On the existence and stability of approximate solutions of perturbed vector equilibrium problems

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Received 7 May 2006
Available online 16 January 2007
Submitted by J.A. Ball

Abstract

In this paper we consider several concepts of approximate minima of a set in normed vector spaces and we provide some results concerning the stability of these minima under perturbation of the underlying set with a sequence of sets converging in the sense of Painlevé–Kuratowski to the initial set. Then, we introduce the concept of approximate solution for equilibrium problem governed by set-valued maps and we study the stability of these solutions. The particular case of linear continuous operators is considered as well. © 2006 Elsevier Inc. All rights reserved.

Keywords: Painlevé–Kuratowski convergence; Approximate minima; Stability; Equilibrium problems

1. Introduction

Several problems in optimization such as fixed point problems, vector optimization problems, Nash economic equilibrium problems, variational inclusion problems, complementarity problems can be studied as particular cases of equilibrium problems. Therefore, the topic of equilibrium problems has many useful applications in economics and applied mathematics and for this reason it has received an important amount of attention from many researcher who contributed to the important development of the field. We mention here, without any claim of being exhaustive, the works [4,13,14] and the references therein. In this paper we consider a special class of vector equilibrium problems governed by set-valued maps acting between normed vector spaces. Let $X, Y$ be normed vector spaces, $A$ be a subset of $X$ and $F : A \times A \rightarrow Y$ be a set-valued
map s.t. $0 \in F(a, a)$ for every $a \in A$. One of the problems which we deal with is to find $a \in A$ s.t. $F(a, u) \subset Y \setminus -\text{int } K$ for every $u \in A$, where $K \subset Y$ is a convex closed cone with nonempty interior. We are interested in the study of a concept of approximate solution and in the study of the stability of solutions when $A$ and $F$ are perturbed. Since a solution of the above problem can be seen as a minimal point with respect to $K$ for a certain set, the background of these questions comes from similar problems in vector optimization which we shall consider in the first part of the paper. In fact, the stability of minimal points sets is widely studied in the recent literature: see [2,7,20,22,26] and the references therein. Here we are interested in the nonconvex case. In this framework the stability of solutions of some more general equilibrium problem is studied in [10] under metrically assumptions on set-valued maps. The study of approximate solutions we present has two main motivations: on one hand we want to obtain convergence result for the sets of approximate minima and, on the other hand, we want to point out existence conditions for the solutions of the equilibrium problems using some concepts of weak solutions.

The paper is organized as follows. The notions of approximate minima of a set are introduced in Section 2, along with some useful results used throughout the paper. The convergence results for approximate minima are given in Section 3, where the case of solutions with respect to some enlargements cones is studied as well. In Section 4 we deal with a type of approximate solutions for equilibrium problems with set-valued maps and we apply the results of the preceding section in order to study the stability of these solutions under perturbations of underlying problems. The section ends with the particular case when the set-valued maps are replaced by linear continuous operators.

2. Preliminaries

Let $X$ be a normed vector space and let $K \subset X$ be a proper pointed convex cone which induces a partial order relation $\leq_K$ in $X$ by $x_1 \leq_K x_2$ if and only if $x_2 - x_1 \in K$. If $A \subset X$ is a nonempty set, then a point $a \in A$ is called a minimum point for $A$ with respect to $K$ if $(A - a) \cap -K = \{0\}$. More general, if $K$ is not pointed, then a point $a \in A$ is called a minimum point for $A$ with respect to $K$ if $(A - a) \cap -K \subset K$. If $\text{int } K \neq \emptyset$, then a point $a \in A$ is called weak minimum point of $A$ with respect to $K$ if $(A - a) \cap -\text{int } K = \emptyset$, i.e. it is a minimum point for $A$ with respect to $\text{int } K \cup \{0\}$. Several concepts of approximate solutions have been introduced and studied in the literature (see, e.g., [5,12,18,19,24]). We start by considering two of these concepts. If $k^0 \in K \setminus \{0\}$ is a given element and $\varepsilon > 0$ is a real number, a point $a \in A$ is called an $(\varepsilon, k^0)$ minimum point of $A$ with respect to $K$ if $(A - a + \varepsilon k^0) \cap -K = \emptyset$ and an $(\varepsilon, k^0)$ weak minimum point of $A$ with respect to $K$ if $(A - a + \varepsilon k^0) \cap -\text{int } K = \emptyset$. We denote by $\text{Min}(A, K)$, $\text{WMin}(A, K)$, $(\varepsilon, k^0)$, and $\text{WMin}(A, K)$ the set of minimal points of $A$ with respect to $K$, the set of weak minimum points of $A$ with respect to $K$, the set of $(\varepsilon, k^0)$ minimum points of $A$ with respect to $K$, and the set of $(\varepsilon, k^0)$ weak minimum points of $A$ with respect to $K$, respectively. It is easy to see that the sets $\text{WMin}(A, K)$, $(\varepsilon, k^0) - \text{WMin}(A, K)$ are closed when $A$ is closed. Note that the above definitions have sense if the cone $K$ is formally replaced by any other subset of $X$. If $\text{int } K \neq \emptyset$, then it is clear that for every $\varepsilon > 0$ and for every $k^0 \in K \setminus \{0\}$, $\text{Min}(A, K) \subset \text{WMin}(A, K)$ and $(\varepsilon, k^0) - \text{Min}(A, K) \subset (\varepsilon, k^0) - \text{WMin}(A, K)$. Moreover, if $k^0 \in \text{int } K$ then for every $0 < \varepsilon < \delta$, $(\varepsilon, k^0) - \text{WMin}(A, K) \subset (\delta, k^0) - \text{Min}(A, K)$. It is also clear that, $(\varepsilon, k^0) - \text{WMin}(A, K) = \text{WMin}(A, K - \varepsilon k^0)$ and, in fact, these notions of approximate minimum envisage a perturbation of the underlying cone. Of course, the new ordering set $(K - \varepsilon k^0)$ does not give a genuine order relation, since it is no longer a cone.
We recall that a convex set $B$ is said to be a base for the cone $K$ if $0 \notin \text{cl} \ B$ and $K = \text{cone} \ B$, where cl denotes the topological closure and cone $B := [0, \infty) B$ is the cone generated by $B$.

**Lemma 2.1.** Let $K$ be a convex pointed cone. The following relations hold:

(i) if $k^0 \in K \setminus \{0\}$ then $K + (0, \infty)k^0 \subset K \setminus \{0\}$;

(ii) if $\text{int} \ K \neq \emptyset$ and $k^0 \in \text{int} \ K$ then $\text{int} \ K = K + (0, \infty)k^0$.

**Proof.** (i) It is obvious that $K + (0, \infty)k^0 \subset K$ because $K$ is a convex cone and since $K$ is pointed, $0 \notin K + (0, \infty)k^0$, whence the conclusion.

(ii) Since $(0, \infty)k^0 \subset \text{int} \ K$ and $K + \text{int} \ K = \text{int} \ K$, it follows that $K + (0, \infty)k^0 \subset \text{int} \ K$. Take $v \in \text{int} \ K$. Thus $v - K$ is a neighborhood of 0, whence there exists $\rho > 0$ s.t. $D(0, \rho) \subset v - K$, where $D(0, \rho)$ denotes the closed ball centered at 0 with radius $\rho$. Taking into account that $\rho\|k^0\|^{-1}k^0 \in D(0, \rho)$ one has $\rho\|k^0\|^{-1}k^0 - v \in -K$ i.e. $v \in \rho\|k^0\|^{-1}k^0 + K \subset (0, \infty)k^0 + K$ and the proof is complete. $\square$

**Lemma 2.2.**

(i) If $k^0 \in K \setminus \{0\}$ and $a \in \text{Min}(A, K)$ then $a \in (\varepsilon, k^0) - \text{Min}(A, K)$ for every $\varepsilon > 0$.

(ii) If $\text{int} \ K \neq \emptyset$ and $k^0 \in \text{int} \ K$ then the following are equivalent:

(a) $a \in \text{WMin}(A, K)$;
(b) $a \in (\varepsilon, k^0) - \text{Min}(A, K)$ for every $\varepsilon > 0$;
(c) $a \in (\varepsilon, k^0) - \text{WMin}(A, K)$ for every $\varepsilon > 0$.

**Proof.** (i) If would exist a positive $\varepsilon$ and an element $u \in K$ s.t. $-u \in A - a + \varepsilon k^0$, then, according to the item (i) in previous lemma, one has

$$A - a \ni -\varepsilon k^0 - u \in - (0, \infty)k^0 - K \subset -K \setminus \{0\},$$

against the minimality of $a$.

(ii) For the implication from (a) to (b), we proceed as above using the item (ii) of Lemma 2.1. For the converse, we suppose, by contradiction, that there exists $u \in \text{int} \ K$ s.t. $-u \in A - a$. Using again Lemma 2.1(ii), we can find a positive $\gamma$ with $u - \gamma k^0 \in K$. Then $-u + \gamma k^0 \in A - a + \gamma k^0$, whence $(A - a + \gamma k^0) \cap -K \neq \emptyset$, a contradiction. The last part is similar. $\square$

At the end of this section we present a variational optimality condition for an approximate minimum point using an approach developed in [11]. One uses the notation $X^*$ for the topological dual of $X$, $K^* := \{y^* \in Y^* \mid y^*(y) \geq 0, \forall y \in K\}$ for the dual of $K$ and $N(A, a)$ for the basic normal cone of Mordukhovich to $A$ at $a$. We recall (see [23, Definition 1.1]) that using the Fréchet subdifferential ($\partial F$) one can define the Fréchet normal cone to a closed set $A \subset X$ at a point $a \in A$ in the following way:

$$N_{\partial F}(A, a) := \partial F I_A(a),$$

where $I_A$ is the indicator function of $A$. Now, if $X$ is an Asplund space, the basic normal cone to $A$ at $a$ is

$$N(A, a) = \{x^* \mid \exists x_n \overset{A}{\rightharpoonup} a, \ x_n^* \overset{w}{\rightharpoonup} x^*, \ x_n^* \in N_{\partial F}(A, x_n)\}.\]
Theorem 2.1. Assume that \( X \) is an Asplund space, \( A \) is closed, \( \text{int} \, K \neq \emptyset \) and \( k^0 \in K \setminus \{0\} \). Let \( a \in (\varepsilon, k^0) - \text{Min}(A, K) \). Then there exists \( \tilde{a} \in A \), \( \|a - \tilde{a}\| \leq \sqrt{\varepsilon} \) s.t. for every \( e \in \text{int} \, K \) there exist \( u^* \in K^*, \ u^*(e) = 1, \ x^* \in X^*, \ \|x^*\| \leq 1 \) s.t.

\[-u^* \in \sqrt{\varepsilon}u^*(k^0)x^* + N(A, \tilde{a}).\]

Proof. Obviously, the identity function \( id \) on \( X \) is Lipschitz and since \( A \) is a closed set in a Banach space it is a complete metric space endowed with the distance given by the norm. Thus, it is easy to see that we are in the conditions of the vectorial variant of Ekeland variational principle in [14, Corollary 3.10.14]. Applying this result we get an element \( \tilde{a} \in A \) s.t. \( \|a - \tilde{a}\| < \sqrt{\varepsilon} \) and having the property that it is minimal element over \( A \) for the function

\[ h(x) := id(x) + \sqrt{\varepsilon}\|x - \tilde{a}\|k^0. \]

Take \( e \in \text{int} \, K \) and consider the functional \( s_e : X \to \mathbb{R}, \ s_e(x) = \inf \{\lambda \in \mathbb{R} \mid \lambda e \in y + K\} \). According [14, Corollary 2.3.5], \( \tilde{a} \) is a minimal point over \( A \) for the scalar function \( s_e \circ h \). Applying Theorem 3.36 from [23] we deduce

\[ 0 \in \partial(s_e \circ h)(\tilde{a}) + N(A, \tilde{a}). \]

It remains to use Lemma 2.1 from [11] and the classical calculus rules for convex function to deduce that there exist \( u^* \in K^*, \ u^*(e) = 1, \ x^* \in X^*, \ \|x^*\| \leq 1 \) s.t.

\[-u^* \in \sqrt{\varepsilon}u^*(k^0)x^* + N(A, \tilde{a}), \]

i.e. the conclusion. \( \Box \)

3. Convergence for sets of approximate minima

Let \( A, (A_n)_{n \in \mathbb{N}} \) be nonempty sets in \( X \). We shall use the following notations:

\[
\begin{align*}
\liminf A_n &= \{x \in X \mid \exists (x_n), \ x_n \in A_n, \ \forall n \in \mathbb{N}, \ x_n \to x\}, \\
\limsup A_n &= \{x \in X \mid \exists (x_n), \ x_n \in A_n, \ \forall k \in \mathbb{N}, \ x_n \to x\}, \\
w - \limsup A_n &= \{x \in X \mid \exists (x_n), \ x_n \in A_n, \ \forall k \in \mathbb{N}, \ x_n \wghto x\}
\end{align*}
\]

where \( \wghto \) denotes the weak topology of \( X \).

Definition 3.1.

(a) One says that \( A \) is the Painlevé–Kuratowski limit of \( (A_n) \) and notes \( A_n \xrightarrow{P-K} A \) if the next conditions hold:

\[ P - K_- : A \subset \liminf A_n \quad \text{and} \quad P - K_+ : \limsup A_n \subset A. \]

(b) One says that \( A \) is the Mosco limit of \( (A_n) \) and notes \( A_n \xrightarrow{M} A \) if the next conditions hold:

\[ M_- : A \subset \liminf A_n \quad \text{and} \quad M_+ : w - \limsup A_n \subset A. \]

It is clear that both above defined limits are unique and closed in the norm topology when they exist. For further details, see [3].

In the sequel we consider that the sets \( A, (A_n)_{n \in \mathbb{N}} \) which appear are closed unless otherwise stated. The next result lists some convergence result for approximate minimal sets.
Proposition 3.1. Suppose that \( \text{int} K \neq \emptyset \).

(i) Take \( k^0 \in \text{int} K \) and consider two real numbers \( 0 < \delta < \varepsilon \). If \( A_n \xrightarrow{P-K} A \) then \( \limsup(\delta, k^0) - \text{WMin}(A_n, K) \subset (\varepsilon, k^0) - \text{Min}(A, K) \).

(ii) Take \( k^0 \in K \setminus \{0\} \) and consider a sequence \((\varepsilon_n)_{n \in \mathbb{N}}\) of positive numbers converging towards 0. Then for every closed subset \( A \) of \( X \) one has \( \limsup(\varepsilon_n, k^0) - \text{Min}(A, K) \subset \text{WMin}(A, K) \). If, moreover, \( k^0 \in int K \) then \((\varepsilon_n, k^0) - \text{Min}(A, K) \xrightarrow{P-K} \text{WMin}(A, K) \).

(iii) As above, take \((\varepsilon_n)_{n \in \mathbb{N}}\) of positive numbers converging towards \( \varepsilon \). If \( A_n \xrightarrow{P-K} A \) then \( \limsup(\varepsilon_n, k^0) - \text{WMin}(A_n, K) \subset (\varepsilon, k^0) - \text{WMin}(A, K) \).

Proof. (i) Let \( a \in \limsup(\delta, k^0) - \text{Min}(A_n, K) \). Following the definition, this means that there exists a subsequence \((n_k)_{k \in \mathbb{N}}\) of the sequence of natural numbers and \( a_k \in (\delta, k^0) - \text{Min}(A_{n_k}, K) \) for every \( k \in \mathbb{N} \) s.t. \( a_k \to a \). Using the hypotheses we have that \( a \in A \). Suppose that \( a \notin (\varepsilon, k^0) - \text{Min}(A, K) \), that is there exists \( u \in -K \setminus u \in A - a + \varepsilon k^0 \). Thus there exists \( b \in A \) s.t. \( u = b - a + \varepsilon k^0 \). Since \( A \) is the Painlevé–Kuratowski limit of \( (A_n) \) one can find for every \( n \in \mathbb{N} \) an element \( b_n \in A_n \) s.t. \( b_n \to b \). Then, for every \( k \),

\[
\begin{align*}
    b_k - a_k + \delta k^0 &= b - a + \varepsilon k^0 + b_k - b - (a_k - a) + (\delta - \varepsilon)k^0.
\end{align*}
\]

But \((\varepsilon - \delta)k^0 \in int K \), so there exists \( \rho > 0 \) s.t. \( D(0, \rho) \subset (\varepsilon - \delta)k^0 - \text{int} K \). For a sufficiently large \( k \), \((b_k - b) - (a_k - a) \in D(0, \rho)\), whence \((b_k - b) - (a_k - a) + (\delta - \varepsilon)k^0 \in \text{int} K \). By use of the above relation we conclude that for \( k \) large enough, \( b_k - a_k + \delta k^0 \in \text{int} K \), a contradiction.

(ii) For the first part of the conclusion, let us consider a strictly increasing sequence \((n_k)_{k \in \mathbb{N}}\) and for every \( k \), take \( a_k \in (\varepsilon_{n_k}, k^0) - \text{Min}(A, K) \) s.t. \( a_k \to a \). Since \( A \) is closed, \( a \in A \). Suppose now that \( a \notin \text{WMin}(A, K) \), i.e. there exist \( u \in \text{int} K, u \in A - a \). Then

\[
0 \in A - a - u = A - a + \varepsilon_{n_k} k^0 + (-u - \varepsilon_{n_k} k^0) = A - a_k + \varepsilon_{n_k} k^0 + a_k - a - u - \varepsilon_{n_k} k^0.
\]

Recalling that \( a_k - a \to 0 \) and \( \varepsilon_{n_k} \to 0 \) we have that for \( k \) large enough, \( a_k - a - u - \varepsilon_{n_k} k^0 \in \text{int} K \subset K \) and this is a contradiction. If \( k^0 \in \text{int} K \), as we already observed, \( \text{WMin}(A, K) \subset (\varepsilon, k^0) - \text{Min}(A, K) \) for every positive \( \varepsilon \), whence in this case the inclusion \( \text{WMin}(A, K) \subset \liminf(\varepsilon_n, k^0) - \text{Min}(A, K) \) is immediate.

(iii) As above, take \( a_k \in (\varepsilon_{n_k}, k^0) - \text{WMin}(A_{n_k}, K) \) for any \( k \in \mathbb{N} \) s.t. \( a_k \to a \in A \). By contradiction, if \( a \notin (\varepsilon, k^0) - \text{WMin}(A, K) \) there exist \( u \in \text{int} K, b \in A, u = b - a + \varepsilon k^0 \). Then there exist \( (b_n) \subset A_n, b_n \to b \). Thus,

\[
0 \in A_{n_k} - b_{n_k} = A_{n_k} - a + a - b_{n_k} = A_{n_k} - a_{n_k} + a_{n_k} - a - b_{n_k} + b + a = A_{n_k} - a_{n_k} + \varepsilon_{n_k} k^0 + (a_{n_k} - a) + (b - b_{n_k}) + (a - b - \varepsilon k^0) + \varepsilon k^0 - \varepsilon_{n_k} k^0.
\]

For \( k \) large enough this gives that \((A_{n_k} - a_{n_k} + \varepsilon_{n_k} k^0) \cap \text{int} K \neq \emptyset \), a contradiction. \( \square \)

From the item (i) of the above results we reobtain in the next corollary a well-known assertion concerning the \( P - K_+ \) convergence of weak minimal sets (see [21,22]).

Corollary 3.1. Suppose that \( \text{int} K \neq \emptyset \). If \( A_n \xrightarrow{P-K} A \) then \( \limsup \text{WMin}(A_n, K) \subset \text{WMin}(A, K) \).
Proof. Let \( a \in \limsup W\text{Min}(A_n, K) \). Thus there exists a subsequence \((n_k)_{k \in \mathbb{N}} \) and \( a_k \in W\text{Min}(A_{n_k}, K) \) for every \( k \in \mathbb{N} \) s.t. \( a_k \to a \). Accordingly, \( a \in A \). Take a positive, arbitrary but fixed, \( \varepsilon \) and consider a positive \( \delta \) smaller than \( \varepsilon \). Then, from Lemma 2.2, we have that \( a_k \in (\delta, k^0) - \text{Min}(A_{n_k}, K) \) for every \( k \), and from Proposition 3.1, \( a \in (\varepsilon, k^0) - \text{Min}(A, K) \). Since \( \varepsilon \) was arbitrarily chosen, we conclude (again from Lemma 2.2), that \( a \in W\text{Min}(A, K) \). □

In general, the results concerning the \( P - K \) convergence of the minimal sets require more special assumptions (see [7,26]). We want to present such a result in a nonconvex setting (but for sets of approximate minima) and to this end we consider a compactness assumption inspired by the quoted papers.

Proposition 3.2. Let \( k^0 \in K \setminus \{0\} \) and \( \varepsilon > 0 \). Suppose that \( A_n \xrightarrow{P-K} A \) and that for every strictly increasing sequence of natural numbers \((n_k)_{k} \), for every convergent sequence \( a_{n_k} \in A_{n_k} \) (for all \( k \)) and every sequence \( b_{n_k} \in A_{n_k} \cap (a_{n_k} - K) \) (for all \( k \)), \( (b_{n_k}) \) admits a convergent subsequence. Then \( (\varepsilon, k^0) - \text{Min}(A, K) \subseteq \liminf (\varepsilon, k^0) - \text{Min}(A_n, K) \).

In particular, the same conclusion holds if one supposes that

(i) \( X \) is finite-dimensional, the sequence of sets \((A_n)\) is uniformly bounded and \( A_n \xrightarrow{P-K} A \), or

(ii) \( X \) is a reflexive Banach space, the sequence of sets \((A_n)\) is uniformly bounded and \( A_n \xrightarrow{M} A \).

Proof. Take \( a \in (\varepsilon, k^0) - \text{Min}(A, K) \). Since \( a \in A \subseteq \liminf A_n \), one can find \((a_{n_k})\) s.t. \( a_{n_k} \in A_{n_k} \) for every \( n \) and \( a_n \to a \). If \( a \notin \liminf (\varepsilon, k^0) - \text{Min}(A_n, K) \), there exists \((n_k)\) strictly increasing, s.t. \( a_{n_k} \notin (\varepsilon, k^0) - \text{Min}(A_{n_k}, K) \) for every \( k \in \mathbb{N} \). This means that for every \( k \) one can find \( b_{n_k} \in A_{n_k} \) and \( u_{n_k} \in -K \) s.t. \( u_{n_k} = b_{n_k} - a_{n_k} + \varepsilon k^0 \). Then

\[
  b_{n_k} = a_{n_k} - u_{n_k} - \varepsilon k^0 \subset A_{n_k} \cap (a_{n_k} - K - \varepsilon k^0) \subset A_{n_k} \cap (a_{n_k} - K).
\]

In our assumptions, we deduce that \((b_{n_k})\) admits a convergent subsequence and the limit \( b \) of this subsequence belongs to \( A \) because \( \limsup A_n \subset A \). Then the corresponding subsequence of \((u_{n_k})\) is also convergent to an element \( u \) which belongs to \(-K\) (since \( K \) is closed). Passing to the limit we obtain \(-u = b - a + \varepsilon k^0\), in contradiction with the \((\varepsilon, k^0)\) minimality of \( a \). □

The second (compactness) assumption is essential in the above result. For example, take \( X = \mathbb{R} \), \( K = \mathbb{R}_+ \), \( k^0 = 1 \), \( \varepsilon = 2^{-1} \), \( A = \{0\} \) and \( A_n = \{-n, 0\} \) for every \( n \geq 1, n \) even and \( A_n = \{0\} \) for every \( n \geq 1, n \) odd. Then \( A_n \xrightarrow{P-K} A \), but \((\varepsilon, k^0) - \text{Min}(A, K) = \{0\} \) is not included in \( \liminf (\varepsilon, k^0) - \text{Min}(A_n, K) \), because \((\varepsilon, k^0) - \text{Min}(A_n, K) = \{-n\} \) for every \( n \geq 1, n \) even. It is easy to see that the assumption we use in the proof does not hold for \( a_{2k} = 0 \) and \( b_{2k} = -2k \).

Let us mention that if every set \( A_n \) satisfies the domination property (i.e. for every \( x \in A_n \) there is \( a \in \text{Min}(A_n, K) \) s.t. \( x - a \in K \)) then the compactness condition in the previous result can be modified as follows: for every strictly increasing sequence of natural numbers \((n_k)\), for every convergent sequence \( a_{n_k} \in A_{n_k} \) (for all \( k \)) and every sequence \( b_{n_k} \in \text{Min}(A_{n_k}, K) \cap (a_{n_k} - K) \) (for all \( k \)), \((b_{n_k})\) admits a convergent subsequence.

Let us briefly consider another two notions of approximate solutions. Take \( \delta \geq 0 \). We say that \( a \in A \) is a \( \delta \) minimum (and we write \( a \in \text{Min}(A_n, K) \) if \( (A - a) \cap (-K \setminus D(0, \delta)) = \emptyset \). Similarly, we say that \( a \in A \) is a \( \delta \) weak minimum (and we write \( a \in \delta - \text{WMin}(A_n, K) \) if \( (A - a) \cap (-\text{int } K \setminus D(0, \delta)) = \emptyset \). It is clear that \( 0 - \text{Min}(A, K) = \text{Min}(A, K) \) and \( 0 - \text{WMin}(A, K) = \text{WMin}(A, K) \). Note that, in general, the notions of approximate minima already introduced are
independent, as simple examples can show. Nevertheless, in the case of a $K$-monotone norm, $-K - \varepsilon k^0 \subseteq -K \setminus D(0, \varepsilon \|k^0\|)$ and whence, $\varepsilon \|k^0\| - \operatorname{Min}(A, K) \subset (\varepsilon, k^0) - \operatorname{Min}(A, K)$.

**Proposition 3.3.**

(i) The following equivalences hold: $a \in \operatorname{Min}(A, K)$ (respectively $a \in \operatorname{WMin}(A, K)$) if and only if $a \in \delta - \operatorname{Min}(A, K)$ (respectively $a \in \delta - \operatorname{WMin}(A, K)$) for every $\delta > 0$.

(ii) If $A_n \xrightarrow{P-K} A$ and $\delta > 0$, then $\limsup \delta - \operatorname{WMin}(A_n, K) \subset \delta - \operatorname{WMin}(A, K)$.

**Proof.** (i) We prove only the non-brackets equivalence. If $a \in \operatorname{Min}(A, K)$, since for any positive $\delta$, we have $-K \setminus D(0, \delta) \subset -K \setminus \{0\}$, if follows that $a \notin \delta - \operatorname{Min}(A, K)$. For the converse, if $a \notin \operatorname{Min}(A, K)$, then there exists $u \in -K \setminus \{0\}$, $u \in A - a$. Taking $\delta$ s.t. $0 < \delta < \|u\|$, we have that $u \notin -K \setminus D(0, \delta)$, a contradiction.

(ii) Let $(n_k)$ be strictly increasing and $a_{n_k} \in \delta - \operatorname{WMin}(A_{n_k}, K)$ s.t. $a_{n_k} \to a \in A$. If $a \notin \delta - \operatorname{WMin}(A, K)$, we can find $u \in A$, $u - a \in -\operatorname{int} K \setminus D(0, \delta)$. From hypotheses, there exist $(u_n) \to u$, $u_n \in A_n$ for every $n$. Then for $k$ large enough,

$$u_{n_k} - a_{n_k} = u_{n_k} - u + u - a + a - a_{n_k} \notin \operatorname{int} K \setminus D(0, \delta)$$

and this contradiction completes the proof. \qed

A result concerning the stability of weak minimal sets with respect to the perturbed cone is the following. Similar result for minimal sets or even for $\delta$ minimal sets can be proved as well in the same way.

**Theorem 3.1.** Suppose that $(K_n)_n$ and $K$ are the closed convex cones with nonempty interiors s.t. $\limsup X \setminus \operatorname{int} K_n \subset X \setminus \operatorname{int} K$. If $A_n \xrightarrow{P-K} A$, then $\limsup \operatorname{WMin}(A_n, K_n) \subset \operatorname{WMin}(A, K)$.

**Proof.** Let $a_k \in \operatorname{WMin}(A_{n_k}, K_{n_k})$ for every $k$, $a_k \to a \in A$. Take $u \in A$. Since there exists $(u_n)_n \subset K_n$, $u_n \to u$, we have that $u_{n_k} - a_k \to u - a$. But for every $k$, $u_{n_k} - a_k \in Y - \operatorname{int} K_{n_k}$ and from hypotheses, $u - a \in Y - \operatorname{int} K$. Since $u$ was arbitrarily chosen in $A$, we obtain that $A - a \subset Y - \operatorname{int} K$, i.e. $a$ is a weak minimum for $A$. \qed

Let us mention that in [20, Theorem 2.1], under additional assumption that $\bigcup_{n=1}^{\infty} \operatorname{WMin}(A_n, K_n)$ is relatively compact, the authors obtained a stronger convergence for the sets of weak minima, i.e., $d(\operatorname{WMin}(A_n, K_n), B) \geq d(\operatorname{WMin}(A, K), B)$ for every bounded subset $B$ of $X$ (here $d(C, D)$ denotes the distance between the sets $C$ and $D$).

As one can see from the above results and their proofs, in general, it is easier to handle with weak minimum points than with minimum points. The extra difficulties in the latter case come from the fact that the set $(Y \setminus K) \cup \{0\}$ is not closed. On the other hand, if $\operatorname{int} K = \emptyset$ one cannot speak about the weak minima. So, it is natural to consider the situations when a minimal point with respect to $K$ can be viewed as a weak minimal point with respect to another ordering cone constructed as an enlargement of $K$.

First, we characterize the convex sets $A$ for which one has the equality between the set of weak minimal points and the set of minimal points. In [15], the notion of rotund set is introduced: a nonempty convex set $A \subset X$ is said to be rotund when its boundary does not contain line segments. From the definition one can see that a convex set $A$ which is rotund and is not a
singleton must have nonempty interior. The “only if” part of the next result is Proposition 4.3 from [22]. For the reader’s convenience, we present the proof for this part as well.

**Proposition 3.4.** Let $A$ be a convex, closed set with nonempty interior. Then $A$ is rotund if and only if for every closed convex cone $K$ with nonempty interior one has $W\text{Min}(A,K) = \text{Min}(A,K)$.

**Proof.** Let us observe that in the case $X = \mathbb{R}$, the assertion is obvious. Suppose first that $A$ is rotund and $K$ is a closed convex cone with $\text{int} K \neq \emptyset$. Suppose, without losing the generality that $0 \in W\text{Min}(A,K) \setminus \text{Min}(A,K)$. Then there exists $y \in A \cap (-K \setminus (-\text{int} K \cup \{0\}))$. Then, from hypothesis, there exists $\alpha \in (0,1)$ s.t. $\alpha y \in \text{int} A$. For a sufficiently small $k \in -\text{int} K$, one has $\alpha y + k \in A \cap -\text{int} K$, a contradiction.

For the converse, suppose (without losing the generality) that the segment $[0,u]$ lies in the boundary of $A$. Then the compact convex set $[0,u]$ has no interior points with $A$ and following a well-known separation theorem, there exist $x^* \in X^* \setminus \{0\}$ and a real $\alpha$ s.t. for all $a \in A$ and for all $v \in [0,u]$, we have $x^*(a) \geq \alpha \geq x^*(v)$. Since $0 \in [0,u] \subset A$, we deduce that $x^*(v) = \alpha = 0$ for every $v \in [0,u]$, that is $[0,u] \subset \{x^* = 0\}$ and $A \subset [x^* \geq 0]$, where $[x^* = 0] := \{x \in X \mid x^*(x) = 0\}$ and the other notation is similar. Consider now $y^* \in X^* \setminus \{0\}$ s.t. $x^*$ and $y^*$ are linearly independent and $y^*(u) < 0$ (note that such an $y^*$ always exists because the dimension of $X$ is greater than 1 and, moreover, $y^*(v) < 0$ for every $v \in (0,u)$) and take $K := [x^* \geq 0] \cap \{y^* \geq 0\}$. It is clear that $K$ is a closed convex cone, $[0,u] \subset A \cap -K = A \cap [x^* \leq 0] \cap \{y^* \leq 0\}$ and $u \notin K$. This proves that $0 \notin \text{Min}(A,K)$. It is also clear that $\text{int} K = [x^* > 0] \cap \{y^* > 0\}$ and $A \cap -\text{int} K = \emptyset$. It remains to prove that $\text{int} K$ is nonempty in order to deduce that $0 \in W\text{Min}(A,K)$, in contradiction with our assumption. Suppose that $[x^* > 0] \subset \{y^* \leq 0\}$, which means that $[x^* \geq 0] \subset [y^* \leq 0]$. But since $x^*$ and $y^*$ are linearly independent, there exists $z \in X$ s.t. $x^*(z) = 0$ and $y^*(z) > 0$, a contradiction. The proof is complete. \hfill $\square$

Using the “only if” part of the above result and Corollary 3.1 the authors have obtained in [22, Theorem 4.4] a result concerning $P - K_+$ convergence of the minimal sets. In the same manner, we have the next result.

**Corollary 3.2.** If $K$ is a closed convex pointed cone with nonempty interior, $k^0 \in K \setminus \{0\}$ and $A$ is a convex rotund set, then $a \in \text{Min}(A,K)$ if and only if $a \in (\varepsilon,k^0) - \text{Min}(A,K)$ for every $\varepsilon > 0$.

**Proof.** It results from the above result and Lemma 2.2. \hfill $\square$

Outside the convex case, we present some results concerning the same problem of characterizing a minimal point as a weak minimal point. In fact, we shall see some situations where a minimal point can be seen as a proper minimal point considering the perturbations of the ordering cone with larger cones defined by some explicit formulae. For a given closed convex pointed cone $K$ (not necessarily with nonempty interior), the following conical $\varepsilon$-enlargement ($\varepsilon > 0$) is studied in the literature (see [16,17,25]):

$$K_\varepsilon = \{u \in X \mid d(u,K) < \varepsilon \|u\|\} \cup \{0\},$$

where $d(x,A) = \inf_{a \in A} \|x - a\|$ denotes the distance from the point $a$ to the set $A$. It is clear that the so-defined $K_\varepsilon$ is a cone (since for every $x \in X$ and $t \geq 0$, $d(tx,K) = td(x,K)$) which
contains $K$ and $K_\varepsilon \setminus \{0\}$ is open. It is also clear that for $\varepsilon \geq 1$, $K_\varepsilon = X$. It is shown in [2, Proposition 3.2.1] that a closed pointed convex cone $K$ admits a convex $\varepsilon$-enlargement if and only if $K$ has a bounded base. One says that $a \in A$ is an $\varepsilon$-strong solution (in the sense of this enlargement) if $a$ is a minimum point of $A$ with respect to $K_\varepsilon$ for a certain $0 < \varepsilon < 1$ (in fact $a$ is a weak minimum because $K_\varepsilon \setminus \{0\}$ is open). In some situations described below, a minimum point with respect to $K$ is an $\varepsilon$-strong solution.

**Proposition 3.5.** Suppose that $X$ is finite-dimensional, $a \in A$ and cone$(A - a)$ is closed. Then $a \in \text{Min}(A, K)$ if and only if there exists $\varepsilon > 0$ s.t. $a \in \text{Min}(A, K_\varepsilon) = \text{WMin}(A, K_\varepsilon)$.

**Proof.** The “if” part is obvious. Let us prove the “only if” part, supposing that $(A - a) \cap -K = \{0\}$. Then it is easy to see that cone$(A - a) \cap -K = \{0\}$. Since cone$(A - a)$ is closed and $K$ has a compact base ($X$ is finite-dimensional) then following a cone separation result (see [6]), there exists a convex pointed cone $S$ s.t. $K \setminus \{0\} \subset \text{int} S$ and cone$(A - a) \cap -S = \{0\}$. Since the last equality is equivalent with $(A - a) \cap -S = \{0\}$, for our purpose it is enough to prove that there exists $\varepsilon > 0$ s.t. $K_\varepsilon \setminus \{0\} \subset \text{int} S$. We proceed by contradiction: suppose that for every positive $\varepsilon$ there exist $u_\varepsilon \in K_\varepsilon \setminus \{0\}$ s.t. $u_\varepsilon \notin \text{int} S$. Accordingly, for every natural number $n \neq 0$ one can find $u_n \notin \text{int} S$, $u_n \neq 0$ with $d(u_n, K) < n^{-1}\|u_n\|$. Thus $d(\|u_n\|^{-1}u_n, K) < n^{-1}$. But, on a subsequence $\|u_n\|^{-1}u_n \to u \neq 0$. Since the distance function is continuous, $u \in K$ and since $X \setminus \text{int} S$ is closed, then $u \notin \text{int} S$ and this is a contradiction. This completes the proof. □

Another type of enlargements can be defined using the so-called Henig dilating cones. The definition and some properties of these cones are given below.

**Lemma 3.1.** (See [14, Lemma 3.2.51].) Let $K \subset X$ be a closed convex cone with a base $B$ and take $\delta = d(0, B) > 0$. For $\varepsilon \in (0, \delta)$, consider $B_\varepsilon = \{x \in X \mid d(x, B) \leq \varepsilon\}$ and $K_\varepsilon = [0, \infty)B_\varepsilon$, the cone generated by $B_\varepsilon$. Then

(i) $K_\varepsilon$ is a closed convex cone for every $\varepsilon \in (0, \delta)$;
(ii) if $0 < \gamma < \varepsilon < \delta$, $K \setminus \{0\} \subset K_\gamma \setminus \{0\} \subset \text{int} K_\varepsilon$;
(iii) $K = \bigcap_{\varepsilon \in (0, \delta)} K_\varepsilon = \bigcap_{n \in \mathbb{N}} K_{\varepsilon_n}$ where $(\varepsilon_n) \subset (0, \delta)$ converges to 0.

In the next proposition $T_B(A, a)$ denotes the Bouligand tangent cone to $A$ at $a$ (see [1] for details).

**Proposition 3.6.** Suppose that $X$ is finite-dimensional, $a \in A$ and cone$(A - a)$ is closed. Then the following are equivalent:

(i) $a \in \text{Min}(A, K)$;
(ii) there exists $\varepsilon > 0$ s.t. $a \in \text{Min}(A, K_\varepsilon)$;
(iii) there exists $\delta > 0$ s.t. $T_B(A + K_\delta, a) \cap -\text{int} K_\delta = \emptyset$.

**Proof.** Again, it is obvious that (ii) implies (i). For proving that (i) implies (ii) we proceed as in the proof of Proposition 3.5 to obtain the existence of a convex pointed cone $S$ s.t. $K \setminus \{0\} \subset \text{int} S$ and cone$(A - a) \cap -S = \{0\}$. It remains to prove that there exists $\varepsilon > 0$ s.t. $K_\varepsilon \subset S$. Since the compact base $B$ of $K$ is contained in int$S$, there exists $\varepsilon \in (0, \delta)$ (notation from the above lemma) s.t. the $\varepsilon$-enlargement $B_\varepsilon$ of $B$ is also contained in int$S$. This ensures that int $K_\varepsilon \setminus \{0\} \subset$...
exists a proper convex cone $C$ we obtain that the cone $C \cap R_a$ is necessarily closed. We present now a result characterizing the sets which generate closed cones.

Of course, in the setting of Proposition 3.6 we obtain that $a$ is a proper minimal point in the sense of Henig. We recall (see [14, p. 110]) that $a \in A$ is called Henig proper minimal if there exists a proper convex cone $C \subset X$ s.t. $K \setminus \{0\} \subset \text{int } C$ and $a \in \text{Min}(A, C)$. In fact, in our result we obtain that the cone $C$ has the special form of a Henig dilating cone.

In both Propositions 3.5 and 3.6 we have used the condition that the set $A - a$ generates a closed cone. We can indicate the example of the cardioid in $A$ and this proves that $a = 0$ and applying (ii) from the above lemma we get that $a \in \text{Min}(A, K^e)$. \hfill $\square$

Theorem 3.2. Let $A \subset X$ be a nonempty closed set.

(i) If $0 \notin A$, then $\text{cl cone } A = \text{cone } A \cup A^\infty$.

(ii) If $0 \in A$, then $\text{cl cone } A = \text{cone } A \cup A^\infty \cup A^0$.

Proof. (i) Let us consider first the situation $0 \notin A$. Obviously, cone $A \subset \text{cl cone } A$. Also from the definition of $A^\infty$ we have $A^\infty \subset \text{cl cone } A$. For the converse, take $(u_n) \subset \text{cone } A$, $u_n \to u$ and prove that $u \in \text{cone } A \cup A^\infty$. Indeed, if $u = 0$, then, obviously, $u \in \text{cone } A$. Suppose that $u \neq 0$; then, for every natural $n$ there exists $t_n \geq 0$ and $a_n \in A$ s.t. $u_n = t_n a_n$. If the sequence $(a_n)$ is unbounded, then one has a subsequence $(a_{n_k})$ with $\|a_{n_k}\| \to \infty$. Then $t_{n_k} \to 0$. Without relabeling one can suppose that $t_{n_k} \downarrow 0$, whence $u \in A^\infty$. Suppose that $(a_n)$ is bounded. Since $0 \notin A$, and $A$ is closed, then there exists a positive $\gamma$ s.t. $\|a_n\| \geq \gamma$ for all $n$. We conclude that $(t_n)$ is bounded, hence it has a subsequence $(t_{n_k})$ convergent towards a nonnegative number $t$. If $t = 0$, then $u_{n_k} \to 0 = u$ (an excluded situation for this stage of the proof). Thus, $t > 0$ and...
\[ \|a_{nk} - t^{-1}u\| = t^{-1}\|ta_{nk} - u\| = t^{-1}\|ta_{nk} - u + (t - t_{nk})a_{nk}\| \leq t^{-1}\|ta_{nk} - u\| + |t - t_{nk}|t^{-1}\|a_{nk}\| \rightarrow 0. \]

Consequently, that \(a_{nk} \rightarrow t^{-1}u\), and since \(A\) is closed, \(u \in \text{cone } A\). The proof of this part is complete.

(ii) Again the inclusion cone \(A \cup A^\infty \cup A^0 \subset \text{cl cone } A\) is obvious. Take \(u \in \text{cl cone } A\) and keep the notation from the first part of the proof. In contrast with the previous case, a new situation can appear: the sequence \((a_n)\) converges to 0. Then \(t_n \rightarrow \infty\), so \(u \in A^0\). This completes the proof. \(\square\)

Thus, one obtains the next characterization result.

**Corollary 3.3.** Let \(A \subset X\) be a nonempty closed set.

(i) If \(0 \notin A\), cone \(A\) is closed if and only if \(A^\infty \subset \text{cone } A\).

(ii) If \(0 \in A\), then cone \(A\) is closed if and only if \(A^\infty \cup A^0 \subset \text{cone } A\).

4. Applications to vector equilibrium problems

Let \(G, (G_n)_{n \in \mathbb{N}} : X \rightharpoonup Y\) be set-valued maps with nonempty values from \(X\) to \(Y\). As usual, we denote the graph of \(G\) by

\[ \text{Gr } G = \{ (x, y) \in X \times Y \mid y \in G(x) \}. \]

For a subset \(P\) of \(X\) the direct image through \(G\) is \(G(P) := \bigcup_{x \in P} G(x)\). In [10], starting from the case of linear operators studied in [27], several assumptions on a sequence of set-valued maps \(\{G_n; \ (n \in \mathbb{N})\}\) are introduced and their role in obtaining stability results for equilibrium problems in a nonconvex setting is studied. Here we list some of these notions needed in the sequel.

(A1) \(\text{Gr } G_n \xrightarrow{P-K} \text{Gr } G\).

(A2) For every \((x, y) \in \text{Gr } G\), there exist a neighborhood \(V\) of \(y\) and \(n_0 \in \mathbb{N}\) s.t. for all \(n \geq n_0\), \(x_n' \rightharpoonup x\) and \(x_n'' \rightharpoonup x\), there exists \((l_n) \subset (0, \infty)\) with \(\lim_{n \rightarrow \infty} l_n \|x_n' - x_n''\| = 0\) s.t.

\[ G_n(x_n') \cap V \subset G_n(x_n'') + l_n\|x_n' - x_n''\|U_Y. \]

(A3) There exist \(\alpha > 0\) and \(n_0 \in \mathbb{N}\) s.t. for all \(n \geq n_0\), \(x, x' \in X\) and \(y \in G_n(x), z \in G_n(x')\),

\[ \|y - z\| \geq \alpha\|x - x'\|. \]

Note that (A2) describes a kind of uniform Lipschitz behavior of \((G_n)_n\) on \(\text{Gr } G\) (but is weaker than the request that the set-valued maps to be uniformly pseudo-Lipschitz in each point of \(\text{Gr } G\)), while (A3) is a condition of uniform growth.

The vector equilibrium problems which we envisage are the following ones: let \(A \subset X\) be a nonempty closed set, \(F : X \times X \rightharpoonup Y\) be a set-valued map s.t. \(0 \in F(a, a)\) for every \(a \in A\), and \(Q \subset Y\) be a closed convex pointed cone with nonempty interior.

**VEP**\((F, A, Q)\): find \(\bar{a} \in A\) s.t. \(F(\bar{x}, u) \subset (Y \setminus -Q) \cup \{0\}\) for every \(u \in A\), and

**WVEP**\((F, A, Q)\): find \(\bar{a} \in A\) s.t. \(F(\bar{x}, u) \subset Y \setminus \text{int } Q\) for every \(u \in A\).
In the latter problem (the weak variant of the first one) the set $Y \setminus \text{int } Q$ is closed and this fact is essential for proving existence results of solutions for this problem: see, for instance [14, Section 3.8], [8], among others. In order to remind a result in this direction, we recall first that a set-valued map $G: X \rightrightarrows Y$ is called lower semicontinuous in a point $x \in X$ if for every open set $V$ in $Y$ with $G(x) \cap V \neq \emptyset$ there exists a neighborhood $U$ of $x$ such that $G(u) \cap V \neq \emptyset$ for every $u \in U \cap \text{Dom } G$. $G$ is said to be lower semicontinuous on a subset $A \subseteq X$ if it is lower semicontinuous at every point in $A \cap \text{Dom } G$.

**Theorem 4.1.** (See [8, Theorem 3.1].) Let $A$ be a nonempty, convex, closed subset of $X$, $Q$ be a convex closed cone with nonempty interior in $Y$ and $F: A \times A \rightrightarrows Y$ be a set-valued map with nonempty values. Suppose that:

(i) for every $u \in A$ the set-valued map $x \mapsto F(x, u)$ is lower semicontinuous on $A$;
(ii) for each finite set $\{u_1, u_2, \ldots, u_n\} \subseteq A$, $\text{conv}\{u_1, u_2, \ldots, u_n\} \subseteq \{a \in A \mid \exists i = 1, n, F(a, u_i) \subseteq Y \setminus \text{int } Q\}$;
(iii) there exists $A_0 \subseteq X$ a compact subset and $u_0 \in A_0 \cap A$ s.t. for every $x \in A \setminus A_0$, $F(x, u_0) \cap \text{int } Q \neq \emptyset$.

Then $W\text{VEP}(F, A, Q)$ has a solution.

Note that this result still holds if we replace $Y \setminus \text{int } Q$ with any other closed set. Note also that if $A$ is compact, then the assumption (iii) can be dropped and, in contrast, the closedness of $A$ and $Y \setminus \text{int } Q$ and the lower semicontinuity of $F(\cdot, u)$ are still in force in the proof (based on Ky Fan’s lemma).

The aim of this section is to present some concepts of approximate solutions for the described problems and to study the stability of these solution with respect to the perturbations of multifunction $F$ and/or the perturbation of $A$. The basis of defining approximate solutions for $\text{VEP}(F, A, Q)$ and $W\text{VEP}(F, A, Q)$ consists of the following simple observations: $\tilde{a}$ is a solution of $\text{VEP}(F, A, Q)$ if and only if $0 \in \text{Min}(F(\tilde{a}, A), Q)$ and $\tilde{a}$ is a solution of $W\text{VEP}(F, A, Q)$ if and only if $0 \in \text{WMin}(F(\tilde{a}, A), Q)$. Accordingly, for an $\varepsilon > 0$ and for a $k^0 \in Q \setminus \{0\}$, we say that $a$ is an $(\varepsilon, k^0)$ solution for $\text{VEP}(F, A, Q)$ if $0 \in (\varepsilon, k^0) - \text{Min}(F(\tilde{a}, A), Q)$. In the same way, $a$ is an $(\varepsilon, k^0)$ solution for $W\text{VEP}(F, A, Q)$ if $0 \in (\varepsilon, k^0) - \text{WMin}(F(\tilde{a}, A), Q)$ and the concepts of $\delta$ solutions can be defined similarly. Following Theorem 2.1 we have the following necessary optimality condition for a point to be approximate solution for $\text{VEP}(F, A, Q)$.

**Proposition 4.1.** Assume that $Y$ is an Asplund space, int $Q \neq \emptyset$ and $k^0 \in Q \setminus \{0\}$. Let $a$ be an $(\varepsilon, k^0)$ solution of $\text{VEP}(F, A, Q)$. If $F(a, A)$ is closed, then there exists $u \in F(a, A)$, $\|u\| \leq \sqrt{\varepsilon}$ s.t. for every $e \in \text{int } Q$ there exist $v^* \in Q^*$, $u^*(e) = 1$, $y^* \in Y^*$, $\|y^*\| \leq 1$ s.t.

$$-v^* \in \sqrt{\varepsilon}v^*(k^0)y^* + N(F(a, A), u).$$

For easy reference we shall denote by $\tilde{F}$ the set-valued map $\tilde{F}: X \rightrightarrows Y$, $\tilde{F}(x) = F(x, A) = \bigcup_{a \in A} F(x, a)$. Consider now a sequence $\{F_n; \ (n \in \mathbb{N})\}$ of set-valued maps acting between $X \times X$ and $Y$ s.t. $0 \in F_n(a, a)$ for every $a \in A$ and for every $n \in \mathbb{N}$. Hence, we are interested in the stability of solutions of $\text{VEP}(F, A, Q)$ and $W\text{VEP}(F, A, Q)$ under perturbations of $F$ and $A$.

**Proposition 4.2.** Suppose that $\{\tilde{F}_n; \ (n \in \mathbb{N})\}, \tilde{F}$ satisfy (A1) and (A2). If for every $n$, $a_n$ is a solution of $W\text{VEP}(F_n, A, Q)$, then every cluster point of $(a_n)$ is a solution of $W\text{VEP}(F, A, Q)$. 
Proof. Without loosing the generality, we can suppose that \( a_n \to a \in A \). First we prove that 
\[
\tilde{F}_n(a_n) \overset{P-K}{\longrightarrow} \tilde{F}(a), \text{ i.e. } F_n(a_n, A) \overset{P-K}{\longrightarrow} F(a, A).
\]

Let \( y \in \tilde{F}(a) \); since \( \text{Gr} \tilde{F} \subset \text{lim inf} \text{Gr} \tilde{F}_n \), we obtain that there exists a sequence \((x_n, y_n) \to (a, y), (x_n, y_n) \in \text{Gr} \tilde{F}_n \) for all \( n \). Applying (A2) at the point \((a, y)\), there exists a neighborhood \( V \) of \( y \) and \((l_n) \subset (0, \infty) \) with \( \lim_{n \to \infty} l_n \|x_n - a_n\| = 0 \), such that for \( n \) large enough,
\[
y_n \in \tilde{F}_n(x_n) \cap V \subset \tilde{F}_n(a_n) + l_n \|x_n - a_n\| U_Y
\]

(we used that \( y_n \to y \)); hence we can find \( y'_n \in \tilde{F}_n(a_n) \) s.t., for all \( n \) large enough,
\[
\|y_n - y'_n\| \leq l_n \|x_n - a_n\| \to 0
\]

and this implies that \( y'_n \to y \); since \( y'_n \in \tilde{F}_n(a_n) \) we obtain that \( y \in \text{lim inf} \tilde{F}_n(a_n) \). Consider now \( y \in \text{lim sup} \tilde{F}_n(a_n) \); there exists \((n_k) \) s.t. \( y_{n_k} \to y \) with \( y_{n_k} \in \tilde{F}_n(a_{n_k}) \) for all \( n_k \).

But \((a_{n_k}, y_{n_k}) \in \text{Gr} \tilde{F}_n \) and \((a_{n_k}, y_{n_k}) \to (a, y)\); using (A1) we have that \((a, y) \in \text{Gr} \tilde{F} \), i.e., \( y \in \tilde{F}(a) \). Thus, the announced convergence is proved. Now, since \( 0 \in \text{WMin}(F_n(a_n, A), Q) \), for every \( n \), it follows that \( 0 \in \text{lim sup} \text{WMin}(F_n(a_n, A), Q) \). Applying Corollary 3.1, we deduce that \( 0 \in \text{WMin}(F(a, A), Q) \), that is \( a \) is a solution of \( \text{WVEP}(F, A, Q) \). \( \square \)

In some sense the assumption in the preceding result are minimal to ensure \( \tilde{F}_n(a_n) \overset{P-K}{\longrightarrow} \tilde{F}(a) \) (see Theorem 2.1 from [10]).

**Theorem 4.2.** Let \( k^0 \in \text{int} Q \) and \( 0 < \delta < \varepsilon \). Suppose that \( \text{Gr} F \subset \text{lim inf} \text{Gr} F_n \{F_n; (n \in \mathbb{N}), F\} \) satisfy (A2) and (A3), \( \text{lim sup} F_n(A \times A) \subset F(X \times X) \) and \( A_n \overset{P-K}{\longrightarrow} A \). If for every \( n \), \( a_n \) is a \((\delta, k^0)\) solution of \( \text{WVEP}(F_n, A_n, Q) \), then every cluster point of \( (a_n) \) is an \((\varepsilon, k^0)\) solution of \( \text{VEP}(F, A, Q) \).

**Proof.** Suppose again that \( a_n \to a \in A \). The first step is to show that under the assumptions we made \( F_n(a_n, A_n) \overset{P-K}{\longrightarrow} F(a, A) \) (see also Corollary 2.3 in [10]). Let \( y \in F(a, A) \); there exists \( x \in A \) with \( y \in F(a, x) \) i.e. \((a, x, y) \in \text{Gr} F \). Since \( \text{Gr} F \subset \text{lim inf} \text{Gr} F_n \) and \( A \subset \text{lim inf} A_n \) we obtain that there exist a sequence \((a_n, x_n, y_n) \to (a, x, y), (a_n, x_n, y_n) \in \text{Gr} F_n \) for all \( n \) and a sequence \( y'_n \to y, \quad x'_n \in A_n \) for all \( n \). Applying (A2) at the point \((a, x, y)\), there exists a neighborhood \( V \) of \( y \) and \((l_n) \subset (0, \infty) \) with \( \lim_{n \to \infty} l_n \|x'_n - x_n\| = 0 \), such that for \( n \) large enough,
\[
y_n \in F_n(a_n, x_n) \cap V \subset F_n(a_n, x'_n) + l_n \|x'_n - x_n\| U_Y
\]

hence we can find \( y'_n \in F_n(a_n, x'_n) \) s.t., for all \( n \) large enough,
\[
\|y_n - y'_n\| \leq l_n \|x'_n - x_n\| \to 0
\]

and this implies that \( y'_n \to y \). Since \( y'_n \in F_n(a_n, x'_n) \subset F_n(a_n, A_n) \) we obtain that \( y \in \text{lim inf} F_n(a_n, A_n) \), thus \( F(a, A) \subset \text{lim inf} F_n(a_n, A_n) \).

Let now \( y \in \text{lim sup} F_n(a_n, A_n) \); there exists \((n_k) \) s.t. \( y_{n_k} \to y \) and for each \( n_k \), \( y_{n_k} \in F_{n_k}(a_{n_k}, x_{n_k}) \) for some \( x_{n_k} \in A_{n_k} \); since \( \text{lim sup} F_n(A \times A) \subset F(X \times X) \), we have \( y \in F(X \times X) \), hence there exists \((u, x) \in X \times X \) with \((u, x, y) \in \text{Gr} F \). Using that \( \text{Gr} F \subset \text{lim inf} \text{Gr} F_n \), there exists \((u'_n, x'_n, y'_n) \to (u, x, y), (u'_n, x'_n, y'_n) \in \text{Gr} F_n \) for all \( n \). Since \( y_{n_k} \to y \) and \( y'_n \to y \); one can use (A3) to find \( \alpha > 0 \), \( k_0 \in \mathbb{N} \) s.t. for each \( k \geq k_0 \) we have
\[
\alpha \|a_{n_k}, x_{n_k}\| - \|u'_{n_k}, x'_{n_k}\| \leq \|y_{n_k} - y'_n\| \to 0
\]
But since \( x'_{n_k} \to x \) and \( u'_{n_k} \to u \), this implies that \( x_{n_k} \to x \) and \( u = a \). This means that \( x \in A \) and \( y \in F(a, A) \), whence \( \limsup F_n(a_{n}, A_n) \subset F(a, A) \). Now, since \( 0 \in (\delta, k^0) - \text{WMin}(F_n(a_{n}, A_n), Q) \), for every \( n \), it follows that \( 0 \in \limsup (\delta, k^0) - \text{WMin}(F_n(a_{n}, A_n), Q) \). Applying Proposition 3.1, we deduce that \( 0 \in (\varepsilon, k^0) - \text{Min}(F(a, A), Q) \), i.e. \( a \) is an \((\varepsilon, k^0)\) solution of \( \text{VEP}(F, A, Q) \). \( \square \)

The above result can be considered also a kind of existence result for \( \text{VEP}(F, A, Q) \) using approximations with weak solutions. Of course, similar results to the preceding ones can be written using the others items of Propositions 3.1, 3.3 or Theorem 3.1. Let us observe that, having in mind Proposition 3.6, one can see that \( a \) is a solution of \( \text{VEP}(F, A, Q) \) if and only if it is a solution of \( \text{WVEP}(F, A, Q^\varepsilon) \) for some \( \varepsilon \) provided that \( Y \) is finite-dimensional and cone \( F(a, A) \) is closed.

In order to illustrate the applicability of Theorem 4.2 let us consider the following definition (see [28]): a nonempty set \( C \subseteq X \) is called \( \sigma \)-compact if there is a sequence \((C_n)_{n \in \mathbb{N}}\) of compact sets satisfying \( C = \bigcup_{n=1}^{\infty} C_n \). Note that, at least in the finite-dimensional setting, the class of \( \sigma \)-compact sets contains every closed set and every open set as well (for an open \( D \) take \( C_n = \{ x \in D \mid \|x\| \leq n, d(x, X \setminus D) \leq n^{-1} \} \)). Hence for a finite-dimensional vector space every \( G_\delta \) set and every \( F_\sigma \) set is \( \sigma \)-compact. In this context, as an example of set which is not \( \sigma \)-compact we can indicate the set of irrational numbers in \( \mathbb{R} \).

Observe that, without loss of generality, the sequence \((C_n)\) in the definition of \( \sigma \)-compact set can be taken to be nondecreasing in the sense of inclusion (replace \( C_n \) by \( \bigcup_{1 \leq i \leq n} C_i \) ) and, if \( C \) is convex, one can consider that every \( C_n \) is convex as well (replace \( C_n \) by \( \text{conv} C_n \) ). So we can suppose that \( \bigcup_{n=1}^{\infty} C_i \xrightarrow{p-K} \text{cl } C \). From Theorem 4.2 we have the following consequence.

**Corollary 4.1.** Let \( k^0 \in \text{int } Q \) and \( 0 < \delta < \varepsilon \). Suppose that \( A \) is \( \sigma \)-compact \((A = \bigcup_{n=1}^{\infty} A_n\), where \((A_n)\) is a nondecreasing sequence of compacts\). If \( a_n \) is a \((\delta, k^0)\) solution of \( \text{WVEP}(F, A_n, Q) \), then every cluster point of \((a_n)\) is an \((\varepsilon, k^0)\) solution of \( \text{VEP}(F, \text{cl } A, Q) \).

We turn now our attention to another important particular case, supposing that \( F_n = T_n \) for every \( n \) and \( F = T \), where \( \{T_n; \ (n \in \mathbb{N}), \ T\} \) are linear continuous operators from \( X \times X \) into \( Y \). In this case:

- \((A1)\) is equivalent to \((P1)\): \( T_n(u, v) \to T(u, v) \) for every \((u, v) \in X \times X \) (see [27] for the use of this condition).
- \((A2)\) is ensured if the next condition \((P2)\) holds: the sequence of norms \((\|T_n\|)_n\) is bounded; note that, taking into account the uniform boundedness principle, \((P1)\) implies \((P2)\) provided that \( X \) is a Banach space.
- \((A3)\) is equivalent to \((P3)\): there exist \( \alpha > 0 \) and \( n_0 \in \mathbb{N} \) s.t. for all \( n \geq n_0 \), \((u, v) \in X \times X \), \( \|T_n(u, v)\| \geq \alpha \| (u, v) \| \) (see also [27]).

According to these observations we obtain the next corollary.

**Corollary 4.2.** Let \( X \) be a Banach space, \( k^0 \in \text{int } Q \) and \( 0 < \delta < \varepsilon \). Suppose that \( \{T_n; \ (n \in \mathbb{N}), \ T\} \) satisfy \((P1)\) and \((P3)\), \( \limsup T_n(A \times A) \subset T(X \times X) \) and \( A \xrightarrow{p-K} A \). If \( a_n \) is a \((\delta, k^0)\) solution of \( \text{WVEP}(T_n, A_n, Q) \), then every cluster point of \((a_n)\) is an \((\varepsilon, k^0)\) solution of \( \text{VEP}(T, A, Q) \).
References