Inference on Reliability
\(P(Y < X)\) in the Levy Distribution

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Abstract—In this paper, we consider the estimation of the parameter of the Levy distribution, the right-tail probability, and \(P(Y < X)\). We also consider a test of hypothesis. © 2005 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

We consider the Levy distribution with the following probability density function,

\[
f(x; \sigma) = \frac{\sigma}{2\pi x^{3/2}} \exp\left(-\frac{\sigma}{2x}\right), \quad x > 0, \quad \sigma > 0.
\]

(See [1]).

The Levy distribution is a special case of the inverted gamma distribution with the shape parameter 1/2 and scale parameter 2/\(\sigma\) (see [1]). An inverse Gaussian density with parameters (\(\lambda, \mu\)) converges to the Levy density as \(\theta \to 0\) where \(\theta = \lambda/\mu\) and \(\sigma = \lambda\) (see [1]). The Levy distribution does not have moments of all orders, but it has been very useful in analysis of stock prices. Montroll and Shlesinger [2] have used Levy distribution in physics and so did Jurlewicz and Weron [3].

The problem of estimating and drawing inference about the probability that a random variable \(Y\) is less than an independent random variable \(X\) arises in reliability. When \(Y\) represents the random value of a stress that a device will be subjected to in service and \(X\) represents the strength that varies from item to item in the population of devices, then the reliability \(R\), i.e., the probability that a randomly selected device functions successfully, is equal to \(P(Y < X)\). The same problem also arises in the context of statistical tolerance where \(Y\) represents, say, the diameter of a shaft and \(X\) the diameter of a bearing that is to be mounted on the shaft. The probability that the bearing fits without interference is then \(P(Y < X)\). In biometry, \(Y\) represents a
patient’s remaining years of life if treated with drug A and X represents the patient’s remaining years when treated with drug B. If the choice is left to the patient, person’s deliberations will center on whether \( P(Y < X) \) is less than or greater than 1/2. McCool [4] considered the problem when \( X \) and \( Y \) are two independent Weibull random variables. Nadarajah [5,6] considered the estimation of \( P(Y < X) \) for lifetime distributions and extreme value distributions. Ali et al. [7] studied the right-tail probability for the power distribution.

In this paper, we consider estimation of the scale parameter, the right-tail probability in a Levy distribution. We also consider the problem of point estimation and interval estimation of \( P(Y < X) \) where \( X \) and \( Y \) are independent Levy random variables with different scale parameters, and a test of hypothesis.

2. ESTIMATION OF \( \sigma \) AND THE RIGHT-TAIL PROBABILITY

From the Levy density (1) and the result of [1], we have the following lemma.

**Lemma 1.** If \( X \) is distributed as Levy in (1), then \( 1/X \) follows a gamma distribution with the shape parameter \( 1/2 \) and scale parameter \( 2/\sigma \).

Assume \( X_1, X_2, \ldots, X_n \) is a random sample from (1). Then, the maximum likelihood estimate of \( \sigma \) is given by

\[
\hat{\sigma} = \frac{n}{\sum_{i=1}^{n} (1/X_i)}.
\]

From the moment of inverted gamma random variable in [8], we have the following lemma.

**Lemma 2.** If a random variable \( X \) follows the gamma distribution with shape parameter \( a > 0 \) and scale \( b > 0 \), then the \( k \) th moment of \( Y = 1/X \) is \( E(Y^k) = (\Gamma(a - k))/\Gamma(a) \cdot 1/b^k \), if \( a > k \).

From Lemma 2, we can obtain the mean and variance of \( \overline{\sigma} \) as follows,

\[
E(\overline{\sigma}) = \frac{n}{n-2} \sigma \quad \text{and} \quad \text{Var}(\overline{\sigma}) = \frac{2n^2}{(n-2)^2(n-4)}\sigma^2, \quad \text{if } n > 4. \quad (2)
\]

Since the Levy density is a member of the exponential family, \( \sum_{i=1}^{n}(1/X_i) \) is a complete sufficient statistic. Hence, the estimator \( \overline{\sigma} = (n-2)/(\sum_{i=1}^{n}(1/X_i)) \) is a UMVUE of \( \sigma \) and its variance

\[
\text{Var}(\overline{\sigma}) = \frac{2}{n-4}\sigma^2, \quad \text{if } n > 4. \quad (3)
\]

From the results (2) and (3), we have the following lemma.

**Lemma 3.**

(a) The estimators \( \overline{\sigma} \) and \( \hat{\sigma} \) are MSE-consistent.

(b) The UMVUE \( \overline{\sigma} \) is more efficient than the MLE \( \hat{\sigma} \).

Since conditions of regularity are satisfied, the Cramer-Rao lower bound (CRLB) for the variance of an unbiased estimator of \( \sigma \) is

\[
\text{CRLB} = \frac{2\sigma^2}{n}. \quad (4)
\]

Next, we consider confidence intervals for the scale parameter \( \sigma \) for the Levy distribution. Based on the pivot quantity \( \sigma \cdot \sum_{i=1}^{n}(1/X_i) \), which has a chi-square distribution with \( n \) degrees of freedom, we obtain a \((1 - \alpha)\) 100% confidence interval,

\[
\left( \frac{\chi^2_{n,1-\alpha/2}}{\sum_{i=1}^{n} (1/X_i)}, \frac{\chi^2_{n,\alpha/2}}{\sum_{i=1}^{n} (1/X_i)} \right),
\]
for $\sigma$, where $\int_{-\infty}^{\infty} g(t) dt = \alpha$, $g(t)$ is the chi-square density with $n$ degrees of freedom. Its length is

$$L_1 = \left( \chi_{n,\alpha/2}^2 - \chi_{n,1-\alpha/2}^2 \right) \frac{1}{\sum_{i=1}^{n} (1/X_i)}.$$

Under some regularity conditions and CRLB,

$$\frac{\hat{\sigma} - \sigma}{\hat{\sigma} \cdot \sqrt{2/n}}$$

has an asymptotic standard normal distribution. So, $(\hat{\sigma} - z_{\alpha/2} \hat{\sigma} \sqrt{2/n}, \hat{\sigma} + z_{\alpha/2} \hat{\sigma} \sqrt{2/n})$ is an $(1 - \alpha) 100\%$ confidence interval for $\sigma$, where $\Phi(z_{\alpha/2}) = 1 - \alpha/2$ and $\Phi$ is the standard normal distribution function. Its asymptotic length is

$$AL_1 = 2z_{\alpha/2} \hat{\sigma} \cdot \sqrt{\frac{2}{n}}.$$

From (2), we obtain expected lengths of $L_1$ and $AL_1$ in Lemma 4 below.

**Lemma 4.** $E(L_1) = (\chi_{n,\alpha/2}^2 - \chi_{n,1-\alpha/2}^2) \cdot (\sigma/(n - 2))$ and $E(AL_1) = 2^{3/2}z_{\alpha/2}(\sqrt{n}/(n - 2))\sigma$.

From Lemma 4, Table 1 provides the numerical values of $E(L_1)$ and $E(AL_1)$.

**Table 1.** Numerical values of $E(L_1)$ and $E(AL_1)$ for $\alpha = 0.05$. (Numerical values are divided by $\sigma$.)

<table>
<thead>
<tr>
<th>$n$</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E(L_1)$</td>
<td>2.1537</td>
<td>1.6338</td>
<td>1.3665</td>
<td>1.1969</td>
<td>1.0782</td>
</tr>
<tr>
<td>$E(AL_1)$</td>
<td>2.1913</td>
<td>1.6515</td>
<td>1.3773</td>
<td>1.2051</td>
<td>1.0844</td>
</tr>
</tbody>
</table>

From Table 1, we observe the expected length of $L_1$ based on the pivot quantity is shorter than expected length of the asymptotic interval, $E(AL_1)$ when $n = 10(5)30$, and $\alpha = 0.05$.

From [1], it is known that the cdf of the Levy distribution can be expressed as

$$F(x; \sigma) = 2 \left( 1 - \Phi \left( \frac{\sigma}{\sqrt{x}} \right) \right), \quad x > 0,$$

where $\Phi(x)$ is the cdf of the standard normal distribution. So, the right-tail probability of the Levy distribution is

$$R(t) = P(X > t) = 2\Phi \left( \frac{\sigma}{\sqrt{t}} \right) - 1, \quad t > 0.$$

We now define the following estimators of right-probability $R(t)$ based on MLE and on UMVUE of $\sigma$,

$$\hat{R}(t) = 2\Phi \left( \frac{\hat{\sigma}}{\sqrt{t}} \right) - 1, \quad t > 0,$$

and

$$\hat{R}(t) = 2\Phi \left( \frac{\hat{\sigma}}{\sqrt{t}} \right) - 1, \quad t > 0.$$

Expectations and variances of the above estimators of right-tail probability cannot be expressed explicitly. Table 2 provides numerical values of MSE of the two estimators based on results (2) and (3) based on two estimators $\hat{\sigma}$ and $\hat{\sigma}$.

**Table 2.** The numerical values of MSE of $\hat{R}(t)$ and $\hat{R}(t)$ when $R(t) = 0.383$, $t = 100$, and $\sigma = 25$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>10</th>
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<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{R}(t)$</td>
<td>0.00857</td>
<td>0.00518</td>
<td>0.00368</td>
<td>0.00285</td>
<td>0.00232</td>
</tr>
<tr>
<td>$\hat{R}(t)$</td>
<td>0.00703</td>
<td>0.00451</td>
<td>0.00331</td>
<td>0.00262</td>
<td>0.00216</td>
</tr>
</tbody>
</table>
From Table 2, \( \hat{R}(t) \) is more efficient in the sense of MSE than \( \tilde{R}(t) \) when \( R(t) = 0.383, t = 100, \) and \( \sigma = 25. \) Because \( R(t) \) is a monotone function of \( \sigma \) and because of the second result of Lemma 3, \( \hat{R}(t) \) is more efficient in the sense of MSE than \( \tilde{R}(t) \) for all sample sizes.

3. INFECTION ON \( P(Y < X) \)

We consider inference on \( P(Y < X) \) for the Levy distribution in similar manner as McCool [4] considered the inference of \( P(Y < X) \) for the Weibull distribution. Let the random variables \( X \) and \( Y \) be distributed as Levy distribution with scale parameters \( \sigma_x \) and \( \sigma_y, \) respectively. Then, from formulas 3.381(1) and 3.455(2) in [9] and using the facts that \( 1/X \) and \( 1/\sigma \) are distributed as Gamma \((1/2, 2/\sigma_x)\) and Gamma \((1/2, 2/\sigma_y)\), respectively, we can obtain the following lemma.

**LEMMA 5.** \( R = P(Y < X) = \frac{2}{\pi} \sin^{-1} \frac{1}{\sqrt{1 + \rho}}, \) by [10, formula 15.16]), where \( F(a, b; c; x) \) is the hypergeometric function.

We know that \( X - Y \) is symmetric if \( X \) and \( Y \) are iid random variables (that is, \( \rho = \sigma_y/\sigma_x = 1) \), then from Lemma 5, \( P(Y < X) = 0.5. \) Because \( R \) is monotone decreasing in \( \rho, \) inference on \( \rho \) is equivalent to inference on \( R. \) We hereafter confine our attention to the parameter \( \rho \) (see [4]).

Assume \( X_1, X_2, \ldots, X_m \) and \( Y_1, Y_2, \ldots, Y_n \) be independent random samples from Levy distributions with parameters \( \sigma_x \) and \( \sigma_y, \) respectively. Let

\[
\hat{\rho} = \frac{\sigma_y}{\sigma_x} = \frac{\frac{\sum_{i=1}^{m} (1/X_i)}{m}}{\frac{\sum_{i=1}^{n} (1/Y_i)}{n}}.
\]

Then, from Lemma 2, \( \sum_{i=1}^{m} (1/X_i) \sim \text{Gamma}(m/2, 2/\sigma_x) \) and \( \sum_{i=1}^{n} (1/Y_i) \sim \text{Gamma}(n/2, 2/\sigma_y). \)

\[
E(\hat{\rho}) = \frac{n}{n-2} \rho, \quad \text{and} \quad \text{Var}(\hat{\rho}) = \frac{2n^2 (m-n-2)}{m(n-2)^2(n-4)} \rho^2, \quad \text{where} \ \rho = \frac{\sigma_y}{\sigma_x} > 0. \tag{5}
\]

Let

\[
\hat{\rho} = \frac{\sigma_y}{\sigma_x} = \frac{\frac{\sum_{i=1}^{m} (1/X_i)}{m}}{\frac{\sum_{i=1}^{n} (1/Y_i)}{n}}.
\]

Then from result (5),

\[
E(\hat{\rho}) = \frac{m}{m-2} \rho, \quad \text{and} \quad \text{Var}(\hat{\rho}) = \frac{2m(m-n-2)}{(m-2)^2(n-4)} \rho^2. \tag{6}
\]

From results (5) and (6), we obtain the following lemma.

**LEMMA 6.**

(a) \( \hat{\rho} \) and \( \hat{\rho} \) are MSE-consistent estimators of \( \rho. \)

(b) \( \hat{\rho} \) is more efficient in the sense of MSE than \( \tilde{\rho} \) if \( m > n > 4. \) In case the sample sizes are equal, \( \text{MSE}(\hat{\rho}) = \text{MSE}(\tilde{\rho}). \)

**REMARK 1.** We note that if \( n > m, \) then \( P(Y < X) = 1 - P(X < Y). \) Then, we can estimate \( P(Y < X) \) based on \( P(X < Y) \) by choosing \( \hat{\rho} = \sigma_x/\sigma_y. \)

We now consider the estimator \( \hat{\rho} \) of \( \rho \) only. Define \( Z = \sum_{i=1}^{m} (1/X_i), \) \( W = \sum_{i=1}^{n} (1/Y_i), \) and \( U = Z/W. \) Then, \( \hat{\rho} = n/m(\sum_{i=1}^{m} (1/X_i)/(\sum_{i=1}^{n} (1/Y_i))) = n/mU. \) From the independence of \( Z \) and \( W \) and using [8, formula 3.381(4)], we obtain the pdf of \( U \) as follows,

\[
f_U(u) = \frac{1}{B(m/2, n/2)} u^{m/2-1} (u + \rho)^{-m-n+1}/2, \quad u > 0, \tag{7}
\]
where \( B(x, y) \) is the beta function. From the pdf (7) of \( U \), we can easily find the pdf of \( \hat{\rho} \). Define \( T = \sigma_x/\sigma_y U \). Then, from the pdf (7) of \( U \), the density of \( T \) can be obtained as follows,

\[
f_T(t) = \frac{1}{B(m/2, n/2)} t^{m/2-1}(1 + t)^{-(m+n)/2}, \quad t > 0,
\]

and so \( T \) is a pivot quantity.

Let \( B \equiv T/(1 + T) \). Then, \( B \) follows a beta distribution with parameters \( m/2 \) and \( n/2 \). Based on the pivot quantity \( T \), we consider a confidence interval of \( \rho = \sigma_y/\sigma_x \). From the beta distribution, for a given \( 0 < \alpha < 1 \), there exists \( 0 < b_\alpha < 1 \), such that \( \alpha = \int_0^{b_\alpha} 1/(B(m/2, n/2))x^{m/2-1}(1 - x)^{n/2-1} \, dx \), where \( B(m/2, n/2) \) is the beta function. Hence, a \((1 - \alpha)\) 100% confidence interval for \( \rho \) can be obtained as follows,

\[
\left( \frac{m}{n} \frac{1 - b_{1-\alpha/2} \hat{\rho}}{b_{1-\alpha/2}} , \frac{m}{n} \frac{1 - b_{\alpha/2} \hat{\rho}}{b_{\alpha/2}} \right).
\]

Its length \( L_2 \) is given by

\[
E(L_2) = \frac{m}{n-2} \left( \frac{1}{b_{\alpha/2}} - \frac{1}{b_{1-\alpha/2}} \right) \rho.
\]

From Slutsky’s theorem and Lemma 6(a) based on the unbiased estimator \((n-2)/n)\hat{\rho} \) of \( \rho \),

\[
\frac{((n-2)/n)\hat{\rho} - \rho}{\sqrt{2(m+n-2)/(m(n-4))}\hat{\rho}^2}, \quad n > 4,
\]

has a limiting standard normal distribution. Therefore,

\[
\left( \frac{n-2}{n} \hat{\rho} - z_{\alpha/2} \hat{\rho}, \frac{n-2}{n} \hat{\rho} + z_{\alpha/2} \hat{\rho} \right), \quad n > 4,
\]

is an asymptotic \((1 - \alpha)\) 100% confidence interval for \( \rho \). Its length \( AL_2 \) is given by

\[
2z_{\alpha/2} \hat{\rho} \sqrt{\frac{2(m+n-2)}{m(n-4)}}, \quad n > 4,
\]

and from the result (5) its expected length is given by

\[
E(AL_2) = \frac{2n}{n-2} z_{\alpha/2} \sqrt{\frac{2(m+n-2)}{m(n-4)}} \rho, \quad n > 4.
\]

Table 3 shows the values of expected lengths of \( L_2 \) and \( AL_2 \).

From Table 3, we observe that when \( m \) and \( n \) are 10(5)30, expected lengths of the asymptotic 95% confidence interval for \( \rho \) is shorter than the expected lengths of \( \rho \) based on the pivot quantity except for \((m, n) = (15, 10), (20, 10), (25, 15), (30, 10), \) and \((30, 15)\). So, \( E(L_2) \) and \( E(AL_2) \) do not dominate each other uniformly.

We now consider a test for the null hypothesis \( H_0 : \sigma_x = \sigma_y \) against \( H_1 : \sigma_x \neq \sigma_y \). Let \( \Theta = \{(\sigma_x, \sigma_y) \mid \sigma_x > 0, \sigma_y > 0\} \) and \( \theta = (\sigma_x, \sigma_y) \). Then, the joint pdf of \((X_1, X_2, \ldots, X_m, Y_1, Y_2, \ldots, Y_n)\) is

\[
L(\theta) = f_0(x, y) = \left( \frac{\sigma_y}{2\pi} \right)^{m/2} \prod_{i=1}^m x_i^{-3/2} \exp \left( -\frac{\sigma_x}{2} \sum_{i=1}^m \frac{1}{x_i} \right) \left( \frac{\sigma_y}{2\pi} \right)^{n/2} \prod_{i=1}^n y_i^{-3/2} \exp \left( -\frac{\sigma_y}{2} \sum_{i=1}^n \frac{1}{y_i} \right).
\]
Table 3. Numerical values of $E(L_2)$ and $E(AL_2)$ for $\alpha = 0.05$. (Units of $E(L_2)$ and $E(AL_2)$ are divided by $\rho$.)

<table>
<thead>
<tr>
<th>m</th>
<th>n</th>
<th>$b_{\alpha/2}$</th>
<th>$b_{1-\alpha/2}$</th>
<th>$E(L_2)$</th>
<th>$E(AL_2)$</th>
</tr>
</thead>
<tbody>
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<td>10</td>
<td>10</td>
<td>0.21201</td>
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<td>2.92488</td>
</tr>
<tr>
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<td>20</td>
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<td>2.57673</td>
</tr>
<tr>
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<td>25</td>
<td>0.10654</td>
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<td>30</td>
<td>0.27640</td>
<td>0.63896</td>
<td>1.83294</td>
<td>1.69605</td>
</tr>
</tbody>
</table>

Differentiating with respect to $\sigma_x$ and $\sigma_y$, we obtain the MLE

$$\hat{\sigma}_x = \frac{m}{\sum_{i=1}^{m} (1/X_i)}$$

and

$$\hat{\sigma}_y = \frac{n}{\sum_{i=1}^{n} (1/Y_i)}.$$

If $\sigma_x = \sigma_y = \sigma$, then the MLE of $\sigma$ is

$$\hat{\sigma} = \frac{m + n}{\sum_{i=1}^{m} (1/X_i) + \sum_{i=1}^{n} (1/Y_i)}.$$

The likelihood ratio test function is given by

$$\Lambda(x, y) = \left(\frac{m + n}{m}\right)^{m/2} \left(\frac{m + n}{n}\right)^{n/2} \left(1 + \frac{1}{U}\right)^{-m/2} (1 + U)^{-n/2},$$

where

$$U = \frac{\sum_{i=1}^{m} (1/X_i)}{\sum_{i=1}^{n} (1/Y_i)}.$$

Therefore, $\Lambda(x, y) < c$ is equivalent to $U < c_1$ or $U > c_2$. Under $H_0: \sigma_x = \sigma_y$, i.e., $\rho = 1$, the statistic $B_0 = U/(1 + U)$ has a beta distribution with $m/2$ and $n/2$, from the result (7). Since
$B_0$ is a monotone increasing function of $U$, so $U < c_1$ or $U > c_2$ is equivalent to $B_0 < b_1$ or $B_0 > b_2$.

**Remark 2.** For $\alpha = 0.05$ values of $b_1$ and $b_2$ can be found in $b_{\alpha/2}$ and $b_{1-\alpha/2}$ in Table 3 when $m$ and $n$ are $10(5)30$.

**References**