Abstract—In the present paper a new stability analysis and stabilization of continuous-time uncertain switched linear systems is considered. This approach is based on the comparison, the overevaluating principle, the application of Borne-Gentina criterion and the Kotelyanski conditions. The stability conditions issued from vector norms correspond to a vector Lyapunov function. Indeed, the switched system to be controlled will be represented in the Companion form. A comparison system relative to regular vector norms is used in order to get the simple arrow form of the state matrix that yields to a suitable use of Borne-Gentina criterion for the establishment of sufficient conditions as function of the uncertain parameters for global asymptotic stability.

Index Terms—Continuous-time uncertain switched linear systems, Global asymptotic stability, Vector norms, Borne-Gentina criterion, Arrow form state matrix, Arbitary switching, State and static output feedback controller.

I. INTRODUCTION

Switched systems are a class of hybrid systems consisting of a family of subsystems and a switching law that specifies which subsystem will be activated along the system trajectory at each instant of time. Switched systems deserve investigation for theoretical development as well as for practical applications. To switch between different system structures is an essential feature of many control systems, for example, in chemical processes, flexible manufacturing systems, automotive industry, aircraft, large-scale power systems, computer-controlled systems and communication networks can be modeled as switched systems [1], [2], [3], [4], [5].

On the other hand, from the practical viewpoint, it is important to investigate switched systems which contain uncertain parameters. Recently, several works considered stability analysis and stabilization of switched linear systems with polytopic uncertainty [6], [7], [8].

Stability analysis is one of the fundamental problems in the study of switched systems. In this aspect, the Lyapunov approach and its variants still play an important role. Stability under arbitrary switching is guaranteed by the existence of a Common Lyapunov function. The parameter dependent Lyapunov functions (PDLF) are introduced [9], [10] in order to reduce the conservatism related to uncertainty problems.

Despite the development of approaches used in order to study the stability problems for uncertain switched systems, those available methods are avoiding general nonlinear polytopic uncertainties parameters.

This paper intends to present a new stability conditions for continuous-time uncertain switched linear systems, based on the aggregations techniques, the vector norm concept [11], [12], [13], [14] and the application of the Borne-Gentina criterion [15].

This proposed approach could be a constructive solution for stabilization problems and aim to design state and static output feedback controllers with guaranteed stability under arbitrary switching for such systems.

The remainder of this paper is organized as follows: the problem of the studied switched systems is formulated in Section. In section 3, sufficient stability conditions of these continuous-time uncertain switched linear systems based on vector norms approach are presented, a validation on examples is drawn, and finally, some concluding remarks are summarized in section 4.

II. UNCERTAIN SWITCHED SYSTEMS DESCRIPTION AND PROBLEM FORMULATION

The continuous-time switched linear system is formed by $N$ subsystems described by the following state equation [16]:

$$\dot{x}(t) = \sum_{i=1}^{N} \zeta_i(t)A_i x(t)$$  \hspace{1cm} (1)

where $t \in [t_0, t_f]$, $x(t) \in \mathbb{R}^n$ is the state vector of the system, $A_i (i = 1, ..., N)$ is a matrix of appropriate dimensions denoting the subsystems, and $N \geq 1$ denotes the number of subsystems. The switching function $\zeta_i$ is an exogenous function which depends only on the time and not on the state, it is defined through:
$$
\zeta_i(t) = \begin{cases} 
1 & \text{if } A_i \text{ is active} \\
0 & \text{otherwise} 
\end{cases} \quad \text{and} \quad \sum_{i=1}^{N} \zeta_i(t) = 1 \quad (2)
$$

When the uncertainty model is present, the exact plant model $A_i$ is unknown. Then, its model is described by:

$$
\begin{bmatrix}
1 & 0 & \ldots & 0 & \beta_i \\
0 & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \alpha_{n-1} & \beta_{n-1} \\
\gamma'_{il} & \cdots & \gamma^n_{il} & \gamma^n_{il}
\end{bmatrix}
$$

$$
M_{il} = P^{-1} A_{il} P =
\begin{bmatrix}
\alpha_1 & 0 & \ldots & 0 & \beta_1 \\
0 & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \alpha_{n-1} & \beta_{n-1} \\
\gamma'_{il} & \cdots & \gamma^n_{il} & \gamma^n_{il}
\end{bmatrix}
$$

III. STABILITY CONDITIONS PRESENTATION

The switched systems has been considered all throughout this paper are represented in the canonical controllability base [17], described by the state matrix as Companion is transformed into a system characterized by state matrix in the arrow form [18]. This particular form allows the application of the Borne-Gentina criterion [15].

In [13], a change of base for the system given by (4) under the arrow form gives:

$$
\dot{z} = \sum_{i=1}^{N} \zeta_i(t) \sum_{l=1}^{N_l} \mu_{il}(t) A_{il} z(t) \quad (10)
$$

and finally:

$$
\dot{z} = M z(t) \quad (11)
$$

with:

$$
M = \sum_{i=1}^{N} \zeta_i M_i =
\begin{bmatrix}
\alpha_1 & 0 & \ldots & 0 & \beta_1 \\
0 & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \alpha_{n-1} & \beta_{n-1} \\
\sum_{i=1}^{N} \zeta_i \sum_{l=1}^{N_l} \mu_{il} \gamma'_{il} & \cdots & \sum_{i=1}^{N} \zeta_i \sum_{l=1}^{N_l} \mu_{il} \gamma^n_{il} & \sum_{i=1}^{N} \zeta_i \sum_{l=1}^{N_l} \mu_{il} \gamma^n_{il}
\end{bmatrix}
$$

where $z = Px$, $M_{il}$ is a matrix in the arrow form and $P$ is the corresponding passage matrix:
In such conditions, if \( p(y) \) denotes a vector norm of \( y \), satisfying component to component the equality:

\[
p(y) = \left\| y_1, y_2, \ldots, y_n \right\|
\]

\[(13)\]

it is possible by the use of the aggregation techniques [18] to define a comparison continuous-time system \( z(t) \in \mathbb{R}^n \) such that:

\[
\dot{z}(t) = M_c z(t)
\]

\[(14)\]

In this expression, the comparison matrix \( M_c \) for continuous-time is deduced from the matrix \( M \) by substituting only the off-diagonal elements by their absolute values, it can be written:

\[
M_c = \begin{bmatrix}
\alpha_1 & 0 & \ldots & 0 & \left\| \beta_1 \right\| \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots \\
0 & \ldots & 0 & \alpha_{n-1} & \left\| \beta_{n-1} \right\|
\end{bmatrix}
\]

\[(15)\]

By applying the Borne-Gentina criterion [15] to the previous system, we can deduce the following theorem for the stability of continuous-time switched linear systems with polytopic uncertainty.

**Theorem 1.** The continuous-time uncertain switched linear system described by (4) is globally asymptotically stable if there exists \( \alpha_j < 0 \) \((j = 1, 2, \ldots, n-1)\),

\[
\alpha_j < \alpha_q \quad \forall j \neq q \quad i = 1, 2, \ldots, N \quad l = 1, 2, \ldots, N_l \quad \text{such as:}
\]

\[
-\max \left\{ \sum_{i=1}^{N_j} \mu_{ij}(t) \gamma_{ij}^0 \right\} + \sum_{j=1}^{n} \max \left\{ \sum_{i=1}^{N_j} \mu_{ij}(t) \gamma_{ij}^0 \right\} \left\| \beta_j \right\| \alpha_j^{-1} > 0
\]

\[(16)\]

Proof of this theorem is given in Appendix A.

**Remark 1.**

We note that the stability conditions proposed are very useful in many switching control problems. Suppose that we have on hand an open-loop system:

\[
\dot{x}(t) = \sum_{i=1}^{N} \zeta_i(t) \sum_{l=1}^{N_l} \mu_{il}(t) (A_{il} x(t) + B_{il} u(t))
\]

\[(17)\]

where \( x(t) \) is the state, \( u(t) \) is the control input, \( A_{il} \), \( B_{il} \) are vertex matrices of appropriate dimension, \( \zeta_i(t) \) is the switching function and the weighting factors \( \mu_{il}(t) (l = 1, 2, \ldots, N_l) \) of the polytopic uncertain parameters. We also suppose that we can design a set of state feedback controllers \( u(t) = -K_{il} x(t) \), \( i = 1, 2, \ldots, N \) and \( l = 1, 2, \ldots, N_l \).

We suppose that the linear models of the switched system are set in the controllable form given by:

\[
A_{il} = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 \\ -a_{il}^n & -a_{il}^{n-1} & \ldots & -a_{il}^1 \end{bmatrix} \quad \text{and} \quad B_{il} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}
\]

\[(18)\]

So, the continuous-time uncertain switched system with the closed-loop is given by:

\[
\dot{x}(t) = \sum_{i=1}^{N} \sum_{l=1}^{N_l} \mu_{il}(t) (A_{il} - B_{il} K_{il}) x(t)
\]

\[(19)\]

As an application of theorem 1, we consider the following example.

**Example 1.** We consider the continuous-time uncertain switched linear system with described by two subsystems \((i = 2)\) and two extreme points \((l = 2)\).

So, the two subsystems are defined as such:

\[
A_1() = \mu_{11}(.) A_{11} + \mu_{12}(.) A_{12} \\
A_2() = \mu_{21}(.) A_{21} + \mu_{22}(.) A_{22}
\]

The vertex matrices are defined by:

\[
A_{11} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -3 & 1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -7 & 4 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 3 & 5 \end{bmatrix}
\]

\[
A_{22} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 4 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}
\]

We can analytically prove that the two subsystems are unstable regardless the values of the weighting factors \( \mu_{il} \), therefore we will stabilize this system with a state feedback controller characterized by the following parameters:

\[
K_{11} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \end{bmatrix}, \quad K_{12} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \end{bmatrix}, \quad K_{21} = \begin{bmatrix} k_{21} & k_{22} & k_{23} \end{bmatrix}
\]

and \( K_{22} = \begin{bmatrix} k_{21} & k_{22} & k_{23} \end{bmatrix} \)

So, the closed-loop system can be written as follows:

\[
A_{il}' = A_{il} - B K_{il} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -3 & 1 - k_{11}^3 \end{bmatrix}
\]
According to [17], the minimal overvaluing matrix relatively to the regular vector norm \( p : p(x) = [x_1, [x_2], [x_3]]^T \) is such as:

\[
M_{11} = \begin{pmatrix}
\alpha_1 & 0 & \beta_1 \\
\gamma_{11} & \alpha_2 & \beta_2 \\
\gamma_{12} & \gamma_{12} & \gamma_{12}
\end{pmatrix}, \quad M_{12} = \begin{pmatrix}
\alpha_1 & 0 & \beta_1 \\
0 & \alpha_2 & \beta_2 \\
\gamma_{12} & \gamma_{12} & \gamma_{12}
\end{pmatrix}, \quad M_{21} = \begin{pmatrix}
0 & \alpha_2 & \beta_2 \\
\gamma_{21} & \gamma_{21} & \gamma_{21}
\end{pmatrix}
\]

\[
\gamma_{11} = -P_{11}(\alpha_1) = -\left((\alpha_1)^3 + (5 + k_{11}) + (k_{12}^2 + 7)\alpha_1 + (-4 + k_{12}^3)\right)(\alpha_1)^2 \]

\[
\gamma_{12} = -P_{12}(\alpha_2) = -\left((\alpha_2)^3 + (5 + k_{12}^3) + (k_{21}^3 + 7)\alpha_2 + (-4 + k_{21}^3)\right)(\alpha_2)^2 \]

\[
\gamma_{21} = -P_{21}(\alpha_1) = -\left((\alpha_1)^3 + (5 + k_{11}) + (k_{22}^3 + 7)\alpha_1 + (-4 + k_{22}^3)\right)(\alpha_1)^2 \]

Then, the stability conditions deduced from theorem 1 are:

i) \( \alpha_1, \alpha_2 < 0 \)

ii) \(-\max\{\mu_{11}\gamma_{11}^{\gamma_1} + \mu_{12}\gamma_{12}^{\gamma_1} + \mu_{21}\gamma_{21}^{\gamma_1} + \mu_{22}\gamma_{22}^{\gamma_1}\}
+ \max\{\mu_{11}\gamma_{11}^{\gamma_1} + \mu_{12}\gamma_{12}^{\gamma_1} + \mu_{21}\gamma_{21}^{\gamma_1} + \mu_{22}\gamma_{22}^{\gamma_1}\}\) $\beta_1 \alpha_1^{-1}$

When we take \( \alpha_1 = -1, \alpha_2 = -1.5 \) and we suppose that for particular constraints the choice of \( K_d \) is imposed such that the pole placement is different for the three subsystems by taking: \( K_{11} = [1 11 10], K_{12} = [4 2 9], K_{21} = [0 11 8] \) and \( K_{22} = [0 15 9] \), and if we choose the uncertain parameters such as: \( \mu_{11} = 0.3, \mu_{12} = 0.7, \mu_{21} = 0.4 \) and \( \mu_{22} = 0.6 \), condition (ii) is verified numerically:

\[-\max(-3.7, -4.7) - 1.322 \max([-0.125], [-1.725]) \]

\[-2 \max([0],[0]) = 1.423 > 0 \]

When \( t_f \) is fixed to 8s, the switched time \( t_k = 4s \) the original state vector \( x(t_0 = 0) = [7 8 10]^T \), the evolution of states with respect to time is given by figure 1.

![Evolution of state vector for example 1](image)

For second order switched systems, \( \beta_1 = 1 \) is positive and when \( \gamma_{ij}^{\gamma_1} \) are positive too, then theorem 1 can be simplified to the following corollary:
Corollary 1. The continuous-time uncertain switched linear system of second order is globally asymptotically stable if there exist $\alpha < 0$ such as:

i) $\sum_{i=1}^{N_i} \mu_i(t) P_i(\alpha) < 0$, $i = 1, 2, ..., N$ and $l = 1, 2, ..., N_l$ (20)

ii) $\sum_{i=1}^{N_i} \mu_i(t) P_i(0) > 0$, $i = 1, 2, ..., N$ and $l = 1, 2, ..., N_l$ (21)

Remark 2.

The stability conditions proposed are very useful in many switching control problems. Suppose that we have on hand an open-loop system:

$$
\dot{x}(t) = \sum_{i=1}^{N} \zeta_i(t) \sum_{l=1}^{N_l} \mu_{il}(t) (A_{il}x(t) + B_{il}u(t)) \\
y(t) = \sum_{i=1}^{N} \zeta_i(t) C_i x(t)
$$

(22)

where $x(t)$ is the state, $u(t)$ is the control input, $A_{il}$, $B_{il}$ are vertex matrices of appropriate dimension, $\zeta_i(t)$ is the switched function and the weighting factors $\mu_{il}(t) (l = 1, 2, ..., N_l)$ of the polytopic uncertain parameters. We also suppose that we can design a set of state feedback controllers $u(t) = -K_{il}y(t)$, $i = 1, 2, ..., N$ and $l = 1, 2, ..., N_l$.

We suppose that the linear models of the switched system are set in the controllable form given by:

$$
A_{il} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ -a_{il}^{n-1} & -a_{il}^{n-2} & \cdots & -a_{il}^0 \end{bmatrix}, \\
B_{il} = B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}
$$

(24)

and: $C_i = \begin{bmatrix} C_i^1 \\ \vdots \\ C_i^{n-1} \\ C_i^n \end{bmatrix}$

So, the switched continuous-time system in the closed-loop is given by:

$$
\dot{x}(t) = \sum_{i=1}^{N} \zeta_i(t) \sum_{l=1}^{N_l} \mu_{il}(t) (A_{il} - BK_{il}C_i)x(t)
$$

(25)

In the following, we will treat the next example by using this corollary 1.

Example 2. We consider the continuous-time uncertain switched linear system of second order with described by two subsystems ($i = 2$) and two extreme points ($l = 2$).

So, the two subsystems are defined such as:

$A_1(\cdot) = \mu_1(\cdot) A_{11} + \mu_2(\cdot) A_{12}$

and: $A_2(\cdot) = \mu_1(\cdot) A_{21} + \mu_2(\cdot) A_{22}$

$A_{11} = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}, A_{12} = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}, A_{21} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$

and $A_{22} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$

$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C_i^1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $C_i^n = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

It is simple to see that the two subsystems are unstable; our task is to stabilize the switched system with a static output feedback controller characterized by the parameters $K_i$ and $K_2$, holding to the conditions given by corollary 1.

So, the closed-loop system can be written as follows:

$A_{11}^c = A_{11} - BK_{11}C_1 = \begin{bmatrix} 0 & 1 \\ 3 & -2 - K_1 \end{bmatrix}$

$A_{12}^c = A_{12} - BK_{12}C_1 = \begin{bmatrix} 0 & 1 \\ -2 - K_1 & 1 - K_1 \end{bmatrix}$

$A_{21}^c = A_{21} - BK_{21}C_2 = \begin{bmatrix} 0 & 1 \\ -2 - K_2 & 1 - K_2 \end{bmatrix}$

and: $A_{22}^c = A_{22} - BK_{22}C_2 = \begin{bmatrix} 0 & 1 \\ 1 - K_2 & 2 - 2K_2 \end{bmatrix}$

According to [18], the minimal overvaluing matrix relatively to the regular vector norm $p$ given by (20) is such as:

$P_{11}(\alpha) = \begin{bmatrix} \alpha^2 + (2 + 2K_1)\alpha - 3 + K_1 \\ \gamma_{11} \end{bmatrix}$

$P_{12}(\alpha) = \begin{bmatrix} \alpha^2 + \alpha(-1 + 2K_1) + 2 + K_1 \\ \gamma_{12} \end{bmatrix}$

$P_{21}(\alpha) = \begin{bmatrix} \alpha^2 + \alpha(-1 + 2K_2) + 2 + K_2 \\ \gamma_{21} \end{bmatrix}$

$P_{22}(\alpha) = \begin{bmatrix} \alpha^2 + \alpha(-2 + 2K_2) + 1 + K_2 \\ \gamma_{22} \end{bmatrix}$

$M_{11} = P^{-1} A_{11}^{c} P = \begin{bmatrix} \gamma_{11} \alpha & 1 \\ \gamma_{12} \end{bmatrix}, M_{12} = P^{-1} A_{12}^{c} P = \begin{bmatrix} \gamma_{11} \alpha & 1 \\ \gamma_{12} \end{bmatrix}$

$M_{21} = P^{-1} A_{21}^{c} P = \begin{bmatrix} \gamma_{21} \alpha & 1 \\ \gamma_{22} \end{bmatrix}$

$M_{22} = P^{-1} A_{22}^{c} P = \begin{bmatrix} \gamma_{21} \alpha & 1 \\ \gamma_{22} \end{bmatrix}$
The stability conditions for example 2 given by corollary 1 are the following:

i) \( \alpha < 0 \)
ii) \( \mu_{11}(,) P_{11}(,\alpha) + \mu_{12}(,) P_{12}(,\alpha) < 0 \)
iii) \( \mu_{21}(,) P_{21}(,\alpha) + \mu_{22}(,) P_{22}(,\alpha) < 0 \)
iv) \( \mu_{11}(,) P_{11}(0) + \mu_{12}(,) P_{12}(0) < 0 \)
v) \( \mu_{21}(,) P_{21}(0) + \mu_{22}(,) P_{22}(0) < 0 \)

When we take \( \alpha = -1.5 \), the nonlinear uncertainties parameters displays the data conditions such as: \( \mu_{11}(,) = \rho(,) \), \( \mu_{12}(,) = 1 - \rho(,) \), \( \mu_{21}(,) = \rho(,) \) and \( \mu_{22}(,) = 1 - \rho(,) \) with \( \rho(,) \) is a general nonlinearity such as \( 0 \leq \rho(,) \leq 1 \), conditions (ii), (iii), (iv) and (v) allows deducing the following stability conditions:

\[
K_1 > \max_{0 \leq \rho \leq 1} \left[ \frac{-9.5 \rho(,) + 5.75}{2}, 5 \rho(,) - 2 \right]
\]
\[
K_2 > \max_{0 \leq \rho \leq 1} \left[ \frac{-0.5 \rho(,) + 6.25}{2}, -\rho(,) - 1 \right]
\]

So, the stability domain found by the controller parameter \( K_2 \) as a function of the controller parameter \( K_1 \) for different values chosen of the nonlinear incertitude parameter \( \rho(,) = 0 \) and 0.5 is given respectively by figure 2 and figure 3.

Fig 2. Stability domain given for example 2 obtained from corollary 1 for \( \rho(,) = 0 \)

Fig 3. Stability domain given for example 2 obtained from corollary 1 for \( \rho(,) = 0.5 \)

IV. CONCLUSION

This paper has investigated new stability conditions for uncertain continuous-time switched linear systems. These conditions were deduced from stability studies of overvaluing systems built on vector norms and the application of Borne-Gentina criterion.

The main advantages of this approach are that it can be applied to a very large class of switched systems with bounded general nonlinear polytopic uncertainties.

As a validation, this approach is used in order to determine a stability domain of the conditions obtained according to controllers with state and static output feedback parameters.

REFERENCES

The matrix $M_C$ has its off-diagonal elements positive and the ones non constant are isolated in the last row. Thus, by referring to results obtained in [13], the conditions of theorem 1 can be deduced from the Kotelyanski conditions [15]. These conditions require having the principal minors of alternated signs, the $\alpha_j$ are chosen all negative:

\[
\alpha_1 < 0, \quad \alpha_1 \alpha_2 > 0, \quad \ldots
\]

\[
(-1)^{n-1} \prod_{j=1}^{n-1} \alpha_j > 0,
\]

\[
(-1)^n \det(M_C) > 0
\]

The $n-1$ first conditions are checked because the $\alpha_j$ are negative, however the last condition yields to:

\[
(-1)^n \det(M_C) = (-1)^n \times
\]

\[
\begin{array}{cccc}
\alpha_1 & 0 & \cdots & 0 \\
0 & \ddots & \cdots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \alpha_{n-1}
\end{array}
\]

\[
\times
\begin{array}{c}
|\beta_1| \\
\vdots \\
|\beta_{n-1}|
\end{array} > 0
\]

\[
(-1)^n \det(M_C) = (-1)^n \times
\]

\[
\begin{array}{cccc}
\alpha_1 & 0 & \cdots & 0 \\
0 & \ddots & \cdots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \alpha_{n-1}
\end{array}
\]

\[
\times
\begin{array}{c}
|\beta_1| \\
\vdots \\
|\beta_{n-1}|
\end{array} > 0
\]

Then the theorem is obtained by dividing this condition by

\[
(-1)^{n-1} \prod_{q=1}^{n-1} \alpha_q
\] such that:

\[
- \max \left\{ \sum_{i=1}^{N_l} \mu_i^{n-1} \right\} - \sum_{j=1}^{n-1} \max \left\{ \sum_{i=1}^{N_l} \mu_i^{n-1} \right\} |\beta_j| \alpha_j^{-1} > 0
\]