HARMONIC BALANCE APPROACH TO PREDICT PERIOD-DOUBLING BIFURCATIONS IN NEARLY SYMMETRIC CNNS

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This paper further investigates on a basic issue that has received attention in the recent literature, namely, the robustness of complete stability of standard Cellular Neural Networks (CNNs) with respect to small perturbations of the nominal symmetric interconnections. More specifically, a class of third-order CNNs with a nominal symmetric interconnection matrix is considered, and the Harmonic Balance (HB) method is exploited for addressing the possible existence of period-doubling bifurcations, and complex dynamics, for small perturbations of the nominal interconnections. The main result is that there are indeed parameter sets close to symmetry for which period-doubling bifurcations are predicted by the HB method. Moreover, the predictions are found to be reliable and accurate by means of computer simulations.

Keywords—Cellular Neural Networks, complete stability, robustness, limit cycles, period-doubling bifurcations, complex dynamics, harmonic balance.

1. Introduction

Standard Cellular Neural Networks (CNNs) with a symmetric neuron interconnection matrix are known to enjoy the fundamental property of being completely stable. This means that each trajectory converges towards a stationary state (an equilibrium point) in the long run behavior. Complete stability is a needed property whenever a CNN is employed for image processing or to solve several other signal processing tasks in real time.

Since the symmetry of the interconnection matrix is not a generic property, the fundamental problem arises as to whether complete stability is robust with respect to small perturbations of the nominal symmetric interconnections. This problem is found to be of importance also in view of the practical CNN applications,
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since in any actual electronic CNN realization it is not realistic to assume exactly symmetric interconnections, due to errors introduced by tolerances in the CNN chip fabrication.\textsuperscript{5}

Unfortunately, the answer to the above problem is in general negative. In fact, it has been found in recent papers that there may exist Hopf bifurcations originating stable limit cycles even arbitrarily close to some nominal symmetric interconnection matrix.\textsuperscript{6,7} More generally, the results in those papers highlight the importance to study and characterize the types of bifurcations which may occur close to symmetry and are responsible for the loss of complete stability. The understanding of such bifurcation mechanisms is believed to be a basic step to arrive at a design procedure for CNNs ensuring not only complete stability in the nominal symmetric case, but also its robustness with respect to sufficiently small perturbations caused by the actual implementation.

More specifically, it has been shown that, when the Jacobian matrix at some equilibrium point of the nominal symmetric neural network has a rank deficiency equal to 2, then there exist arbitrarily small perturbations of the interconnections leading to a Hopf bifurcation originating the birth of a stable limit cycle.\textsuperscript{7} Along this line of reasoning, it is also natural to ask whether for rank deficiencies larger than 2 there may exist more complicated dynamics close to symmetry, such as chaotic attractors. Besides the application interest related to the robustness issue, this question has a theoretical importance, too. In fact, if the answer were affirmative, it follows that quite unexpectedly there may exist a disordered unpredictable behavior close to a thoroughly ordered nominal behavior where all trajectories converge towards equilibrium points.

A recent note\textsuperscript{8} has given a first answer to the previous question, by showing the occurrence of a sequence of period-doubling bifurcations leading to the birth of a complex attractor in a class of three-cell CNNs. All these bifurcations and complex dynamics are located very close to some nominal third-order symmetric CNN for which the rank deficiency of the Jacobian at some equilibrium point is equal to 3, and they have been discovered by means of computer simulations through an exhaustive search in parameter space.\textsuperscript{a}

In this paper we consider a class of nominal symmetric third-order CNNs which includes CNNs previously studied.\textsuperscript{8} For such a class we analytically investigate the possible existence of period-doubling bifurcations, for small perturbations of the interconnections, by using the Harmonic Balance (HB) method. It is well known that the HB principle is an approximate analytical technique that has been widely and successfully employed to predict limit cycles, period-doubling bifurcations, and

\textsuperscript{a}We recall that roughly speaking a period-doubling (or ‘flip’) bifurcation describes the phenomenon according to which a limit cycle of some period $T$ loses its stability as some parameter is varied, and it originates the birth of another stable limit cycle with double period $2T$ (supercritical case).\textsuperscript{9,10} The importance of this type of bifurcation is due to the fact that it is usually related to a cascade of period-doubling bifurcations, a phenomenon which constitutes one of the commonest route to chaos in nonlinear dynamical systems.
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... chaos in several classes of nonlinear circuits and systems. The main result is that the HB analysis is indeed able to reliably predict the existence of period-doubling bifurcations close to symmetry for the considered class of CNNs. Bifurcations are predicted not only for the parameters experimentally found in the previously quoted paper, thus giving a theoretical foundation to this study, but also for other sets of CNN parameters. Moreover, the HB method directly leads to an analytical approximation of the main features of the bifurcated limit cycles, such as amplitude and period.

Notation

\( x = (x_1, \ldots, x_n)^t \): column vector of \( \mathbb{R}^n \);

\( A = [a_{ij}] \): matrix in \( \mathbb{R}^{n \times m} \);

\( \|A\|_M = \max_{i,j} |a_{ij}| \): norm of matrix \( A \);

\( \|y(t)\|_2 \): \( L_2 \) norm of signal \( y(t) \);

\( E_n \): \( n \times n \) identity matrix;

\( e_1 \): vector \([1 \ 0 \ \cdots \ 0]^t \in \mathbb{R}^n \);

\( A^t \): transpose of \( A \);

\( A^{-1} \): inverse of \( A \);

\( A_0 \): region \( \{ x \in \mathbb{R}^3 : \|x\|_M < 1 \} \);

\( A_1 \): region \( \{ x \in \mathbb{R}^3 : |x_2| < 1, |x_3| < 1 \} \).

2. CNN Model and Problem Formulation

Let us consider the nominal third-order CNN described by the differential equations

\[
\dot{x} = -x + G(x) \quad (N)
\]

with \( x \in \mathbb{R}^3 \), and \( G(x) = (g(x_1), g(x_2), g(x_3))^t : \mathbb{R}^3 \to \mathbb{R}^3 \), where \( g(\rho) : \mathbb{R} \to \mathbb{R} \) is the standard piecewise linear (PWL) neuron activation defined as

\[
g(\rho) = \frac{1}{2}(\rho + 1) - |\rho - 1|.
\]

Note that the constant input for (N) is assumed to be 0.

The CNN (N) has the following symmetric neuron interconnection matrix

\[
E_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

i.e., the neurons are uncoupled in the nominal model.

We associate to (N) the perturbed CNN model

\[
\dot{x} = -x + (E_3 + \Delta T)G(x) + \Delta I,
\]

where \( \Delta T = [\delta T_{ij}] \in \mathbb{R}^{3 \times 3} \) is the nonsymmetric perturbation of the nominal interconnection matrix, and \( \Delta I = [\delta I_j] \in \mathbb{R}^3 \) is the perturbation of the input vector.
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The nominal symmetric CNN (N) is known to be completely stable, i.e., each trajectory of (N) tends to a constant limit as \( t \to \infty \). Moreover, it can immediately be verified that any point \( x^e \in \mathbb{R}^3 \) with \( \| x^e \|_M \leq 1 \) is an equilibrium point of (N). Hence, each equilibrium point satisfying \( \| x^e \|_M < 1 \) is degenerate, namely the Jacobian matrix at \( x^e \) has a rank deficiency equal to 3.

The goal of this paper is to investigate whether the perturbed CNN (P) displays a limit cycle which undergoes a period-doubling bifurcation close to the nominal symmetric CNN (N), i.e., for small perturbation coefficients \( |\delta T_{ij}| \) and \( |\delta I_i| \). Furthermore, we are interested in discovering situations where the bifurcated limit cycle has a large amplitude, which is comparable with the saturation level of the neuron activations. Under these circumstances, it is expected that there is the presence of a cascade of period-doubling bifurcations giving origin to a large complex attractor for the CNN (P).

Only few methods are available to study the existence of period-doubling bifurcations. One possibility, which is empirical in nature, is to rely on computer simulations of the dynamical system. Another possibility is to deal with two-dimensional discrete-time systems (Poincaré maps) describing the behavior on suitable sections in the state space. The latter method is exact, but Poincaré maps are usually very complex to analytically characterize. In this paper, we exploit a method based on the Harmonic Balance (HB) principle, in order to predict the existence of period-doubling bifurcations of the CNN (P). This HB method is approximate in nature, but it has already been successfully applied to several classes of nonlinear systems, including the celebrated Chua’s circuit.

It is of importance to remark that the HB method has been mainly developed for a class of systems of the Luré type, which are characterized by a linear system interconnected in feedback with a unique scalar nonlinearity. As such, those systems differ from model (N) (or (P)), which instead have three nonlinear neuron activation functions \( g \). In order to overcome this difficulty, first of all we study a modified perturbed CNN (P1) where only one neuron activation is nonlinear. This enables us to recast (P1) into the form of a Luré system, and apply the HB method to predict the existence of period-doubling bifurcations for (P1) (see Section 3). Then, we embed the cycles and period-doubling bifurcations of (P1) also within a CNN (P) which is close to the nominal symmetric CNN (N) (Section 4).

Before proceeding, we report in Section 2.1 the background material needed for the subsequent development.

2.1. Harmonic Balance (HB) method

In this section we summarize the HB method for the analysis of limit cycles in a class of nonlinear systems.

The considered systems consist of the feedback interconnection of a linear time-invariant subsystem \( \mathcal{L} \) and a nonlinear subsystem \( \mathcal{N} \), as depicted in Fig. 1. This

\(^b\) Convergence can also be directly checked by explicitly solving the three uncoupled equations (N).
feedback structure is usually referred to as Lur’e form for its connection with the classical Lur’e problem, and it is possessed by many systems displaying rich nonlinear dynamics, such as Duffing, Van der Pol, and Chua oscillators. Subsystem $\mathcal{L}$ is assumed to have a strictly proper rational transfer function

$$L(s) = \frac{p(s)}{q(s)},$$

where $p(s)$ and $q(s)$ are suitable polynomials of the complex variable $s$. Subsystem $\mathcal{N}$ is assumed time-invariant memoryless and described by the scalar function $n(\cdot)$. Therefore, the overall system obeys the differential equation

$$q(D)y(t) + p(D)n[y(t)] = 0,$$

where $D$ is the differential operator.

The HB method can be used for the prediction of equilibrium points and limit cycles of system $S$. In the HB framework, equilibria and limit cycles of $S$ are solutions in which the output $y(t)$ is constant or periodic, respectively.

The output equilibrium points (OEP) are constant output solutions $y(t) = y^e$, which are obtained by solving the equation

$$y^e + n(y^e)L(0) = 0.$$  

In the case of prediction of limit cycles, the HB method looks for a harmonic approximation of the periodic output $y(t)$ up to some order. For instance, the classical first order approximation can be taken to be of the form

$$y_0(t) = A + B \cos \omega t.$$  

The corresponding periodic output of $\mathcal{N}$ can be written as

$$n[y_0(t)] = N_0 A + N_1 B \cos \omega t + \ldots,$$  

where $N_0$ and $N_1$ are constants referred to as the bias gain and first harmonic gain, respectively, of subsystem $N$. These two gains, which are usually named the describing function terms, have the following expressions:

$$N_0 = N_0(A, B) = \frac{1}{2\pi A} \int_{-\pi}^{\pi} n[y_0(t)] \, d\omega t$$

$$N_1 = N_1(A, B) = \frac{1}{\pi B} \int_{-\pi}^{\pi} n[y_0(t)] \cos \omega t \, d\omega t.$$  

Note that both $N_0$ and $N_1$ are independent of the frequency $\omega$, since $N$ is assumed to be memoryless. Similarly, one can define higher frequency complex gains $N_k$, $k = 2, 3, \ldots$, which describe the remaining terms of (5).

A Predicted Limit Cycles (PLC) (of first order) is a signal $y_0(t)$ as in (4), where $A$, $B$, and $\omega$ solve the following algebraic equations:

$$A[1 - N_0(A, B)L(0)] = 0,$$

$$1 - N_1(A, B)L(j\omega) = 0.$$  

Such equations follow by equating the constant and first harmonic terms in the output of $L$, given the input signal $n[y_0(t)]$ in (5), with the corresponding terms in $y_0(t)$. Equations (8) and (9) must be solved with respect to the $A$, $B$ and $\omega$. In this respect, we note that the frequency $\omega$ can be obtained simply by solving the real equation $\text{Im}[L(j\omega)] = 0$.

The reliability of a PLC has been investigated since long time. Indeed, the existence of a true limit cycle in some neighbourhood of the predicted one can be rigorously proven if restrictions are placed on the subsystems of $S$. These restrictions basically require that the so-called filtering hypothesis holds, i.e., $L$ has low gain at frequencies greater than $\omega$ and $N$ does not generate large higher harmonics. A quantity usually employed to quantify such hypothesis is the distortion, which describes the amount of the neglected higher harmonics along the system loop and can be expressed as:

$$D = D(A, B, \omega) = \frac{\|\bar{y}_0(t) - y_0(t)\|_2}{\|y_0(t)\|_2}.$$  

Here, the symbol $\| \cdot \|_2$ denotes the $L_2$ norm on the period $2\pi/\omega$, and $\bar{y}_0(t)$ is the steady-state periodic output of the system that is obtained when the closed loop is broken just before the nonlinearity (see Fig. 1) and the signal $y_0(t)$ in (4) is injected into $N$. Clearly, small values of the distortion $D$ indicate that the PLC is reliable.

In addition to determining OEPs and PLCs, the HB method has been also successfully employed to predict the occurrence of limit cycle bifurcations. Here, we are interested in the case of period-doubling bifurcation. Suppose that the transfer function and/or the nonlinear function in Fig. 1 depend on a real parameter $\mu$, i.e., $L(s) = L(s; \mu)$ and/or $n(\cdot) = n(\cdot; \mu)$, and that system $S$ possesses a PLC given by

$$y_0(t; \mu) = A + B \cos \omega t$$  

(11)
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for a range of values of the parameter \( \mu \). Obviously, for each \( \mu \) the PLC parameters \( A, B, \) and \( \omega \) are obtained by solving the following equations

\[
A[1 - N_0(A, B; \mu)L(0; \mu)] = 0, \quad (12)
\]
\[
1 - N_1(A, B; \mu)L(j\omega; \mu) = 0, \quad (13)
\]

where \( N_0(A, B; \mu) \) and \( N_1(A, B; \mu) \) are as in (6) and (7), respectively, with \( y_0(t; \mu) \) in place of \( y_0(t) \).

If the PLC undergoes a period-doubling bifurcation at some critical value \( \mu_c \) of the parameter \( \mu \), then we expect to be able to detect a second nearby PLC for values of \( \mu \) near \( \mu_c \) which contains a subharmonic term at half the frequency. In a first order HB method, this second PLC can be assumed to have the following expression:

\[
y(t; \mu) = A + B \cos \Omega t + \varepsilon \cos \left( \frac{\Omega}{2} t + \varphi \right) + \ldots \quad (14)
\]

where \( \varepsilon \) denotes a small positive coefficient. Note that, if \( A, B, \) and \( \Omega \) are close to \( A, B, \) and \( \omega \) of equation (11), respectively, then for small \( \varepsilon \) the signal in (14) is close to the PLC in (11) and is of approximately twice the period. Moreover, it is expected that the values of the bifurcation parameter \( \mu \) for which such a limit cycle occurs will be given by a smooth function \( \mu(\varepsilon) \) such that \( \mu(0) = \mu_c \).

Motivated by these considerations, in the HB method a Predicted Period-Doubling bifurcation (PPD) is said to occur when a periodic signal of the form (14) exists along system \( \mathcal{S} \) for \( \varepsilon \to 0 \). Practically, this means to first write down the Fourier series expansion of the nonlinearity output \( n[y(t; \mu)] \) as follows

\[
n[y(t; \mu)] = N_0 A + N_1 B \cos \Omega t + N_2 \varepsilon \cos \left( \frac{\Omega}{2} t + \varphi \right) + \ldots \quad (15)
\]

where

\[
N_0 = \frac{1}{2\pi A} \int_{-\pi}^{\pi} n[y(t; \mu)] d\Omega \quad (16)
\]
\[
N_1 = \frac{1}{\pi B} \int_{-\pi}^{\pi} n[y(t; \mu)] e^{-j\Omega t} d\Omega \quad (17)
\]
\[
N_2 = \frac{1}{\pi \varepsilon} \int_{-\pi}^{\pi} n[y(t; \mu)] e^{-j(\frac{\Omega}{2} t + \varphi)} d\frac{\Omega}{2} t \quad (18)
\]

and then to equate for \( \varepsilon \to 0 \) the constant and the harmonic terms of frequency \( \Omega/2 \) and \( \Omega \) in the output of \( \mathcal{L} \), given the input signal \( n[y(t; \mu)] \) in (15), with the corresponding terms in \( y(t; \mu) \).

To express such an equality, we find it convenient to introduce the following complex gain

\[
N_4 = N_2(A, B, \varphi; \mu) = \frac{1}{\pi} \int_{-\pi}^{\pi} n[y_0(t; \mu)] \cos \left( \frac{\omega t}{2} + \varphi \right) e^{-j(\frac{\Omega}{2} t + \varphi)} d\frac{\Omega}{2} t \quad (19)
\]
where \( n'[y_0(t; \mu)] \) denotes of the derivative \( n(\cdot) \) along the PLC in (11). Taking into account that for \( \varepsilon \to 0 \) we have \( A \to A, B \to B, \) and \( \Omega \to \omega, \) it is not difficult to check that the PLC (11) undergoes a PPD bifurcation at \( \mu = \mu_c \) if the following algebraic equations

\[
\begin{align*}
A[1 - N_0(A, B; \mu_c) L(0; \mu_c)] &= 0 \\
1 - N_1(A, B; \mu_c) L(j \omega; \mu_c) &= 0 \\
1 - N_2(A, B, \varphi; \mu_c) L(j \frac{\omega}{2}; \mu_c) &= 0
\end{align*}
\]

are solved for some \( \varphi. \)

Clearly, the first two equations are exactly those in (12) and (13) and therefore simply define the existence of the PLC, while the last one expresses the occurrence of a PPD bifurcation for such a PLC. In this respect, we note that this complex equation can be reduced to a real one. Indeed, it is not difficult to verify that equation (19) can be rewritten as

\[
N_2 = F_0 + F_1 e^{-j2\varphi},
\]

where \( F_0 \) and \( F_1 \) are the first two terms of the Fourier series expansion of \( n'[y_0(t; \mu)]: \)

\[
F_0 = F_0(A, B; \mu) = \frac{1}{2\pi} \int_{-\pi}^{\pi} n'[y_0(t)] d\omega t \\
F_1 = F_1(A, B; \mu) = \frac{1}{2\pi} \int_{-\pi}^{\pi} n'[y_0(t)] e^{-j\omega t} d\omega t.
\]

Hence, the last equation in (20) can be rewritten as

\[
|L^{-1} \left( j \frac{\omega}{2}; \mu_c \right) - F_0(A, B; \mu_c)| = |F_1(A, B; \mu_c)|,
\]  

which can be checked without determining the actual value of \( \varphi. \)

Summing up, the procedure to check whether a PPD bifurcation is occurring at some \( \mu_c, \) consists in first determining a PLC, i.e., solving the first two equations in (20) with respect to \( A, B, \) and \( \omega, \) and then checking if condition (22) is satisfied on the found PLC. Clearly, the reliability of a PPD prediction is measured by the distortion related to the PLC, according to (10).

3. Main Results on Period-Doubling Bifurcations

As discussed in Section 2, the main problem addressed in the paper is to see whether there are perturbed CNNs (P) which are close to (N), in the sense that the perturbation coefficients \( |\delta_{T_{ij}}| \) and \( |\delta_{I_i}| \) are small, and display a period-doubling bifurcation. To tackle this question, we find it convenient to introduce an auxiliary modified perturbed CNN (P1) which possesses only one nonlinear neuron activation. This feature permits to apply the HB method, which we reviewed in Section 2.1, to predict period-doubling bifurcations for a one-parameter family (P1m) of CNNs...
selected from (P1). Finally, by accounting for the relationships between models (P) and (P1m), it is shown that the bifurcations of (P1m) can also be embedded within a CNN (P) which is close to (N).

Let us consider the modified perturbed CNN

\[ \dot{x} = -x + (E_3 + \Delta T)G_1(x) + \Delta I, \tag{P1} \]

where the neuron activations are defined as

\[ G_1(x) = (g(x_1), x_2, x_3)^T, \]

i.e., only the first neuron activation is nonlinear, and the activations of the second and third neuron are linear functions. Note that while the activation \( g(x_1) \) of the first neuron is the same for (P) and (P1), the activations of the second and third neuron of (P) are equal to those of (P1) only for \( x \in \Lambda_1 \). We also remark that (P) and (P1) have the same interconnection matrix \( E_3 + \Delta T \) and input vector \( I \), and that the vector fields defining (P) and (P1) coincide within region \( \Lambda_1 \).

We select a one-parameter family of perturbed systems (P1m),

\[ \dot{x} = -x + (E_3 + \Delta T(\mu))G_1(x) + \Delta I(\mu), \tag{P1m} \]

where \( \mu \) is a real parameter varying in the interval \((1 - \delta \mu, 1 + \delta \mu)\), for some \( \delta \mu > 0 \), and \( \Delta T(\mu), \Delta I(\mu) \) obey the next assumptions.

**Assumption 1.** Matrix \( \Delta T(\mu) \) has the following structure

\[ \Delta T(\mu) = \begin{bmatrix} \mu(1 - \delta_1) - 1 & \delta T_{12} & \delta T_{13} \\ \mu \delta_2 & \delta T_{22} & \delta T_{23} \\ \mu \delta_3 & \delta T_{32} & \delta T_{33} \end{bmatrix}, \tag{23} \]

where \( \delta_1, i = 1, 2, 3, \) and \( \delta T_{ij}, i = 1, 2, 3; j = 2, 3, \) are given constants. Moreover, \( \Delta T(\mu) \) is a nonsingular matrix for \( \mu \in (1 - \delta \mu, 1 + \delta \mu) \), for some \( \delta \mu > 0 \).

**Assumption 2.** We suppose that vector \( \Delta I(\mu) \) is defined as

\[ \Delta I(\mu) = -\Delta T(\mu)x^e, \tag{24} \]

where \( x^e \in \Lambda_0 \) is a given vector.

We point out some features of system (P1m), which are useful for the development.

i) The role of parameter \( \mu \) in Assumption 1 is made clear by Remark 1, where it is shown that, once (P1m) is recast into the form of a Lur’e system, \( \mu \) is the DC gain of subsystem \( \mathcal{L} \) of \( S \) (cf. Fig. 1).
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ii) Assumption 2 means that (P1m) has an equilibrium point \( x^e \) which belongs to the region \( \Lambda_0 \) and which does not depend on parameter \( \mu \). Note that \( x^e \) is also an equilibrium point of (P).

iii) Any given system of family (P1m) is specified by the 9 coefficients \( \delta_i, \ i = 1, 2, 3, \) and \( \delta T_{ij}, \ i = 1, 2, 3; \ j = 2, 3, \) and the components \( x^e_i, \ i = 1, 2, 3, \) of the equilibrium point \( x^e \), in addition to parameter \( \mu \).

iv) The following property establishes a relation between certain solutions of (P1m) and (P), which is an immediate consequence of the definition of model (P1m).

**Property 1.** Suppose that \( \tilde{x}(t), t \geq 0, \) is a solution of (P1) such that \( \tilde{x}(t) \in \Lambda_1, \) for \( t \geq 0. \) Then, \( \tilde{x}(t) \) is also a solution of (P) for \( t \geq 0. \)

v) The next property quantitatively evaluates how matrix \( E_3 + \Delta T(\mu) \) and input vector \( \Delta I(\mu) \) of (P1m) differ from the nominal ones that define the CNN (N).

**Property 2.** Suppose that matrix \( \Delta T(\mu) \) satisfies Assumption 1, and vector \( \Delta I(\mu) \) satisfies Assumption 2. Then,

\[
\| \Delta T(\mu) \|_M \leq \delta \mu + (1 + \delta \mu) \max \{|\delta_1|, |\delta_2|, |\delta_3|\} + \max_{i=1,2,3; j=2,3} |\delta T_{ij}|,
\]

and

\[
\| \Delta I(\mu) \|_M \leq 3 \| \Delta T(\mu) \|_M.
\]

Therefore,

\[
\| \Delta T(\mu) \|_M \to 0; \quad \| \Delta I(\mu) \|_M \to 0,
\]

as \( \delta_i \to 0, \ i = 1, 2, 3, \delta T_{ij} \to 0, \ i = 1, 2, 3; \ j = 2, 3, \) and \( \delta \mu \to 0. \)

By taking into account Property 1 and Property 2, it can easily be realized that the original problem for (P), which is addressed in this paper, can be brought back to the solution of the following problem for (P1m). Namely, the question is to see whether there exist limit cycles for (P1m) which undergo a period-doubling bifurcation, as parameter \( \mu \) is varied, under the following circumstances:

a) The coefficients \( \delta_i, \ i = 1, 2, 3, \delta T_{ij}, \ i = 1, 2, 3; \ j = 2, 3, \) and \( \delta \mu, \) which define model (P1m), are small;

b) The limit cycles belong to region \( \Lambda_1 \) for all parameters \( \mu. \)

Clearly, if such a problem for (P1m) has solution, then the cycles and period-doubling bifurcations of (P1m) are displayed also by a perturbed CNN (P) which is close to the nominal CNN (N).
In what follows we apply the HB method to predict period-doubling bifurcations of (P1m) which satisfy the previous requirements (a) and (b). First of all, let us recast (P1m) into the Lur’e form of system $S$ described in Fig. 1. Define

$$M = \begin{bmatrix} -1 & \delta T_{12} & \delta T_{13} \\ 0 & \delta T_{22} & \delta T_{23} \\ 0 & \delta T_{32} & \delta T_{33} \end{bmatrix}$$

(25)

and

$$H(\mu) = \mu \begin{bmatrix} 1 - \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix}.$$  

(26)

The following holds.

**Proposition 1.** Consider the CNN (P1m), and suppose that $\Delta T(\mu)$ and $\Delta I(\mu)$ satisfy Assumption 1 and Assumption 2, respectively. Then, (P1m) admits a Lur’e representation as in system $S$ of Fig. 1, where the transfer function of $\mathcal{L}$ is

$$L(s; \mu) = e^1(sE_3 - M)^{-1}H(\mu),$$

(27)

the scalar function describing $\mathcal{N}$ is

$$n(\cdot) = g(\cdot + x_1^c) - g(x_1^c),$$

(28)

and the output is

$$y(t) = x_1(t) - x_1^c.$$  

(29)

**Proof:** According to (24)-(26), system (P1m) can be rewritten as

$$\dot{x} = Mx + H(\mu)g(x_1) - Mx_c - H(\mu)x_1^c.$$  

If we perform the change of coordinates

$$z = x - x_c,$$

and take into account that $g(x_1^c) = x_1^c$, it is straightforward to verify that system (P1m) can be written as

$$\dot{z} = Mz + H(\mu)u,$$

$$y = e_1^t z,$$

$$u = g(e_1^t z + x_1^c) - g(x_1^c).$$

(30)

Note that system (30) is in Lur’e form, with $L(s; \mu)$, $n(\cdot)$, and $y$ as in (27), (28) and (30), respectively $\Box$. 


Remark 1

On the basis of (25) and (26), the transfer function $L(s; \mu)$ turns out to be linear with respect to $\mu$. More precisely, it results

$$L(s; \mu) = \frac{\mu}{s+1} [1 + \Delta \Gamma(s)],$$

(31)

where

$$\Delta \Gamma(s) = \Delta_1 + \frac{\Delta_2 s + \Delta_3}{s^2 + \Delta_4 s + \Delta_5},$$

(32)

and $\Delta_i, i = 1, \cdots, 5$, depend on coefficients $\delta_i, i = 1, 2, 3$, and $\delta T_{ij}, i = 1, 2, 3; j = 2, 3$.

It is not difficult to verify that if we set in (P1m) $\delta_i = 0, i = 1, 2, 3$, $\delta T_{ij} = 0, i = 1, 2, 3; j = 2, 3$, and $\delta \mu = 0$, the transfer function in (31) becomes equal to

$$L_0(s) = \frac{1}{s+1}.$$  

(33)

Therefore, for small perturbations $\delta_i, i = 1, 2, 3$, $\delta T_{ij}, i = 1, 2, 3; j = 2, 3$, and $\delta \mu$, it is expected that $L(s; \mu)$ is close to $L_0(s)$ (see also Example 1 in Section 4).

Remark 2

The nonlinearity $n$ in (28) is independent of $\mu$, since by construction the equilibrium point $x^e$ does not depend on $\mu$ (see (24)). Figure 2 depicts $n$ for a given choice of $x^e_1 \in (-1, 1)$.

![Fig. 2. Nonlinearity $n(\rho)$ in (28).](image)

Remark 3 According to (3), the OEPs of system $S$ are the solutions of the equation

$$\frac{1}{\mu(1 + \Delta \Gamma(0))} y^e = g(y^e + x^e_1) - g(x^e_1).$$
Clearly, these solutions can be obtained graphically as the intersections of the nonlinearity \( n \) with the straight line

\[
\frac{1}{\mu(1 + \Delta \Gamma(0))} y,
\]

as depicted in Fig. 3. Hence, by varying \( \mu \) we can have only one solution, infinite solutions or three solutions. Taking into account that parameter \( \mu \) can affect also the stability properties of the equilibria, it can be expected that, by varying \( \mu \), system \( S \) can pass from the case of a unique equilibrium with a stable dynamical behavior, to that of three equilibria with a set of eigenvalues satisfying the typical scenario occurring in Chua’s circuit family, thus making complex dynamics likely to occur.\(^{20,11}\)

As discussed at the end of Section 2.1, the HB method for Lur’e systems as in Fig. 1, amounts to find solutions \( A, B \) and \( \omega \) of the first two equations in (20), and equation (22). Let us now explicitly give the form assumed by these equations for system (P1m), using the Lur’e representation given in Proposition 1. By accounting for (27)-(28) (see also (31) and (32)), the first two equations in (20) and equation (22) become

\[
A[1 - N_0(A, B; \mu_c)(1 + \Delta \Gamma(0; \mu_c))] = 0 \quad (34)
\]
\[
1 - N_1(A, B; \mu_c) \text{Re} \left[ \frac{1}{1 + j\omega}(1 + \Delta \Gamma(j\omega; \mu_c)) \right] = 0 \quad (35)
\]
\[
\text{Im} \left[ \frac{1}{1 + j\omega}(1 + \Delta \Gamma(j\omega; \mu_c)) \right] = 0 \quad (36)
\]
\[
\left| \frac{1 + j\omega/2}{1 + \Delta \Gamma(j\omega/2; \mu_c)} - F_0(A, B; \mu_c) \right| - |F_1(A, B; \mu_c)| = 0. \quad (37)
\]
In particular, the expression for $N_0(A, B)$, $N_1(A, B)$, $F_0(A, B)$ and $F_1(A, B)$ for the nonlinearity $n$ in (28) are reported in Table 1.

Table 1. Describing function terms for the nonlinearity (28).

<table>
<thead>
<tr>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_0(A, B)$</td>
<td>$\frac{1}{\pi A} \left[ \pi A + \left(1 - y_e - A\right) \arccos \left( \frac{1 - y_e - A}{B} \right) - B \sqrt{1 - \left( \frac{1 - y_e - A}{B} \right)^2} \right]$</td>
</tr>
<tr>
<td>$N_1(A, B)$</td>
<td>$\frac{1}{\pi B} \left[ B - B \arccos \left( \frac{1 - y_e - A}{B} \right) + \left(1 - y_e - A\right) \sqrt{1 - \left( \frac{1 - y_e - A}{B} \right)^2} \right]$</td>
</tr>
<tr>
<td>$F_0(A, B)$</td>
<td>$1 - \frac{1}{\pi} \arccos \left( \frac{1 - y_e - A}{B} \right)$</td>
</tr>
<tr>
<td>$F_1(A, B)$</td>
<td>$-\frac{1}{\pi} \sqrt{1 - \left( \frac{1 - y_e - A}{B} \right)^2}$</td>
</tr>
</tbody>
</table>

We have seen that any given system (P1m) is specified by a total of 12 coefficients, in addition to parameter $\mu$. Let us fix the equilibrium point $x^e \in \Lambda_0$ (cf. (24)), and suppose also to fix the scaling parameter $\mu$ to $\mu = 1$ (see Remark 1). Therefore, we actually have at our disposal 9 coefficients $\delta_i$, $i = 1, 2, 3$, and $\delta T_{ij}$, $i = 1, 2, 3; j = 2, 3$, to be chosen such that equations (34)-(37) can be solved with respect to the three unknowns $A$, $B$ and $\omega$. Note that if for some set of coefficients there is a solution $A$, $B$ and $\omega$, then by construction (P1m) has a PPD at $\mu = \mu_e = 1$ (i.e., $\delta \mu = 0$).

It has been found through an optimization searching procedure that there are indeed sets of coefficients $\delta_i$, $i = 1, 2, 3$, and $\delta T_{ij}$, $i = 1, 2, 3; j = 2, 3$, for which (34)-(37) admit solutions $A, B$ and $\omega$. Furthermore, solutions have been obtained even for small values of $\delta_i$, $i = 1, 2, 3$, $\delta T_{ij}$, $i = 1, 2, 3; j = 2, 3$. In Section 4 we present a number of systems defined by such sets of coefficients, together with experimental simulations confirming the reliability of the predictions. Moreover, Section 4 addresses the question of obtaining limit cycles of large size for (P1m), and embedding them within a perturbed CNN (P) which is close to the nominal one (N).

4. Experimental Examples

In this section, we report a number of simulation results confirming that the cycles and bifurcations predicted for (P1m) in Section 3 are reliable. Moreover, the simulations show that it is possible to embed those cycles also within the perturbed CNN (P). In this respect, one key device is related to the choice of the equilibrium point $x^e$ of (P1m) (cf. (24)). Namely, by suitably locating $x^e$ in the region $\Lambda_0$, we impose that a portion of the attraction domain of the limit cycles of (P1m), as
obtained in Section 3, is contained within the region $\Lambda_1$. This permits to embed the cycles and the period-doubling bifurcations also within the original perturbed CNN (P). Clearly, in this construction the cycles involve only one saturated variable ($x_1$). Finally, by changing the location of $x^c$ we are able to obtain bifurcated cycles of (P) with larger size. Eventually, cycles are constructed which involve two or three saturation regions of (P).

4.1. Example 1: Small-size limit cycles

In the first example, we fix the equilibrium point $x^e$ of (P1m) (cf. (24)) close to the positive saturation of variable $x_1$, namely,

$$x^e = \begin{bmatrix} 0.96 \\ 0 \\ 0 \end{bmatrix}.$$  

Let $\mu = 1$. It has been found that for the following set of coefficients defining (P1m),

$$\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} = \begin{bmatrix} 0.0134 \\ -0.0272 \\ 0.0138 \end{bmatrix}; \quad \begin{bmatrix} \delta T_{12} & \delta T_{13} \\ \delta T_{22} & \delta T_{23} \\ \delta T_{32} & \delta T_{33} \end{bmatrix} = \begin{bmatrix} 0.0267 & 0.0148 \\ 0.0010 & 0.0404 \\ -0.0407 & 0.0105 \end{bmatrix},$$  

(38)

equations (34)-(37) can be solved in terms of $A$, $B$ and $\omega$. The solution is

$$A = 0.0317, \quad B = 0.0106, \quad \omega = 0.0326.$$  

This implies that for $\mu = \mu_c = 1$ the first state variable $x_1$ of (P1m) has a PLC centered approximately at $A + x^e_1 = 0.9917$, and with amplitude of about $B = 0.0106$. Moreover, for $\mu = \mu_c = 1$ there is a PPD of this cycle.

Let us see what it actually happens for (P) in relation to the choice of coefficients as in (38), i.e., consider the following one-parameter family of CNNs (P)

$$\dot{x} = -x + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} G(x) + \begin{bmatrix} 0.96 - 0.9471\mu \\ 0.0261\mu \\ -0.0133\mu \end{bmatrix},$$  

(39)

where $\mu \in (0.97, 1.03)$. Note that $\|\delta T_{ij}\|_M < 0.041$, and $\|\delta I_i\|_M < 0.041$.

Figure 4 shows the results of numerical simulations for (39). It is seen that the PLC is quite accurate. Moreover, the simulations show that state variables $x_2$ and $x_3$ along the cycle always remain in the linear region, and the attraction domain of the cycle is in part contained within the region $\Lambda_1$. For $\mu = 1$, the distortion $D$
defined by (10) can be evaluated, and it turns out that $D = 0.016$, i.e., $D \leq 2\%$. We point out that for $\mu = 1$ the period-doubling bifurcation is not present. Nevertheless, as shown in Fig. 4, for a nearby value of $\mu$, namely $\mu = 1.0036$, the period-doubling bifurcation indeed has already occurred. Additionally, for increasing values of $\mu$, the system undergoes a cascade of period-doubling bifurcations, as shown in Fig. 5.

As a final comment, we observe that the coefficients in (38) lead to the following transfer function in (27)

$$L(s; \mu) = \frac{\mu}{s + 1} \left[ 1 + \frac{-0.0131 s^2 - 0.0003 s}{s^2 - 0.0102 s + 0.0013} \right].$$

As discussed in Remark 1, this transfer function is close to $L_0(s)$ in (33). We note however that its Nyquist plot possesses a significant though small topological modification, namely two intersections with the real axis, as shown in Fig. 6, for $\mu = 1$. This shape of the Nyquist plot is a typical requirement for the existence of complex behavior in the HB analysis of Lur’e systems.

4.2. Example 2: Large-size limit cycles

In the second example, we consider a system (P1m) with an equilibrium point which is far from the positive saturation level of $x_1$, i.e.,

$$x_e = \begin{bmatrix} 0.3 \\ -0.1 \\ 0.3 \end{bmatrix}.$$  

Let $\mu = 1$. It has been found that for the following choices for the coefficients defining (P1m),

$$\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} = \begin{bmatrix} 0.0149 \\ 0.0210 \\ 0.0202 \end{bmatrix}; \quad \begin{bmatrix} \delta T_{12} & \delta T_{13} \\ \delta T_{22} & \delta T_{23} \\ \delta T_{32} & \delta T_{33} \end{bmatrix} = \begin{bmatrix} 0.0089 & -0.0277 \\ 0.0092 & 0.0348 \\ -0.0396 & 0.0002 \end{bmatrix},$$

equations (34)-(37) can be solved, and they provide the following values for $A$, $B$ and $\omega$

$$A = 0.5538, \quad B = 0.184, \quad \omega = 0.0326.$$  

Accordingly, let us then consider the one-parameter family of systems (P)

$$\dot{x} = -x + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} G(x) + \begin{bmatrix} 0.9851 \mu - 1 \\ 0.0210 \mu \\ 0.0202 \mu \end{bmatrix} \begin{bmatrix} 0.0089 & -0.0277 \\ 0.0092 & 0.0348 \\ -0.0396 & 0.0002 \end{bmatrix} G(x)$$

$$+ \begin{bmatrix} 0.3092 - 0.2955 \mu \\ -0.0063 \mu - 0.0010 \\ -0.0061 \mu - 0.0040 \end{bmatrix},$$

(41)
Fig. 4. Numerical simulations of system (39) for increasing values of $\mu$. Figures at the left: projection of $x(t)$ onto the $(x_1, x_3)$ plane; figures at the right: projection of $x(t)$ onto the $(x_2, x_3)$ plane. Limit circles of period $T$ and $2T$ are shown. Note that along the cycles it results $|x_2| < 1$ and $|x_3| < 1$. 
Fig. 5. Numerical simulations of system (39) for increasing values of $\mu$. Figures at the left: projection of $x(t)$ onto the $(x_1, x_3)$ plane; figures at the right: projection of $x(t)$ onto the $(x_2, x_3)$ plane. Limit cycles of period $4T$, $8T$ and higher are shown. Note that system (39) undergoes a cascade of period-doubling bifurcations.
where $\mu \in (0.99, 1.01)$. Now, $\|\delta T_{ij}\|_M < 0.04$ and $\|\delta I_i\|_M < 0.02$. Once again, the numerical simulations confirm that the PLC is reliable, see Fig. 7. The figure also shows that the cycle develops within region $A$. It is of interest to note that the size of the limit cycle is now remarkably larger than in Example 1, and it is comparable with the saturation level of the nonlinearities. This can be explained by the fact that $x^e$ is farther from the positive saturation level of $x_1$, with respect to Example 1.

For $\mu = \mu_c = 1$, the system of Example 2 does not show a period-doubling bifurcation. However, such a bifurcation is indeed already occurred for a value of $\mu = 1.0013$, as depicted in Fig. 7. For increasing values of $\mu$, system (41) undergoes a cascade of period-doubling bifurcations, leading to the birth of a complex attractor (Fig. 8). Figure 9 shows a three dimensional view of such an attractor.

4.3. Example 3: Scaling of the perturbation matrices

Let us now consider the one-parameter family of CNN (P)

$$
\dot{x} = -x + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} G(x) + 0.01 \begin{bmatrix} 0.9851\mu - 1 & 0.0089 & -0.0277 \\ 0.0210\mu & 0.0092 & 0.0348 \\ 0.0202\mu & -0.0396 & 0.0002 \end{bmatrix} G(x) + 0.01 \begin{bmatrix} 0.3092 - 0.2955\mu \\ -0.0063\mu - 0.0010 \\ -0.0061\mu - 0.0040 \end{bmatrix},
$$

(42)

where $\mu \in (0.99, 1.01)$. This system has been derived from that of equation (41) by
Fig. 7. Numerical simulations of system (41) for increasing values of $\mu$. Figures at the left: projection of $x(t)$ onto the $(x_1, x_3)$ plane; figures at the right: projection of $x(t)$ onto the $(x_2, x_3)$ plane. Limit cycles of period $T$ and $2T$ are shown.
Fig. 8. Numerical simulations of system (41) for increasing values of $\mu$. Figures at the left: projection of $x(t)$ onto the $(x_1, x_3)$ plane; figures at the right: projection of $x(t)$ onto the $(x_2, x_3)$ plane. Limit cycles of period $4T$ and $8T$, and a complex attractor, are shown.
scaling the perturbation matrices $\Delta T(\mu)$ and $\Delta I(\mu)$ by the same factor $\epsilon = 0.01$. From (24) it is easily verified that system (42) still has an equilibrium point at

$$\begin{bmatrix} 0.3 \\ -0.1 \\ 0.3 \end{bmatrix}.$$  

In addition, one can check that, in the linear region, the orbits of (41) and (42) are the same (the overall effect of $\epsilon$ when the system evolves in the linear region is to perform an expansion on the time variable).

The value of $\epsilon$ allows one to scale the amplitude of the perturbation matrices $\Delta T$ and $\Delta I$, namely, the amplitude is reduced by a factor equal to 100 with respect to Example 2 (this implies that, for $\mu \in (0.99, 1.01)$, $\|\delta T_{ij}\|_M < 0.0004$ and $\|\delta I_i\|_M < 0.0002$). The influence on the dynamics of $\epsilon$ can be seen from Fig. 10, where it can be verified that the orbits of the flow of (42) are indeed left almost unchanged with respect to those of (41).

**4.4. Example 4: Limit cycles involving multiple saturations**

In the final example, we consider the following one-parameter family of CNNs (P),
\[
\dot{x} = -x + \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} G(x) + \begin{bmatrix}
0.2974 - 0.2955\mu \\
-0.0023 - 0.0063\mu \\
0.0109 - 0.0061\mu \\
\end{bmatrix},
\]

where \(\mu \in (0.99, 1.01)\). The CNN has an equilibrium point at

\[
\begin{bmatrix}
0.3 \\
0.275 \\
-0.007 \\
\end{bmatrix}.
\]

Fig. 10. Period-doubling bifurcation and complex attractor for system (42) (cf. Fig. 7 and Fig. 8).

According to the HB method, (43) has a PPD bifurcation for \(\mu = 1\). From Fig. 11, it is seen that, for \(\mu = 1.0016\), a period-doubling bifurcation has already occurred. However, different from the previous examples, the bifurcated limit cycle
Fig. 11. Example of a CNN with a period-doubling bifurcation which involves multiple saturated variables (cf. (43)).

now involves the saturation of all the three state variables. Namely, it can be verified that there are \( \hat{t}_{2i} \) and \( \hat{t}_{3i} \) for which \( |x_2(\hat{t}_{2i})| > 1 \) and \( |x_3(\hat{t}_{3i})| > 1 \), respectively.

We recall that the HB method predicts a period-doubling bifurcation in the case where only the state variable \( x_1 \) can evolve in the saturation region. Nevertheless, when also \( x_2(t) \) and \( x_3(t) \) slightly enter the corresponding saturation regions, their behavior is not too much influenced by the presence of the saturation, as it can be verified by the time-domain simulations. This gives an intuitive explanation of why the PPD is in general still reliable also in the case where there are multiple saturated variables.

5. Conclusion

The paper has studied a class of third-order nominal symmetric Cellular Neural Networks (CNNs) which possess a degenerate equilibrium point, namely, the rank deficiency of the Jacobian at some equilibrium is equal to 3. The Harmonic Balance (HB) method has been used in order to analytically study the presence of period-doubling bifurcations and complex dynamics for small perturbations of the nominal symmetric neuron interconnections. The main result is that the HB method indeed predicts the possible occurrence of such complex dynamics. Moreover, computer simulations have shown that the predictions are accurate and reliable.

We are currently investigating perturbations of larger size nominal symmetric CNNs, which possess an equilibrium point with a rank deficiency equal to 3 and non-zero interconnections between neurons. The results show that the HB method is still able to correctly predict the existence of period doubling bifurcations in this more general situation. Due to space limitation such results will be presented in future work.

The basic message conveyed by this paper, and by the previous ones, where limit cycles generated via Hopf bifurcations in nearly symmetric CNNs were characterized, is that there are unlucky yet possible situations where complete stability of
nominal symmetric CNNs is not robust with respect to perturbations of the interconnections. Clearly, any robust design procedure for CNNs implementation must yield CNN templates such that the equilibrium points are far from the critical situations we have discovered. Future work aims at arriving at a complete scenario of all possible bifurcations which may occur close to symmetry and are responsible of the loss of CNN complete stability.