Computing the variable coefficient telegraph equation using a discrete eigenfunctions method


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Abstract

This paper deals with the construction of discrete numerical solutions of mixed problems for the telegraph equation. After discretization, the two-variables partial difference mixed problem is solved by means of a discrete eigenfunctions method that mimics the advantages of the continuous eigenfunction method while eliminating its computational disadvantages. The solution is based on a closed form solution of the inhomogeneous second order difference equation without increasing the problem’s dimension.

Keywords: Telegraph equation; Discrete eigenfunctions method; Numerical solution; Explicit difference scheme

1. Introduction

The telegraph equation is important for modeling several relevant problems such as signal analysis [1], wave propagation [2], random walk theory [3], etc. The continuous eigenfunctions method provides an infinite series solution involving eigenpairs of variable coefficient Sturm–Liouville problems [4]. Although numerical solutions of Sturm–Liouville problems are widely studied [5], the continuous eigenfunctions method is computationally unpractical, and a numerical alternative is desirable.

The use of a difference scheme and the further algebraic treatment of the resulting discretized problem disregards the advantage of the eigenfunctions expansion of the source term. The aim of this paper is to construct a finite series numerical solution by developing a discrete eigenfunctions method for the discretized problem. For the case of the diffusion equation, this approach has been recently used in [6].

Here we consider the problem

\[ r(x)u_{tt}(x,t) = [p(x)u_x(x,t)]_x - q(x)u(x,t) + F(x,t), \quad 0 < x < 1, t > 0 \]  

\[ a_1 u_x(0, t) + b_1 u(0, t) = 0, \quad t > 0, \]
where

\[ |a_1| + |b_1| > 0, \quad |a_2| + |b_2| > 0. \quad (6) \]

\[ r(x) > 0, p(x) > 0, r(x), p(x), q(x), f(x), g(x) \text{ and } \mathcal{F}(x, t) \quad (7) \]

are continuous real functions and \( p(x) \) is differentiable.

This paper is organized as follows. Section 2 deals with the explicit solution of the inhomogeneous second order difference equation without increasing the problem’s dimension by developing a variation of the parameters method. In Section 3 the problem (1)–(6) is discretized using an explicit scheme, and a discrete eigenfunctions method is developed in order to construct a discrete numerical solution of the discretized problem as a finite series, having as many terms as the number of interior nodes we have in the space-domain discretization. An illustrative example is given in Section 4.

2. Explicit solution of the second order inhomogeneous difference equation

The aim of this section is the construction of an explicit real solution of problem

\[
\begin{aligned}
b(j + 1) + Bb(j) +Cb(j - 1) &= D(j), \quad 2 \leq j \leq N, \\
b(0) &= \hat{\alpha}, \quad b(1) = \hat{\beta}
\end{aligned}
\]

where \( \hat{\alpha}, \hat{\beta}, B, C, D(j) \) and the unknown \( b(j) \) are real numbers, and

\[ C > 0, \quad B^2 \neq 4C. \quad (9) \]

Let \( \{b^1(j)\} \) and \( \{b^2(j)\} \) be a pair of solutions of

\[ b(j + 1) + Bb(j) +Cb(j - 1) = 0, \quad (10) \]

satisfying the nonzero Cassorati determinant condition

\[ b^1(j)b^2(j + 1) - b^1(j + 1)b^2(j) \neq 0, \quad (11) \]

which guarantees the linear independence of \( \{b^1(j)\} \) and \( \{b^2(j)\} \), see [7, page 53]. For the sake of clarity in the presentation, we recall some elementary results about the forward difference operator \( \Delta \) defined by

\[ \Delta v(j) = v(j + 1) - v(j). \quad (12) \]

Given a sequence \( \{z(k)\}_{k=1}^{N} \), the equation

\[ \Delta v(k) = z(k), \quad (13) \]

admits a solution

\[ v(k) = \Delta^{-1}z(k) = \sum_{j=1}^{k-1} z(j) + \gamma, \quad (14) \]

where \( \gamma \) is a constant. The general solution of

\[ b(j + 1) + Bb(j) +Cb(j - 1) = D(j), \quad 2 \leq j \quad (15) \]

can be described as

\[ b(j) = b^1(j)\rho_1 + b^2(j)\rho_2 + b_\rho(j), \quad (16) \]

where \( \rho_1, \rho_2 \) are arbitrary real constants and \( \{b^1(j)\}, \{b^2(j)\} \) are a pair of linearly independent solutions of (10).
Let us seek a particular solution of (15) of the form
\[ b_p(j) = b_1(j)v_1(j) + b_2(j)v_2(j), \quad j \geq 0, \] (17)
where \( v_i(j) \) for \( i = 1, 2 \), must be determined. By (12) and (17) note that
\[ b_p(j) = [v_1(j - 1) + \Delta v_1(j - 1)]b_1(j) + [v_2(j - 1) + \Delta v_2(j - 1)]b_2(j) \]
\[ = v_1(j - 1)b_1(j) + v_2(j - 1)b_2(j) + b_1(j)\Delta v_1(j - 1) + b_2(j)\Delta v_2(j - 1). \] (18)
Note that if we impose the condition
\[ b_1(j)\Delta v_1(j - 1) + b_2(j)\Delta v_2(j - 1) = 0, \] (19)
then (18) takes the form
\[ b_p(j) = v_1(j - 1)b_1(j) + v_2(j - 1)b_2(j). \] (20)
By imposing to \{b_p(j)\} defined by (17) that satisfies (15), and taking into account (20) it follows that
\[ [v_1(j - 1) + \Delta v_1(j - 1)]b_1(j + 1) + [v_2(j - 1) + \Delta v_2(j - 1)]b_2(j + 1) \]
\[ + B[v_1(j - 1)b_1(j) + v_2(j - 1)b_2(j)] + C[v_1(j - 1)b_1(j - 1) + v_2(j - 1)b_2(j - 1)] = D(j). \] (21)
Note that (21) can be written in the form
\[ v_1(j - 1)[b_1(j + 1) + Bb_1(j) + Cb_1(j - 1)] + v_2(j - 1)[b_2(j + 1) + Bb_2(j) + Cb_2(j - 1)] \]
\[ + b_1(j + 1)\Delta v_1(j - 1) + b_2(j + 1)\Delta v_2(j - 1) = D(j), \]
or
\[ b_1(j + 1)\Delta v_1(j - 1) + b_2(j + 1)\Delta v_2(j - 1) = D(j). \] (22)
The algebraic system defined by (19), (22) admits a unique solution for \( \Delta v_1(j) \) and \( \Delta v_2(j) \), for each \( j \), because of (11).
If \( B^2 - 4C > 0 \), the characteristic equation
\[ z^2 + Bz + C = 0, \] (23)
admits a pair of distinct nonzero real solutions \( m_1 \) and \( m_2 \). In this case
\[ b_1(j) = m_1^j; \quad b_2(j) = m_2^j. \] (24)
The algebraic system (19), (22) takes the form
\[ \begin{cases} m_1^j\Delta v_1(j - 1) + m_2^j\Delta v_2(j - 1) = 0 \\ m_1^{j+1}\Delta v_1(j - 1) + m_2^{j+1}\Delta v_2(j - 1) = D(j) \end{cases} \] (25)
Solving (25), one gets
\[ \Delta v_1(j - 1) = \frac{D(j)}{m_1^j(m_1 - m_2)}; \quad \Delta v_2(j - 1) = \frac{-D(j)}{m_2^j(m_1 - m_2)}. \] (26)
From (26) and (14) it follows that
\[ \begin{aligned} v_1(j) &= (m_1 - m_2)^{-1}\sum_{\ell=1}^{j-1} m_1^{-\ell} D(\ell), \quad j \geq 2; \quad v_1(j) = 0, j = 0, 1 \\ v_2(j) &= -(m_1 - m_2)^{-1}\sum_{\ell=1}^{j-1} m_2^{-\ell} D(\ell), \quad j \geq 2; \quad v_2(j) = 0, j = 0, 1. \end{aligned} \] (27)
From (17), (24) and (27) one gets
\[
b_p(j) = (m_1 - m_2)^{-1} \sum_{\ell=1}^{j-1} (m_1^{j-\ell} - m_2^{j-\ell}) D(\ell), \quad j \geq 2
\]
\[
b_p(j) = 0, \quad j = 0, 1.
\] (28)

Under the condition
\[
B^2 < 4C,
\] (29)
a pair of linearly independent solutions of equation (10) is given by [8, page 154]
\[
b^1(j) = C^{\frac{j}{2}} \cos(j\theta); \quad b^2(j) = C^{\frac{j}{2}} \sin(j\theta), \quad j \geq 0, \quad \cos \theta = -\frac{B}{2} C^{-\frac{1}{2}}.
\] (30)

The particular solution of (15) is given by
\[
b_p(j) = C^{\frac{j}{2}} \{(\cos(j\theta) v_1(j) + \sin(j\theta) v_2(j)),
\] (31)
and system (19), (22), takes the form
\[
\begin{aligned}
C^{\frac{j}{2}} \{(\cos(j\theta) \Delta v_1(j - 1) + \sin(j\theta) \Delta v_2(j - 1) = 0
\end{aligned}
\]
\[
C^{\frac{j}{2}} \{(\cos(\beta + j\theta) \Delta v_1(j - 1) + \sin(\beta + j\theta) \Delta v_2(j - 1) = D(j).
\] (32)

Solving (32) for each \( j \), one gets
\[
\begin{aligned}
\Delta v_1(j - 1) = -\frac{D(j) \sin(j\theta)}{C^{\frac{j}{2}} \sin \theta}; \quad \Delta v_2(j - 1) = \frac{D(j) \cos(j\theta)}{C^{\frac{j}{2}} \sin \theta}.
\end{aligned}
\] (33)

Solving (33) by (14), it follows that
\[
\begin{aligned}
v_1(j) = \frac{-1}{\sin \theta} \sum_{\ell=1}^{j-1} \frac{D(\ell)}{C^{\frac{j}{2}} \sin \theta} \sin(\ell\theta), \quad j \geq 2; \quad v_1(j) = 0, \quad j = 0, 1
\end{aligned}
\]
\[
\begin{aligned}
v_2(j) = \frac{1}{\sin \theta} \sum_{\ell=1}^{j-1} \frac{D(\ell)}{C^{\frac{j}{2}} \cos \theta} \cos(\ell\theta), \quad j \geq 2; \quad v_2(j) = 0, \quad j = 0, 1.
\end{aligned}
\] (34)

By (31), (34), the particular solution of (15) is given by
\[
\begin{aligned}
b_p(j) = \frac{C^{\frac{j}{2}}}{\sin \theta} \sum_{\ell=1}^{j-1} \frac{D(\ell)}{C^{\frac{j}{2}} \ell} \{-\cos(j\theta) \sin(\ell\theta) + \sin(j\theta) \cos(\ell\theta)\}, \quad j \geq 2
\end{aligned}
\]
\[
\begin{aligned}
b_p(j) = 0, \quad j = 0, 1.
\end{aligned}
\] (35)

By (16) and (24) and the initial conditions \( b(0) = \hat{\alpha}, b(1) = \hat{\beta} \), the solution of problem (8) under hypothesis \( B^2 > 4C \), is given by
\[
\begin{aligned}
b(j) = \left\{ \begin{array}{ll}
(m_1 - m_2)^{-1} \left\{ m_1^j (\hat{\beta} - \hat{\alpha} m_2) + m_2^j (\hat{\alpha} m_1 - \hat{\beta}) + \sum_{\ell=1}^{j-1} (m_1^{j-\ell} - m_2^{j-\ell}) D(\ell) \right\}, & j \geq 2 \\
(m_1 - m_2)^{-1} \left\{ m_1^j (\hat{\beta} - \hat{\alpha} m_2) + m_2^j (\hat{\alpha} m_1 - \hat{\beta}) \right\}, & j = 0, 1.
\end{array} \right.
\] (36)

Under hypothesis (29), the general solution of (15) is given by
\[
\begin{aligned}
b(j) = C^{\frac{j}{2}} \{\cos(j\theta) \rho_1 + \sin(j\theta) \rho_2\} + \frac{C^{\frac{j}{2}}}{\sin \theta} \sum_{\ell=1}^{j-1} \frac{D(\ell)}{C^{\frac{j}{2}} \ell} \{-\cos(j\theta) \sin(\ell\theta) + \sin(j\theta) \cos(\ell\theta)\}, \quad j \geq 2
\end{aligned}
\]
\[
\begin{aligned}
b(0) = \rho_1, \quad b(1) = C^{\frac{1}{2}} \{\cos \theta \rho_1 + \sin \theta \rho_2\}.
\end{aligned}
\] (37)
By imposing the initial conditions of (8)–(37) one gets
\[ ρ_1 = \hat{a}, \quad C^{1/2} [\cos θ ρ_1 + \sin θ ρ_2] = \hat{β}, \]
or
\[ ρ_1 = \hat{a}; \quad ρ_2 = [\hat{β} - \hat{a} \cos θ](\sin θ C^{1/2})^{-1}. \] (38)

Summarizing, the following result has been established:

**Theorem 2.1.** Consider the initial value problem (8) under hypothesis (9).

(i) If \( B^2 > 4C \), let \( m_1 \) and \( m_2 \) be defined as
\[ m_1 = \frac{-B + \sqrt{B^2 - 4C}}{2}, \quad m_2 = \frac{-B - \sqrt{B^2 - 4C}}{2} \]
then the solution of (8) is given by (36).

(ii) If \( B^2 < 4C \), and \( θ \) is defined by \( θ = \arccos \left( \frac{-BC^{-1/2}}{2} \right) \), then the solution of problem (8) is given by (37) and (38).

3. The discrete eigenfunctions method

Let us begin this section by subdividing the domain \([0, 1] \times [0, \infty[\) into a rectangular mesh of equal rectangles of sides \( Δx = h, Δt = k \). Let \( N \) be a natural number, let
\[ h = \frac{1}{N + 1}, \quad a = \frac{k}{h}, \]
and let us introduce coordinates of a typical mesh point \( P(ih, jk) \), denoting \( u(ih, jk), f(ih), g(ih), p(ih), q(ih), r(ih), F(ih, jk) \) by \( U(i, j), f(i), g(i), p(i), q(i), r(i) \) and \( F(i, j) \) respectively.

Let us consider the finite difference approximations
\[ u_x(ih, jk) \approx \frac{U(i + 1, j) - U(i, j)}{h}, \]
\[ u_{tt}(ih, jk) \approx \frac{U(i, j + 1) - 2U(i, j) + U(i, j - 1)}{k^2}, \]
\[ [p(ih)u_x(ih, jk)]_i \approx \frac{p(i)U(i + 1, j) - [p(i) + p(i - 1)]U(i, j) + p(i - 1)U(i - 1, j)}{h^2}, \]
and the difference scheme approximating (1) by
\[ r(i)[U(i, j + 1) - 2U(i, j) + U(i, j - 1)] = a^2[p(i)U(i + 1, j) - (p(i) + p(i - 1) \]
\[ + h^2 q(i)]U(i, j) + p(i - 1)U(i - 1, j)] + k^2 F(i, j), \quad 1 \leq i \leq N, j \geq 0. \] (43)

With respect to the discretization of the boundary value conditions (2) and (3) using (40), note that there are several cases, depending of the situation where \( a_1 ≠ 0 \) or \( b_1 ≠ 0 \) and \( a_2 ≠ 0 \) or \( b_2 ≠ 0 \). It is easy to check that, for any of the four possible cases, the resulting discretized boundary conditions are
\[ U(0, j) = \alpha U(1, j), \quad U(N + 1, j) = \beta U(N, j), \quad j \geq 0 \] (44)
where
\[ \alpha = \frac{a_1}{a_1 - hb_1}, \quad \beta = \frac{a_2}{a_2 + hb_2}. \] (45)

Discretizing the initial conditions (4) and (5) one gets
\[ U(i, 0) = f(i), \quad 0 \leq i \leq N + 1, \]
\[ U(i, 1) = kg(i) + U(i, 0) = kg(i) + f(i), \quad 0 \leq i \leq N + 1. \] (46)
Let us seek solutions of the homogeneous boundary value problem defined by
\[ r(i)[U(i, j + 1) - 2U(i, j) + U(i, j - 1)] = a^2[p(i)U(i + 1, j) - (p(i) + p(i - 1) + h^2q(i))U(i, j) + p(i - 1)U(i - 1, j)], \]

(48)
together with (44), of the form
\[ U(i, j) = H(i)G(j), \quad 1 \leq i \leq N, \quad j \geq 0. \]

(49)
Substituting (49) into (48), one gets
\[ r(i)[G(j + 1) - 2G(j) + G(j - 1)]H(i) = a^2[p(i)H(i + 1) - (p(i) + p(i - 1) + h^2q(i))H(i) + p(i - 1)H(i - 1)]H(j). \]

(50)
By adding to both sides of (50) the term \( a^2 \lambda r(i)H(i)G(j) \), it follows that
\[ r(i)[G(j + 1) + (a^2 \lambda - 2)G(j) + G(j - 1)]H(i) - a^2[p(i)H(i + 1) - \lambda r(i)H(i) + p(i - 1)H(i - 1)]H(j) = 0, \quad 1 \leq i \leq N. \]

(51)
Note that (51) holds if
\[ G(j + 1) + (a^2 \lambda - 2)G(j) + G(j - 1) = 0, \quad j \geq 1. \]

(52)
and
\[ -p(i)H(i + 1) + (p(i) + p(i - 1) + h^2q(i))H(i) - p(i - 1)H(i - 1) = \lambda r(i)H(i), \quad 1 \leq i \leq N. \]

(53)
From (44) and (49), one gets
\[ H(0) = \alpha H(1), \quad H(N + 1) = \beta H(N). \]

(54)
and (53), (54) defines a discrete Sturm–Liouville problem having \( N \) eigenpairs \( \{ \lambda_\ell, \Phi_\ell \}_{\ell=1}^N \), \( \{ \Phi_\ell(i) : 1 \leq i \leq N \}_{\ell=1}^N \),

where the eigenfunctions \( \{ \Phi_\ell(\cdot) \}_{\ell=1}^N \) are taken to be orthonormal with respect to the weight function \( r(i) \); see [6] or [7, chapter 11]. The eigenpairs of the discrete Sturm–Liouville problem (53) and (54) satisfy
\[ A \Phi_\ell = \lambda_\ell R \Phi_\ell, \]

(55)
where \( A \) is the \( N \times N \) matrix,
\[ A = \begin{pmatrix}
\overline{s}(1) & -p(1) & 0 & \cdots & \cdots & 0 \\
-p(1) & s(2) & -p(2) & \cdots & \cdots & \vdots \\
0 & \ddots & \ddots & \ddots & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & s(N - 1) & -p(N - 1) \\
0 & \cdots & \cdots & 0 & -p(N - 1) & \overline{s}(N)
\end{pmatrix}, \]

(56)
with
\[ s(i) = p(i) + p(i - 1) + h^2q(i), \quad 1 \leq i \leq N \]
\[ \overline{s}(1) = s(1) - \alpha p(0); \quad \overline{s}(N) = s(N) - \beta p(N) \]

(57)
and \( R \) is the diagonal \( N \times N \) matrix
\[ R = \text{diag}(r(1), r(2), \ldots, r(N)). \]

(58)
Note that by (55), the eigenpairs \( \{ (\lambda_\ell, \Phi_\ell(\cdot)) \}_{\ell=1}^N \) can be obtained as the eigenpairs of the algebraic eigenvalue problem
\[ R^{-1} A \Phi_\ell = \lambda_\ell \Phi_\ell, \quad 1 \leq \ell \leq N. \]

(59)
By Theorem 5.1 of [9], all the eigenvalues of the discrete Sturm–Liouville problem (53) and (54) are positive under the hypotheses:

\[ p(i) > 0, \quad 0 \leq i \leq N, \]  
\[ r(i) > 0, \quad 1 \leq i \leq N, \]  

and

\[
\begin{align*}
\text{sig}(a_1) &= -\text{sig}(b_1) \quad \text{if } a_1 \neq 0, b_1 \neq 0 \\
\text{sig}(a_2) &= \text{sig}(b_2) \quad \text{if } a_2 \neq 0, b_2 \neq 0
\end{align*}
\]

\[ q(i) \geq 0, \quad 1 \leq i \leq N \]  

The discrete eigenfunctions method developed in [6] for the diffusion equation suggests seeking a solution of problem (43)–(47) of the form

\[ U(i, j) = \sum_{n=1}^{N} b_n(j) \Phi_n(i), \quad 1 \leq i \leq N, \quad j \geq 0. \]  

As \( \{\Phi_k(\cdot)\}_{k=1}^{N} \) is an orthonormal basis in \( \mathbb{R}^N \), for each fixed \( j \), the discrete function \( F(\cdot, j)/r(\cdot) \) admits a discrete Fourier series expansion of the form (see [7, chapter 11])

\[ \frac{F(i, j)}{r(i)} = \sum_{k=1}^{N} \gamma_k(j) \Phi_k(i), \]  

where

\[ \gamma_k(j) = \sum_{n=1}^{N} F(n, j) \Phi_k(n), \quad 1 \leq k \leq N. \]  

By imposing a condition on \( U(i, j) \) given by (63) that it satisfy (43), and taking into account (64) it follows that

\[
\begin{align*}
 r(i) & \left\{ \sum_{n=1}^{N} b_n(j+1) \Phi_n(i) + \sum_{n=1}^{N} b_n(j-1) \Phi_n(i) - 2 \sum_{n=1}^{N} b_n(j) \Phi_n(i) \right\} \\
&= a^2 \left\{ p(i) \sum_{n=1}^{N} b_n(j) \Phi_n(i+1) - [p(i) + p(i-1) + h^2 q(i)] \sum_{n=1}^{N} b_n(j) \Phi_n(i) \\
&\quad + p(i-1) \sum_{n=1}^{N} b_n(j) \Phi_n(i-1) \right\} + k^2 r(i) \sum_{k=1}^{N} \gamma_k(j) \Phi_k(i),
\end{align*}
\]

or

\[
\begin{align*}
 r(i) & \sum_{n=1}^{N} [b_n(j+1) - 2b_n(j) + b_n(j-1)] \Phi_n(i) \\
&= a^2 \left\{ \sum_{n=1}^{N} [p(i) \Phi_n(i+1) - (p(i) + p(i-1) + h^2 q(i)) \Phi_n(i) + p(i-1) \Phi_n(i-1)] b_n(j) \right\} \\
&\quad + k^2 r(i) \sum_{n=1}^{N} \gamma_n(j) \Phi_n(i).
\end{align*}
\]

As \( (\lambda_k, \Phi_k(\cdot)) \) is an eigenpair of the discrete Sturm–Liouville problem (53) and (54) one gets

\[ p(i) \Phi_n(i+1) - (p(i) + p(i-1) + h^2 q(i)) \Phi_n(i) + p(i-1) \Phi_n(i-1) = -\lambda_n r(i) \Phi_n(i), \quad 1 \leq i \leq N, \]  

(67)
and replacing (67) in (66), it follows that \( b_n(j) \) must verify

\[
\sum_{n=1}^{N} \left( b_n(j + 1) + (a^2 \lambda_n - 2)b_n(j) + b_n(j - 1) \right) \Phi_n(i) = k^2 \sum_{n=1}^{N} \gamma_n(j) \Phi_n(i).
\]

(68)

Note that (68) holds if \( b_n(j) \) verify for each \( n \),

\[
b_n(j + 1) + (a^2 \lambda_n - 2)b_n(j) + b_n(j - 1) = k^2 \gamma_n(j), \quad j \geq 1.
\]

(69)

Note that from (46) and (63), we have

\[
f(i) = U(i, 0) = \sum_{n=1}^{N} b_n(0) \Phi_n(i) = \sum_{n=1}^{N} \tilde{\alpha}_n \Phi_n(i),
\]

(70)

where

\[
b_n(0) = \tilde{\alpha}_n = \sum_{i=1}^{N} r(i) f(i) \Phi_n(i).
\]

(71)

From (47), by expanding

\[
h(i) = kg(i) + f(i),
\]

(72)

in an eigenfunctions expansion of \( \{ \Phi_n(\cdot) \}_{\ell=1}^{N} \), one gets

\[
h(i) = \sum_{n=1}^{N} c_n \Phi_n(i); \quad c_n = \sum_{k=1}^{N} r(k) h(k) \Phi_n(k).
\]

(73)

By (63) we have

\[
h(i) = U(i, 1) = \sum_{n=1}^{N} b_n(1) \Phi_n(i) = \sum_{n=1}^{N} \tilde{\beta}_n \Phi_n(i).
\]

(74)

By (73) and (74) it follows that

\[
b_n(1) = c_n = \tilde{\beta}_n = \sum_{i=1}^{N} r(i)[kg(i) + f(i)] \Phi_n(i),
\]

(75)

and by (71) and (75), the difference equation (69) must satisfy the initial conditions

\[
b_n(0) = \tilde{\alpha}_n, \quad b_n(1) = \tilde{\beta}_n, \quad 1 \leq n \leq N.
\]

(76)

Given a fixed value \( n \), the characteristic equation associated to (69) is

\[
z^2 + (a^2 \lambda_n - 2)z + 1 = 0, \quad a = \frac{k}{h}.
\]

(77)

If \((a^2 \lambda_n - 2)^2 < 4\), or \((a^2 \lambda_n)(a^2 \lambda_n - 4) < 0\), the two characteristic roots are complex. Under hypotheses (60)–(62), for a fixed value of \( h = h_0 \) with \( h_0 = \frac{1}{N+1} \), all the eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_N \) are positive. Hence \( a^2 \lambda_n > 0 \), and condition \( a^2 \lambda_n - 4 < 0 \) holds if

\[
a^2 < \frac{4}{\lambda_n}, \quad 1 \leq n \leq N.
\]

(78)

Condition (78) is satisfied by taking \( k = \Delta t \) small enough, so that

\[
k < \frac{2h_0}{\sqrt{\lambda_{\text{max}}}}, \quad \lambda_{\text{max}} = \max\{\lambda_1, \lambda_2, \ldots, \lambda_N\}.
\]

(79)
Under conditions (60)–(62) and (79), using Theorem 2.1 and taking
\[ \theta_n = \arccos \left(1 - \frac{k^2 \lambda_n}{2h_0^2}\right), \quad 1 \leq n \leq N, \]  
the solution of (69), (71) and (75) is given by
\[
\begin{align*}
 b_n(j) &= \cos(j \theta_n) \hat{a}_n + \sin(j \theta_n) \left[ \frac{\hat{\beta}_n}{\sin \theta_n} - \frac{\hat{\alpha}_n \cos \theta_n}{\sin \theta_n} \right] \\
 &\quad + \frac{k^2}{\sin \theta_n} \sum_{\ell=1}^{j-1} \gamma_n(\ell) \{- \cos(j \theta_n) \sin(\ell \theta_n) + \sin(j \theta_n) \cos(\ell \theta_n)\}, \quad j \geq 2 \\
 b_n(0) &= \hat{a}_n; \quad b_n(1) = \hat{\beta}_n, \quad 1 \leq n \leq N.
\end{align*}
\]

From (63) and (81), one gets the discrete numerical solution of problem (1)–(6), whose construction we summarize in the following algorithm:

3.1. Constructive algorithm

Step 1. Let \( t_0 > 0 \) be the time where the solution is computed.
Step 2. Discretization and eigenpairs computation.
  - Take \( h_0 = \Delta x > 0 \) with \((N + 1)h_0 = 1\).
  - Take \( k = \Delta t \) satisfying (79) with \( J = \frac{L_0}{k} \) integer.
  - Compute \( a = \frac{k}{h_0} \) and \( \alpha = \frac{a_1 - h_0 b_1}{\alpha_2} \) and \( \beta = \frac{a_2}{\alpha_2 + h_0 b_1} \).
  - Compute \( q(i), r(i) \) for \( 1 \leq i \leq N \), and \( p(i) \) for \( 0 \leq i \leq N \).
  - Compute eigenpairs of the algebraic eigenvalue problem (59).
  - Compute \( \lambda_{\max} = \max\{\lambda_1, \ldots, \lambda_N\} \).
  - Compute \( f(i) \) and \( g(i) \) for \( 1 \leq i \leq N \).
Step 3. Computation of Fourier coefficients \( \{b_n(j)\}_{n=1}^N \).
  - Compute \( \hat{a}_n = b_n(0) \) for \( 1 \leq n \leq N \) using (71).
  - Compute \( \hat{\beta}_n = b_n(1) \) for \( 1 \leq n \leq N \) using (75).
  - Compute \( \gamma_n(\ell) \) for \( 1 \leq n \leq N, 1 \leq \ell \leq J - 1 \) using (65).
  - Compute \( b_n(j) \) for \( j = J, 1 \leq n \leq N \) using (81).
Step 4. Compute \( U(i, j) \) for \( 1 \leq i \leq N, j = J \) using (63).

4. Example

Let us consider the following problem:
\[
\begin{align*}
u_{tt}(x, t) &= u_{xx}(x, t) - u(x, t) + 2 \sin(x), \quad 0 < x < \pi, t > 0 \\
u(0, t) &= 0, \quad t > 0 \\
u(\pi, t) &= 0, \quad t > 0 \\
u(x, 0) &= \sin(x), \quad 0 \leq x \leq \pi \\
u_t(x, 0) &= \sin(x), \quad 0 \leq x \leq \pi
\end{align*}
\]

where
\[
\begin{align*}
p(x) &= 1 \\
q(x) &= 1 \\
r(x) &= 1 \\
f(x) &= \sin(x) \\
g(x) &= \sin(x) \\
F(x, t) &= 2 \sin(x) \\
a_1 &= 0, \quad \forall b_1 \neq 0 \\
a_2 &= 0, \quad \forall b_2 \neq 0.
\end{align*}
\]
Fig. 1. Comparison of the exact solution versus the numerical solution obtained using the proposed explicit method, for the values $a = 9.835775483079132 \times 10^{-1}$, $h = \frac{\pi}{30}$ and the time interval $t \in [1, 20]$.

By using the continuous eigenfunction method, the exact solution of problem (82)–(86) is given by

$$u(x, t) = \sin(x) \left( \frac{1}{\sqrt{2}} \sin(\sqrt{2}t) + 1 \right), \quad 0 \leq x \leq \pi, \quad t \geq 0.$$  \hspace{1cm} (87)

The constants given in (45) take the following values:

$$\alpha = \frac{a_1}{a_1 - h b_1} = 0, \quad \beta = \frac{a_2}{a_2 + h b_2} = 0.$$  \hspace{1cm} (88)

Taking the space step size $h_0 = \frac{\pi}{30}$ such as $30 h_0 = \pi$, the maximum of the eigenvalues of the matrix problem (59), where in this case we have $R^{-1} A = A$, is

$$\lambda_{\max} = \max\{\lambda_1, \ldots, \lambda_{29}\} = 4.000010017848869.$$  \hspace{1cm} (89)

In order to guarantee the restriction over the time step size $k$ given by (79), that is:

$$k < \frac{2 h_0}{\sqrt{\lambda_{\max}}} = 1.047196239865710 \times 10^{-1},$$  \hspace{1cm} (90)

we take the value $k = 1.03 \times 10^{-1}$, and $a = \frac{k}{h} = 9.835775483079132 \times 10^{-1}$. 

Fig. 2. Absolute errors’ evolution of the numerical solution as $h = \Delta x$ decreases at time level $t = 1$. 

*By using the continuous eigenfunction method, the exact solution of problem (82)–(86) is given by* 

$$u(x, t) = \sin(x) \left( \frac{1}{\sqrt{2}} \sin(\sqrt{2}t) + 1 \right), \quad 0 \leq x \leq \pi, \quad t \geq 0.$$  \hspace{1cm} (87)
Fig. 1 shows the good properties of the numerical solution given by (63), (71), (75) and (81), because the numerical computations are not only close to the exact solution, but even reproduce very well the behavior of the solution. Computations have been done using the MATLAB software package.

Finally, Fig. 2 illustrates the improvement of the numerical solution as the space step \( h \) decreases. The presented values of \( k \) for each \( h \) are respectively \( k = 6.2 \times 10^{-2} \), \( k = 2 \times 10^{-2} \), \( k = 1 \times 10^{-2} \), \( k = 1 \times 10^{-3} \) and \( k = 0.8 \times 10^{-3} \).

References