergent point of (1), then we have $\omega(x) \cap E = \emptyset$. The main result in [13] is that, if assumption (A) is satisfied, then the LIMIT SET DICHOTOMY holds, namely, for any $x, y \in \mathbb{R}^n$ such that $x < y$ we have either

1) $\omega(x) < \omega(y)$, or
2) $\omega(x) = \omega(y) = \{\xi\} \subset E$.

Furthermore, the LIMIT SET DICHOTOMY implies that (1) is almost convergent (13, Th. 3).

As discussed in [13], assumption (A) requires that there do not exist heteroclinic or homoclinic orbits attracting solutions of (1). This is expected to be satisfied generically in the SCNN parameter space. However, since assumption (A) involves some global dynamical aspects of the dynamics of (1), it is actually hard to check for a particular set of SCNN parameters. In this paper we replace assumption (A) by an alternative assumption. Let $\xi \in E$ be a saddle-type EP of (1). We say that $\xi$ is a frozen saddle if, whenever $\xi \in \omega(x)$, we have $x \in \mathcal{C}$ (and so $\omega(x) = \{\xi\} \subset E$).

Assumption 1: All saddle-type EPs of (1), except the origin, are frozen.

The main result in the paper is as follows.

Theorem 1: Suppose that Assumption 1 is satisfied. Then, the LIMIT SET DICHOTOMY holds for the SCNN (1) and (1) is almost convergent.

Proof: In order to prove the LIMIT SET DICHOTOMY for (1), we can follow a number of steps analogous to those in [13, Sects. 7.2, 7.3] (see also [14]), involving the proof of basic preliminary results as the NONORDERING OF LIMIT SETS, COLIMITING PRINCIPLE, INTERSECTION PRINCIPLE, ABSORPTION PRINCIPLE and LIMIT SET SEPARATION PRINCIPLE [10]. Once the LIMIT SET DICHOTOMY is established, it follows that (1) is almost convergent by a proof as that of [10, Th. 1.16]. The proofs of the NONORDERING OF LIMIT SETS and ABSORPTION PRINCIPLE require significant modifications with respect to those given in [13] (see Appendix A for the details). On the other hand, the proofs of the other preliminary results can be obtained by slight modifications of the proofs in [13], once assumption (A) is replaced by the new Assumption 1 (details are omitted due to space limitations).

Theorem 1 shows that we can obtain convergence results analogous to those in [13] under an assumption that is easier to check in practice with respect to assumption (A). Indeed,
Assumption 1 is related to static properties of the EPs of (1) and, as such, can be checked by means of standard numerical programs for finding the EPs of the pwl SCNN (1). In Sect. V we will present parameter ranges for which the SCNNs (1) satisfy Assumption 1.

V. APPLICATIONS

For simplicity of presentation in the following we let

\[ r = p - d, s = p + d \quad (0 < d < p/2). \]

Two subsets of parameters \( p, d \) are of interest, i.e., \( p - 1 > s - r \), where (1) behaves in a local diffusion (LD) mode, and \( p - 1 < s - r \), where (1) displays a global propagation (GP) behavior [2], [3]. In GP mode (1) has only two sinks ([3, Th. 3]), whereas in LD mode the number of sinks is greater than two and grows rapidly with \( n \) [2, Th. 6].

Let \( E_\xi \) be the set of saddle points \( \xi \in E \) such that there exists \( i \in \{1, \ldots, n\} \) for which we have \( \xi_i > 1, \xi_{i+1} > 1 \) (modulo \( n \)), or \( \xi_i \leq -1, \xi_{i+1} < -1 \) (modulo \( n \)).

**Proposition 1:** Suppose that (1) is in LD mode. Then any \( \xi \in E_\xi \) is a frozen saddle.

**Proof:** Let \( \xi \in E_\xi \) and assume that there exists \( i \in \{1, \ldots, n\} \) such that \( \xi_i > 1 \) and \( \xi_{i+1} > 1 \). Let \( r > 0 \) be such that \( B(\xi, r) \subseteq \Gamma_\xi \), where \( \Gamma_\xi \) is the open region of \( \mathbb{R}^n \) where (1) is affine and \( \xi \in \Gamma_\xi \), and \( B(\xi, r) \) is the open ball with radius \( r \) about \( \xi \). Finally, let \( x \in \mathbb{R}^n \) be such that \( \xi \in \omega(x) \). There exists \( t_k \rightarrow +\infty \) such that \( \Phi_{t_k}(x) \rightarrow \xi \), so that for a large \( k \) we have \( z = \Phi_{t_k}(x) \in B(\xi, r) \). Let us denote by \( z(t) \) the solution of (1) with \( z \) as initial condition. It suffices to show that \( z \in \mathcal{C} \). Let \( \tau = \sup\{\theta \geq 0 : z_i(\theta) \geq 1, z_{i+1}(\theta) \geq 1, \forall t \in [0, \theta]\} \). We have \( z_i > 1 \) and \( z_{i+1} > 1 \) and then \( \tau > 0 \). Suppose for contradiction that \( \tau < +\infty \).

Then, there exists \( j \in \{i, i+1\} \) such that \( z_j(\tau) = 1 \) and \( z_j(\tau) \leq 0 \). Moreover, \( \min\{z_i(\tau), z_{i+1}(\tau)\} = 1 \). If \( j = i \), then \( \dot{z}_j(\tau) = -z_i(\tau) + (z_{i-1}(\tau) + p)(z_i(\tau)) + s_g(z_{i+1}(\tau)) = -1 - r_g(z_{i-1}(\tau)) + (z_i(\tau) + s) \geq -1 - r + p + s > p - 1 > 0 \). If \( j = i+1 \), then \( \dot{z}_j(\tau) = -z_{i+1}(\tau) + (z_i(\tau)) + p)(z_{i+1}(\tau)) + s_g(z_{i+2} \tau) = -1 + r + p + s \geq -1 + r + p - s > 0 \). Thus, \( z_j(\tau) \leq 0 \), which is a contradiction. Since \( z_i(\theta) \geq 1 \) for any \( \theta \geq 0 \), it can be seen that the remaining state variables \( x_j \), \( j \in \{1, \ldots, n\}, j \neq i \), satisfy an \( n \) - 1-dimensional SCNN system (1) with a tridiagonal sign-symmetric interconnection matrix \( A \). Then, there exists a diagonal, positive definite matrix \( Q \in \mathbb{R}^{(n-1)\times(n-1)} \), such that \( Q^{-1}AQ \) is symmetric. Convergence then follows from standard results on convergence of symmetric SCNNs [15]. See the proof of [13, Lemma 1] for more details. The case where \( \xi_i \leq -1 \) and \( \xi_{i-1} \leq -1 \) can be proved analogously.

We have conducted numerical simulations, with a standard routine for finding the saddle-type EPs of the pwl SCNN (1), in order to find parameter ranges \( p, d \) where Assumption 1 is satisfied and Theorem 1 can be applied to yield quasi-convergence of (1). This was done for SCNN dimensions \( n \) up to \( n = 13 \).

Figs. 1, 2, and 3 refer to \( n = 9 \), \( n = 8 \), and \( n = 7 \), respectively. In the simulation the parameter range \( [p, d] \in [1, 4] \times [0, p/2] \) was swept with a step size 0.1. The area close to symmetry \( (d = 0) \) was swept with a smaller step 0.05, due to finer bifurcations occurring in this area. Note that the line \( d = (p - 1)/2 \) divides the parameter space in the LD and GP modes.

It is seen that for \( n = 7, 8, 9 \) in GP mode there do not exist saddle-type EPs of (1) except the origin, hence Assumption 1 is satisfied. We conjecture this is true for all values of \( n \). In the LD mode the situation is instead quite different in the three considered cases. When \( n = 9 \), Assumption 1 is satisfied. In fact (1) has only frozen saddles, except the origin, for all parameters in LD mode (Fig. 1). An analogous situation has been found when \( n = 5, 6 \). When \( n = 7 \), there is a small parameter region close to symmetry (see the grey area in the inset of Fig. 3) where (1) possesses saddle-type EPs, different from the origin, that are not frozen saddles. In a large part of the LD area Assumption 1 continues however to be satisfied (light grey region in Fig. 3). A similar situation holds for \( n = 7, 10, 11, 13 \). Finally, when \( n = 8 \), it is seen that Assumption 1 is never satisfied in the LD parameter range (Fig. 2). In fact there is the presence of non-frozen saddles \( \xi \in \mathcal{E} \) of the type \( (-1, 0, 1, 0, -1, 0, 1, 0) \), where \( 0 \) denotes a linear component of \( \xi \) and 1 (resp., -1) denotes a component of \( \xi \) saturated to 1 (resp., -1). The same behavior is present for \( n = 4, 12 \) and is conjectured to be true for any multiple of 4.

![Fig. 1. Relevant subsets of the parameter space p, d](image)

**VI. CONCLUSION**

The paper has given conditions for convergence of a class of nonsymmetric cooperative SCNNs with pwl activations. The conditions, which involve the new concept of a frozen saddle, are easier to verify in practice with respect to those in a recent paper [13]. By means of numerical methods the paper has established parameter ranges for which the proposed conditions are satisfied and the SCNNs satisfy a LIMIT SET DICHOTOMY and are almost convergent.
APPENDIX A

Proposition 2 (Nonordering of Limit Sets): Suppose that Assumption 1 is satisfied. Then, for any $x \in \mathbb{R}^n$, there cannot exist points $p, q \in \omega(x)$ such that $p < q$ ($\omega(x)$ is unordered).

Proof. As in [13] the state space $\mathbb{R}^n$ can be partitioned in the two positively invariant subsets $\Sigma = \{x \in \mathbb{R}^n : 3i \in \{1, \ldots, n\}, 3\tilde{t} \geq 0 : \Phi_i(x)|_{t} \geq 1 \text{ or } \Phi_i(x)|_{t} \leq -1, t \geq \tilde{t}\}$ and $\Lambda = \mathbb{R}^n \setminus \Sigma$. If we have $x < y \in \mathbb{R}^n$ and $x \in \Lambda$ or $y \in \Lambda$, then there exists $\tilde{t} \geq 0$ such that $\Phi_i(x) \ll \Phi_i(y)$ for any $t \geq \tilde{t}$ [13, Property 3]. Moreover, we have $\Sigma \subseteq C$ [13, Lemma 1].

Suppose for contradiction that we can find $x \in \mathbb{R}^n$ and points $p, q \in \omega(x)$ such that $p < q$. If we have $p \in \Lambda$ or $q \in \Lambda$, then due to Property 3 of [13] there exists $\tilde{t} \geq 0$ such that $\Phi_i(p) \ll \Phi_i(q)$. Moreover, since $\omega(x)$ is positively invariant we have $\Phi_i(\{p, q\}) \subset \omega(x)$, which contradicts [10, Th. 8]. Hence, $p, q \in \Sigma$ and there exist $\xi_p, \xi_q \in E$ such that $\Phi_i(p) \rightarrow \xi_p$ and $\Phi_i(q) \rightarrow \xi_q$ as $t \rightarrow +\infty$. Note that $\xi_p, \xi_q \in \omega(x)$. If we had $\xi_p \neq 0$ or $\xi_q \neq 0$, then due to Assumption 1, $\xi_p$ or $\xi_q$ would be a frozen saddle or a sink, and so $x \in C$. Then it would result $\omega(x) = \{\xi_p\} = \{\xi_q\} = \{p\} = \{q\}$, while we assumed $p < q$. Thus, $\xi_p = 0 = \xi_q$ and we can find $\tilde{t} \geq 0$ such that $\Phi_i(p), \Phi_i(q) \in (-1, 1)^n$ for any $t \geq \tilde{t}$, i.e., $p, q \not\in \Sigma$, which is a contradiction.\[\blacksquare\]

Proposition 3 (Absorption Principle): Suppose that Assumption 1 is satisfied. Let $x, y \in \mathbb{R}^n$. If there exists $u \in \omega(x)$ such that $u < \omega(y)$, then $\omega(x) < \omega(y)$. Similarly, if there exists $v \in \omega(y)$ such that $v > \omega(x)$, then $\omega(y) > \omega(x)$.

Proof. Assume that there exists $u \in \omega(x)$ such that $u < \omega(y)$. If $x \in \Sigma$, then $\omega(x) = \{\xi\}$ for some $\xi \in E$. Thus, we have $\xi = u$ and $\omega(x) = \{u\} < \omega(y)$.

Now, let us consider the case $x \in \Lambda$. We have to distinguish two cases.

(a) Suppose that $u \in \Sigma$. Then, $\Phi_i(u) \rightarrow \xi$ as $t \rightarrow +\infty$ for some $\xi \in E$. However, $\xi \neq 0$. In fact, if $\xi = 0$, then $\Phi_i(u) \in (-1, 1)^n$ for any large $t$, and so $u \not\in \Sigma$. Hence, $\xi \in \omega(x)$ is a sink or a frozen saddle and $\omega(x) = \{\xi\} = \{u\} < \omega(y)$.

(b) If $u \in \Lambda$ we can proceed as in the proof of [13, Lemma 3, point (b)].\[\blacksquare\]