ABSTRACT
A multitarget tracking algorithm, based on representing the set of target states and the set of measurements as realisations of finite random sets, is proposed. A measurement model allowing for missed detections and false returns and a motion model allowing for target death are defined. Target births are incorporated by combining the existing track set at the prior scan with a set of proposed birth tracks. Finite-set statistics are used to derive the global posterior density. The realisation of a practical algorithm requires that this global posterior density is approximated in some way. We propose an approximation in which a set of tentative tracks is maintained with each tentative track represented by an existence probability and a density characterising its state.

1. INTRODUCTION
Multitarget tracking involves estimating, usually recursively, the states of several targets from a set of sensor measurements. The occurrence of false returns, due to clutter, and missed detections means that the number of measurements received does not correspond to the number of targets present. As a result the multitarget tracking problem requires the estimation of an unknown number of quantities from a set of measurements of uncertain origin. The issue of finding a suitable framework for the multitarget tracking problem has been addressed in the work of Mahler [1]. Here, the set of target states and the set of sensor measurements are considered to be realisations of finite random sets. Loosely speaking, a finite random set is a mapping from a probability space into a class of finite subsets of a particular space, usually $\mathbb{R}^n$.

In the random set framework the multitarget state is a set comprising a random number of single target states, each of which is a random vector, and the tracking problem amounts to computing the posterior density, known as the global or multitarget posterior density, of this set-valued quantity. This representation is useful for multitarget tracking as it captures the uncertainty in the number of targets present in addition to the fact that the ordering of the individual target states within the multitarget state is immaterial, i.e., the set $\{x_1, x_2\}$ is equivalent to the set $\{x_2, x_1\}$. In order to make the random set framework accessible to engineers Mahler has formulated a set of rules, known as finite-set statistics, which may be used to obtain the global posterior density using steps analogous to those used when applying Bayes theorem to find the posterior density of a single target state. The finite-set statistics can be applied with only minimal consideration given to the theoretical foundation upon which they are based. The usefulness of this approach becomes readily apparent after an examination of books devoted to the theory of random sets, e.g., [2].

Despite the suitability of the random sets framework for multitarget tracking, its acceptance has been limited due, conceivably, to the unfamiliar mathematics required for the derivation of global posterior densities and the problems involved in arriving at a succinct statistical characterisation of the multitarget track set. The benchmark algorithm for multitarget tracking is the multiple hypothesis tracking (MHT) algorithm [3]. The MHT algorithm computes the posterior probabilities of a set of hypotheses which account for the origins of each measurement in a sequence of scans. In order to account for all possible measurement origins, including the introduction of new targets, the number of hypotheses increases exponentially with the amount of data collected. Even with the use of hypothesis management techniques, the MHT can be computationally expensive. This has motivated the development of the integrated probabilistic data association filter (IPDAF) [4] which is a reformulation of the well-known PDAF [5] without the assumption of track existence. A multitarget tracking algorithm based on the IPDAF can be developed by maintaining a set of tentative tracks, each characterised by a posterior density and an existence probability, and applying the IPDA algorithm to each track. Tracks are confirmed as targets or removed from consideration based on the existence probabilities. This multitarget algorithm works well in a single target environment or when several well-separated targets are present. However, its performance deteriorates when two or more targets are in close proximity. In such cases, the joint
IPDAF (JIPDAF) [6], which extends the joint PDAF [7] in the same way that the IPDAF extends the PDAF, offers a substantial increase in performance.

It was shown in [8] that the random set framework can be used to derive the IPDAF under the assumption that at most one target is present. Under this assumption, the random set modelling the multitarget state \( X_k \) takes on values of \( \emptyset \) (the target does not exist) or \( \{x_k\} \) (the target exists) with probability determined by the existence probability.

While the random sets formalism provides a useful theoretical basis for the multitarget tracking problem and a way of producing, at least in principle, a solution to this problem it is clear that some approximations are required in order to arrive at an algorithm which is computationally feasible and produces a characterisation of the multitarget track state which can be readily interpreted. In particular we propose the following approach:

1. Decide on a suitable representation of the multitarget state: The use of the word suitable here is intended to capture the trade-off required between an accurate, in the sense dictated by random sets theory, representation of the multitarget state, and one which can be readily interpreted by an operator and propagated to future scans.

2. Develop models for the measurement process, target motion and target birth.

3. Derive the global posterior density.

4. Determine how the global posterior density must be approximated to arrive at the representation chosen in 1.

In this paper, we choose to represent the multitarget state as a set of tentative tracks, each track having an existence probability and a posterior density. The existence probability is a measure of our confidence that one of the sequence of measurements hypothesised as giving rise to a particular track follows the assumed target dynamical model. The corresponding posterior density gives probabilistic information regarding the state, e.g., position, velocity, of the target. Together the existence probability and posterior density provide a concise and easily interpreted representation of the status of a particular track. The use of this representation is an attempt to extend the intuitively appealing nature of the results obtained using the random sets framework for the single target case to the considerably more complicated multitarget case.

Our approach results in an algorithm quite similar to the JIPDAF [6]. The main difference between our approach and the JIPDAF is the method used to start new tracks. The JIPDAF gives each birth track an initial existence probability and adds them to the existing track set upon initiation, i.e., birth tracks are treated in the same way as already existing tentative tracks. In our approach, we assume that the number of targets born in a scan has a truncated Possion distribution. Unlike the JIPDAF, birth tracks are not assigned an existence probability and added to the existing track set until a scan has been processed. Although they represent the multitarget state differently, other related approaches include the algorithm by Mori et al. [9] and Washburn’s filter based on point process theory [10]. A different approach which makes use of the random sets formalism is the probability hypothesis density filter (PHDF) [11]. The PHDF may be considered to be a multitarget generalisation of the alpha-beta filter used in single target tracking. Mutitarget tracking algorithms based on particle filtering techniques have recently started to appear in the literature. One such technique uses particle filters to recursively estimate the joint multitarget probability density (JMPD) [13]. Inherent in the JMPD is the idea that the dimension of the multitarget state, i.e. the number of targets present, is unknown and must be estimated along with the individual target states themselves [12]. This is reflected in the particle set by allowing the dimension of the particles to vary. A similar idea was suggested in [14]. A different type of particle filter representation, in which each particle contains the same number of target states and a measure of existence is used to add or remove individual target states, was proposed in [15].

It is not the intention of this paper to suggest that the random sets framework is the only way to approach the multitarget tracking problem. Indeed, much of the work we have done in this area indicates the presence of numerous links between the algorithms mentioned in the previous paragraph and filters based on the random sets formulation. We feel that useful insights could be obtained by further exploration of this subject.

The paper is structured as follows. In Section 2 the measurement, motion and target birth models are given and the global likelihood, global transition density and global prior density are derived. The global posterior density is derived in Section 3. The multitarget tracking algorithm is presented in Section 4. Conclusions are drawn in Section 5.

### 2. RANDOM SETS FORMULATION

The statistical behaviour of a random set \( \Gamma \) is characterised by its belief measure

\[
\beta_T(S) = P(\Gamma \subseteq S)
\]

Although the belief measure of a random set is non-additive it assumes the same position as the probability measure of a random vector. In single target tracking it is customary to work with probability densities which can be obtained from probability measures using the Radon-Nikodým derivative.
For multitarget tracking using random sets an analogous operation, the set derivative, can be used to obtain the global probability density from the belief measure,

$$f_T(X) = \frac{\delta \beta_r(S)}{\delta X} \bigg|_{S=\emptyset}$$

In the following we formulate models for the measurement process and target motion, derive the belief measures for these models and use the set derivative to obtain the global likelihood and global transition density. The derivation of the global likelihood is presented in considerable detail. The derivations of the remaining quantities uses similar principles and are therefore given a shorter treatment. We will make frequent use of the finite-set statistics calculus developed by Mahler [1, Chapter 5].

The following assumptions and notation will be used. Target states are elements in $\mathbb{R}^{n_x}$ and measurements are elements in $\mathbb{R}^{n_y}$ for integers $n_x$ and $n_y$. All targets are assumed to move independently with the same motion model. Measurements are independent, for both targets and clutter, and the observation and detection procedures are assumed to be independent. Clutter is assumed to be uniformly distributed throughout the validation region, the volume of which is denoted as $V_k$. Occassionally different probability densities will be denoted by $p(\cdot)$. In such cases, the particular density under consideration will be clear from the argument.

### 2.1. The measurement model

We assume the presence of $r$ targets, $X_k = \{x_{k,1}, \ldots, x_{k,r}\}$, and the measurements are collected into $Y_k = \{y_{k,1}, \ldots, y_{k,m}\}$ which is modelled as a realisation of the random set

$$\Sigma_k = \Upsilon_k \cup \Lambda_k$$

The set $\Upsilon_k$ contains measurements originating from targets and can be decomposed as

$$\Upsilon_k = \bigcup_{i=1}^r \Upsilon_{k,i}$$

where

$$\Upsilon_{k,i} = \begin{cases} \emptyset & \text{with probability } 1 - P_i^D, \\ \{y_k\} & \text{with probability } P_i^D. \end{cases}$$

The set $\Upsilon_{k,i}, i = 1, \ldots, r$ models observation of the $i$th target which is detected with probability $P_i^D$. The set $\Lambda_k$ contains measurements originating from clutter and can be written as

$$\Lambda_k = \bigcup_{i=1}^t \Lambda_{k,i}$$

where

$$\Lambda_{k,i} = \begin{cases} \emptyset & \text{with probability } 1 - P_{FA}, \\ \{y_k\} & \text{with probability } P_{FA}. \end{cases}$$

The set $\Lambda_{k,i}$ models observation of a hypothetical clutter object which produces a false return with probability $P_{FA}$.

The belief measure of $\Sigma_k$ is

$$\beta_{\Sigma_k}(S | X_k) = \beta_{\Upsilon_k \cup \Lambda_k}(S | X_k) = \beta_{\Upsilon_k}(S | X_k) \beta_{\Lambda_k}(S | X_k)$$

where we have used the assumption that the target measurements are independent of the clutter measurements. Using (1),

$$\beta_{\Upsilon_k}(S | X_k) = \prod_{i=1}^r \beta_{\Upsilon_{k,i}}(S | X_k) = \prod_{i=1}^r P(\Upsilon_{k,i} \subseteq S | X_k)$$

where we have used the assumption that the target measurements are independent of the clutter measurements. Using (1),

$$\beta_{\Upsilon_k}(S | X_k) = \prod_{i=1}^r \beta_{\Upsilon_{k,i}}(S | X_k)$$

Similarly, assuming independent and identically distributed clutter points,

$$\beta_{\Lambda_k}(S) = \left\{ 1 - P_{FA} + P_{FA} \int_S p(c_k) dc_k \right\}^t$$

where $p(c_k)$ is the density of the clutter.

The rules of finite-set statistics can be used to write the global likelihood as

$$f_{\Sigma_k}(Y_k | X_k) = \sum_{Z \subseteq Y_k} f_{\Upsilon_k}(Z | X_k) f_{\Lambda_k}(Y_k \setminus Z)$$

We therefore require the set derivatives of the belief measures in (2) and (3). Let $Z = \{z_1, \ldots, z_j\}$, $j = 0, \ldots, \min(m, r)$ so that $|Y_k \setminus Z| = m - j$. The set derivative of (2) can be found as

$$f_{\Upsilon_k}(Z | X_k) = \sum_{i=1}^r \prod_{t=1}^j \left( 1 - P_i^D \right)$$

where $\theta(i, 1, \ldots, \theta(j, i))$ is the $i$th permutation of $j$ integers from $\{1, \ldots, r\}$ and

$$\{\bar{\theta}(1, i), \ldots, \bar{\theta}(r-j, i)\} = \{1, \ldots, r\} \setminus \{\theta(1, i), \ldots, \theta(j, i)\}$$
Similarly, the set derivative of (3) is
\[
f_{\mathcal{A}_k}(Y_k \setminus Z) = (m-j)\left(\frac{t}{m-j}\right) (P_{FA}/V_k)^{m-j} \\
\times (1-P_{FA})^{t-m+j}
\]
where we have substituted \(p(e_k) = 1/V_k\) since the clutter is uniformly distributed. If we assume that the number \(t\) of clutter objects is large and the false alarm probability \(P_{FA}\) is small we can use the Poisson approximation to obtain,
\[
f_{\mathcal{A}_k}(Y_k \setminus Z) = \lambda^{m-j} \exp(-\lambda V_k) \tag{6}
\]
where \(\lambda\) is the number of clutter points per unit volume. The global likelihood is then found by substituting (5) and (6) into (4),
\[
f_{\Sigma_k}(Y_k|X_k) = \exp(-\lambda V_k) \sum_{j=0}^{\min(m,r)} \lambda^{m-j} \\
\times \sum_{i=1}^{\binom{m}{j}} \prod_{t=1}^{j} \left(1 - P_{D(t,i)}\right) \prod_{t=1}^{j} P_{D(t,i)} \\
\times \sum_{d=1}^{r} \prod_{i=1}^{d} p(y_k,\mu(t,d)|x_k,\theta(t,i)) \tag{7}
\]
where \(\{\mu(1,d), \ldots, \mu(j,d)\}\) is the \(d\)th combination of \(j\) integers from \(\{1, \ldots, m\}\). If we assume, for the sake of notational convenience, that all targets have the same detection probability, \(P_D = P_{D,i}\), \(i = 1, \ldots, r\), we obtain the simplified expression,
\[
f_{\Sigma_k}(Y_k|X_k) = \exp(-\lambda V_k) \sum_{j=0}^{\min(m,r)} \lambda^{m-j} (1 - P_{D})^{j-i} P_D^i \\
\times \sum_{d=1}^{r} \sum_{i=1}^{d} \prod_{t=1}^{d} l_{\mu(t,d)}(x_k,\theta(t,i)) \tag{8}
\]
where \(l_{\mu}(x_k,c) = p(y_k,\mu|x_k,c)\).

2.2. The motion model

The set of targets \(X_k = \{x_{k,1}, \ldots, x_{k,r}\}\) is modelled as a realisation of the random set
\[
\Gamma_k = \Phi_k(X_{k-1}, V_k)
\]
where \(X_{k-1} = \{x_{k-1,1}, \ldots, x_{k-1,l}\}\), \(l \geq r\). The motion model used here does not allow for target birth. Target births are introduced in the global prior density as described in Section 2.3. It is assumed that each target existing at the prior scan has a persistence probability \(\psi^j\), \(j = 1, \ldots, l\). The belief measure is
\[
\beta_{\Gamma_k}(S|X_{k-1}) = \prod_{j=1}^{l} \left(1 - \psi^j + \psi^j \int_S p(x_k|x_{k-1,j}) \, dx_k\right)
\]
Taking the set derivative of the belief measure gives
\[
f_{\Gamma_k}(X_k|X_{k-1}) = \sum_{i=1}^{r} \prod_{t=1}^{i} \left(1 - \psi^{(t,i)}\right) \\
\times \prod_{t=1}^{r} \psi^{(t,i)} p(x_k,t|x_{k-1},\theta(t,i)) \tag{8}
\]
Assuming for the sake of notational convenience that all targets have the same persistence probability, \(\psi^j = \psi\), \(j = 1, \ldots, l\), gives the simplified expression
\[
f_{\Gamma_k}(X_k|X_{k-1}) = (1 - \psi)^{l-r} \psi^r \sum_{i=1}^{l} \prod_{t=1}^{i} p(x_k,t|x_{k-1},\theta(t,i)) \tag{8}
\]
2.3. The global prior density and target birth

The set of targets \(X_{k-1} = \{x_{k-1,1}, \ldots, x_{k-1,l}\}\) at scan \(k-1\), to be termed prior targets, is modelled as a realisation of the random set
\[
\Xi_{k-1} = \Gamma_{k-1} \cup B_{k-1}
\]
where \(\Gamma_{k-1}\) contains tentative tracks formed prior to scan \(k-1\) and \(B_{k-1}\) contains tracks formed at scan \(k-1\), i.e., proposed birth tracks. As discussed above, we assume \(T\) tentative tracks, with the \(l\)th target having an existence probability \(\epsilon_{k-1}\) and density \(p(t|x)\), \(l = 1, \ldots, T\). Birth tracks are initiated using the process of two-point initiation, as described in [6]. This process initiates tracks based on measurements received in consecutive scans, in this case scans \(k-1\) and \(k-2\). We denote the number of proposed birth tracks in the validation region as \(\xi\) and denote the initial density for each track as \(q_1(x)\), \(i = 1, \ldots, \xi\). The number of targets actually born in each scan is assumed to have a truncated Poisson distribution with parameter \(\tau V_k\). When \((\tau V_k)^{\xi}/\xi! \approx 0\), the truncated Poisson distribution is essentially the same as the Poisson distribution and \(\tau\) is approximately equal to the expected number of targets born per unit volume.

The global prior density can be written as
\[
f_{\Xi_{k-1}}(X_{k-1}|Y^{k-1}) = \sum_{Z \subseteq X_{k-1}} f_{\Gamma_{k-1}}(Z|Y^{k-1}) \\
\times f_{B_{k-1}}(X_{k-1} \setminus Z|Y^{k-1}) \tag{9}
\]
Let \( Z = \{ z_1, \ldots, z_l \} \) and \( E = X_{k-1} \setminus Z = \{ e_1, \ldots, e_{L-1} \} \). We have

\[
f_{\Gamma_{k-1}}(Z|Y^{k-1}) = \sum_{l=1}^{\nu(L)} \prod_{t=1}^{T-l} \left( 1 - \theta_k^{(t,i)} \right) \times \prod_{t=1}^{L-l} \left\{ \theta_k^{(t,i)} P_{\theta(t,i)}(z_t) \right\}
\]

(10)

\[
f_{B_{k-1}}(E|Y^{k-1}) = b_{L-1}(\tau V_k) \sum_{\xi=1}^{(L-1)(\xi_k)} \prod_{t=1}^{L-1} q_{\theta(t,i)}(e_t)
\]

where, for \( i = 0, \ldots, \xi \), [16]

\[
b_i(\alpha) = \alpha^i \left\{ \sum_{j=0}^{\xi} \alpha^j / j! \right\}
\]

Substituting (10) and (11) into (9) gives the global prior density as

\[
f_{\Xi_{k-1}}(X_{k-1}|Y^{k-1}) = \frac{\min(L,T)}{\max(0,L-\xi)} \sum_{l=1}^{\nu(L)} \prod_{t=1}^{T-l} \bar{\theta}_k^{(t,i)} \times \prod_{t=1}^{L-l} \left( 1 - \theta_k^{(t,i)} \right) \sum_{i=1}^{L} \prod_{t=1}^{L} P_{\theta(t,i)}(x_{k-1,\mu(t,i)}) \times b_{L-1}(\tau V_k) \sum_{c=1}^{(L-1)(\xi_k)} \prod_{t=1}^{L-1} q_{\theta(t,c)}(x_{k-1,\bar{\mu}(t,c)})
\]

(12)

3. DERIVATION OF THE GLOBAL POSTERIOR DENSITY

The global posterior density can be obtained using the Bayes-like recursion

\[
f_{\Gamma_k}(X_k|Y^k) = f_{\Xi_k}(Y_k|X_k) f_{\Gamma_k}(X_k|Y^{k-1}) / C_k
\]

(13)

where \( C_k = f_{\Xi_k}(Y_k|Y^{k-1}) \) is a normalisation constant. For \( X_k = \{ x_{k,1}, \ldots, x_{k,r} \} \), the prediction density is obtained using the set integral

\[
f_{\Gamma_k}(X_k|Y^{k-1}) = \int f_{\Gamma_k}(X_k|X_{k-1}) f_{\Xi_{k-1}}(X_{k-1}|Y^{k-1}) \delta X_{k-1}
\]

\[
= \sum_{L=\infty}^{T+\xi} f_L(X_k)
\]

(14)

where

\[
f_L(X_k) = 1/L! \int f_{\Gamma_k}(X_k|X_{k-1}^{L-1}) \times f_{\Xi_{k-1}}(X_{k-1}^L|Y^{k-1}) \, dx_{k-1,1} \cdots dx_{k-1,L}
\]

(15)

with \( X_{k-1}^L = \{ x_{k-1,1}, \ldots, x_{k-1,L} \} \). Note that the number \( L \) of prior targets, which includes both tentative tracks and birth tracks, must exceed the number \( r \) of posterior tracks as target births are introduced in the prior density, as shown in Section 2.3, and are not part of the motion model. In the following we derive \( f_L(X_k), L = r, \ldots, T + \xi \) and subsequently the global posterior density. Note that \( X_0 = \emptyset \).

3.1. The global prediction density

Substituting (8) and (12) into (15) gives, after some working,

\[
f_L(X_k) = \min(L,T) \kappa_{L,r,l} \sum_{h=1}^{(L)} \prod_{t=1}^{T-l} \left( 1 - \bar{\theta}_k^{(t,h)} \right) \times \prod_{t=1}^{L} \beta_{k-1}^{(t,h)} \sum_{i=1}^{L} \prod_{t=1}^{L} \bar{\theta}_k^{(t,i)}(x_{k-1,\mu(t,i)}) \times \prod_{t \in T^r} Q_{\mu(\theta(t,i)-t,c)}(x_{k,t})
\]

(16)

where

\[
\kappa_{L,r,l} = b_{L-1}(\tau V_k)(1 - \psi)^{L-r} \psi^r
\]

\[
P_j(x_k) = \int p(x_k|x_{k-1}) p_j(x_{k-1}) \, dx_{k-1}
\]

\[
Q_j(x_k) = \int p(x_k|x_{k-1}) q_j(x_{k-1}) \, dx_{k-1}
\]

\[
T^r = \{ t \in \{1,\ldots,r\} : \theta(t,i) \leq l \}
\]

\[
T^r_i = \{1,\ldots,r\} \setminus T^r_i
\]

The global prediction density is found by substituting (16) into (14).
3.2. The global posterior density

The global posterior density can now be found by substituting (14) and (7) into (13).

\[
\begin{align*}
&f_{\Gamma_{k}}(X_k|Y^k) = \sum_{L=r+\xi}^{T+\xi} \sum_{l=\max(0,L-\xi)}^{\min(L,T)} \sum_{j=0}^{\min(m,r)} \sum_{h=1}^{\alpha_{L,r,l,j,h}} \\
&\quad \times \left( \frac{\xi}{\xi} \sum_{c=1}^{\xi} \sum_{d=1}^{\xi} \sum_{i=1}^{\xi} \prod_{j=1}^{\xi} \mu(t,d)(x_{k,i}) \right) \\
&\quad \times \prod_{t \in T^d} \sum_{u=1}^{\xi} \prod_{t \in T^d} \mu(t,i,u)(x_{k,i}) \\
&\quad \times \sum_{u=1}^{\xi} \prod_{t \in T^d} \mu(t,i,u)(x_{k,i}) ,
\end{align*}
\]

where

\[
\alpha_{L,r,l,j,h} = \kappa_{L,r,l} \exp(-\lambda V_k) \sum_{j=0}^{\min(m,r)} \alpha_{L,r,l,j,h} \\
	imes \prod_{t=1}^{T-1} \left( 1 - \delta_{k-1} \right) \prod_{k=1}^{l} / C_k
\]

\[
N_{u,w} = \{ t \in \{1,\ldots,r-j \} : \delta_{\mu(t,w),u} \leq l \}
\]

\[
N_{u,w} = \{1,\ldots,r-j \} \setminus N_{u,w}.
\]

The constant \( C_k \) is found by setting the set integral of the global posterior density to one:

\[
\int f_{\Gamma_{k}}(X_k|Y^k) \delta X_k = \sum_{r=0}^{T+\xi} g_r = 1,
\]

where

\[
g_r = \sum_{L=r+\xi}^{T+\xi} \sum_{l=\max(0,L-\xi)}^{\min(L,T)} \sum_{j=0}^{\min(m,r)} \sum_{h=1}^{\alpha_{L,r,l,j,h}} \\
\quad \times \left( \frac{\xi}{\xi} \sum_{c=1}^{\xi} \sum_{d=1}^{\xi} \sum_{i=1}^{\xi} \prod_{j=1}^{\xi} \mu(t,d)(x_{k,i}) \right) \\
\quad \times \prod_{t \in T^d} \sum_{u=1}^{\xi} \prod_{t \in T^d} \mu(t,i,u)(x_{k,i}) \\
\quad \times \sum_{u=1}^{\xi} \prod_{t \in T^d} \mu(t,i,u)(x_{k,i}) ,
\]

with

\[
\begin{align*}
\alpha_{L,r,l,j,h} &= \alpha_{L,r,l,j,h} \left( \frac{L-j}{r-j} \right) \\
\lambda_{i,j} &= \int l_i(x_k) P_j(x_k) \, dx_k \\
\gamma_{i,j} &= \int l_i(x_k) Q_j(x_k) \, dx_k.
\end{align*}
\]

4. The multitarget tracking algorithm

We are now in a position to present a multitarget tracking algorithm based on an approximation of the global posterior density. We require a global posterior density of the form,

\[
\hat{f}_{\Gamma_{k}}(X_k|Y^k) = \sum_{i=1}^{r_1} \prod_{t=1}^{T-1} \left( 1 - \epsilon_k \right) \prod_{t=1}^{r_1} \epsilon_k \rho_{\theta(t,i)}(x_{k,t})
\]

where \( S = T + \xi, \epsilon_k, s = 1,\ldots,T \) is the existence probability of the \( st \) tentative track and \( \epsilon_k^{s+T} \), \( s = 1,\ldots,\xi \) is the existence probability of the \( th \) birth track. The posterior densities \( p_s(x_k), s = 1,\ldots,S \) are similarly indexed. We obtain such an approximation to the global posterior density through the retention of appropriate terms in the exact global posterior density.

We begin by computing association weights between the \( st \) target and the \( th \) measurement. The association weights \( \beta_{e,s} \) between the \( st \) target and the \( th \) measurement is found by retaining terms in the set integral of the global posterior density which associate the \( st \) target with the \( th \) measurement and contain only associations between targets and validated measurements. The quantitie \( \beta_{e,s} \) may be interpreted as the probability that measurement \( e \) originated from target \( s \), given that target \( s \) exists. Let \( Y_s^e, s = 1,\ldots,T \) denote the set of measurements in the validation gate of the \( st \) tentative track and \( Y_{V_{T+\xi}, s} = 1,\ldots,\xi \) denote the set of measurements in the validation gate of the \( st \) birth track. For the \( st \) tentative track, \( s = 1,\ldots,T \), we have

\[
\beta_{e,s} = \sum_{r=1}^{S} \sum_{l=1}^{\min(L,T)} \sum_{j=0}^{\min(m,r)} \sum_{h=1}^{\alpha_{L,r,l,j,h}} \\
\quad \times \sum_{c=1}^{\xi} \sum_{d=1}^{\xi} \sum_{i=1}^{\xi} \prod_{j=1}^{\xi} \mu(t,d)(x_{k,i}) \\
\quad \times \prod_{t \in T^d} \sum_{u=1}^{\xi} \prod_{t \in T^d} \mu(t,i,u)(x_{k,i}) \\
\quad \times \sum_{u=1}^{\xi} \prod_{t \in T^d} \mu(t,i,u)(x_{k,i}) ,
\]

where

\[
I_A(s) = \begin{cases} 
1, & s \in A, \\
0, & \text{otherwise},
\end{cases}
\]

\[
\Delta_{d,b,u}(e,s) = \sum_{t \in T^d} \delta_{s-\mu(t,u),h} \delta_{e-\mu(t,d)}
\]
Similarly, for the $s$th birth track, $s = 1, \ldots, \xi$,

$$
\beta_{e,s} = \frac{S}{S} \sum_{r=1}^{S} \sum_{L=r}^{L} \min(L,T) \min(m,r) \left( \begin{array}{c}
\gamma
\end{array} \right) \min(\nu, r) \left( \begin{array}{c}
\nu
\end{array} \right) \sum_{j=0}^{S} \sum_{h=1}^{\alpha_{L,r,j,h}} \left( \begin{array}{c}
\xi
\end{array} \right) \frac{j}{(r, \nu)} \left( \begin{array}{c}
\nu
\end{array} \right) \times \sum_{\xi=1}^{\delta} \sum_{\xi=1}^{\Delta} \Delta_{d,e,u}(e, s) \times \prod_{t \in T_{d}} \left[ I_{V_{\mu(e,t,u)}} \{ \mu(t, d) \} \lambda_{\mu(t,d), \mu(\theta(t,u), h)} \right] \times \prod_{t \in T_{d}} \left[ I_{V_{\mu(e,t,u)-1,c}+T} \{ \mu(t, d) \} \gamma_{\mu(t,d), \mu(\theta(t,u)-1,c)} \right]
$$

(18)

where

$$
\Delta_{d,e,u}(e, s) = \sum_{t \in T_{d}} \delta_{e-\mu(\theta(t,u)-1,c)-T} \delta_{e-\mu(t,d)}
$$

The association weighting for the case where the target exists but has not been detected is found by retaining terms in the set integral of the global posterior density which assume existence of the 4th target but do not associate it with any measurements and contain only associations between targets and validated measurements. This gives

$$
\beta_{0,s} = \sum_{r=1}^{S} \sum_{L=r}^{L} \min(L,T) \min(m,r) \left( \begin{array}{c}
\gamma
\end{array} \right) \sum_{j=0}^{S} \sum_{h=1}^{\alpha_{L,r,j,h}} \left( \begin{array}{c}
\xi
\end{array} \right) \beta_{e,s} \left( \begin{array}{c}
\nu
\end{array} \right) \times \sum_{\xi=1}^{\delta} \sum_{\xi=1}^{\Delta} \Delta_{d,e,u}(e, s) \times \prod_{t \in T_{d}} \left[ I_{V_{\mu(e,t,u)}} \{ \mu(t, d) \} \lambda_{\mu(t,d), \mu(\theta(t,u), h)} \right] \times \prod_{t \in T_{d}} \left[ I_{V_{\mu(e,t,u)-1,c}+T} \{ \mu(t, d) \} \gamma_{\mu(t,d), \mu(\theta(t,u)-1,c)} \right]
$$

(19)

$$
\beta_{0,s} = \frac{S}{S} \sum_{r=1}^{S} \sum_{L=r}^{L} \min(L,T) \min(m,r) \left( \begin{array}{c}
\gamma
\end{array} \right) \sum_{j=0}^{S} \sum_{h=1}^{\alpha_{L,r,j,h}} \left( \begin{array}{c}
\xi
\end{array} \right) \beta_{e,s} \left( \begin{array}{c}
\nu
\end{array} \right) \times \sum_{\xi=1}^{\delta} \sum_{\xi=1}^{\Delta} \Delta_{d,e,u}(e, s) \times \prod_{t \in T_{d}} \left[ I_{V_{\mu(e,t,u)}} \{ \mu(t, d) \} \lambda_{\mu(t,d), \mu(\theta(t,u), h)} \right] \times \prod_{t \in T_{d}} \left[ I_{V_{\mu(e,t,u)-1,c}+T} \{ \mu(t, d) \} \gamma_{\mu(t,d), \mu(\theta(t,u)-1,c)} \right]
$$

(20)

where

$$
C_{u,w,h} = \{ \mu(\theta(t,w), h), t \in N_{u,w} \}
$$

$$
D_{u,w,c} = \{ \mu(\theta(t,w), u) - l, c), t \in N_{u,w} \}
$$

denote, respectively, the sets of tentative and birth tracks existing but undetected.

The existence probability of the $s$th track, $s = 1, \ldots, S$ can then be found by summing the association weights $\beta_{e,s}$ as

$$
\epsilon_{e}^{s} = \sum_{e \in V_{t} \cup \{0\}} \beta_{e,s}
$$

(21)

The posterior density of the $s$th track, $s = 1, \ldots, S$ is given by

$$
p_{e,s}(x_{k}) = 1/\epsilon_{e}^{s} \sum_{e \in V_{t} \cup \{0\}} \beta_{e,s} p_{e,s}(x_{k}),
$$

(22)

where, for $e \in V_{t}$,

$$
p_{e,s}(x_{k}) = \left\{ \begin{array}{ll}
\frac{l_{e}(x_{k})}{p_{e,s}} / \gamma_{e,s}, & s = 1, \ldots, T,
\frac{l_{e}(x_{k})}{Q_{x_{k}}}, & s = T + 1, \ldots, \xi.
\end{array} \right.
$$

This completes the derivation of the proposed multitracking algorithm. In summary, the algorithm works as follows. The weights $\beta_{e,s}$, $s = 1, \ldots, S$, $e \in V_{t} \cup \{0\}$ are computed as shown in (17), (18), (19) and (20). The existence probability and posterior density of each track are then computed as shown in (21) and (22).

### 4.1. Practical considerations

In a practical implementation of the proposed algorithm several issues need to be considered:

- Clearly the densities $p_{e}(x_{k})$ are mixture densities with an exponentially increasing, in $k$, number of components. Such densities cannot be propagated intact but must be approximated. The simplest approximation would be a nearest neighbour-type approximation, letting $\hat{p}_{e}(x_{k}) = p_{E_{e}}(x_{k})$

where

$$
E = \arg \max_{e \in V_{t} \cup \{0\}} \beta_{e,s}
$$

A more robust approximation would represent $p_{e}(x_{k})$ using a mixture with a fixed number of components, as in [17, 18].

- Computations can be greatly reduced by separating targets into clusters and processing each cluster of targets independently. Clusters are formed by grouping targets whose validation gates share measurements.
• The number of tracks under consideration can be managed by merging similar tracks and discarding tracks with a small existence probability.

5. CONCLUSIONS

We have used the random sets framework to develop a multitarget tracking algorithm. In the random sets framework, the multitarget state is characterised probabilistically by its global posterior density. Using the finite-set calculus developed by Mahler, we derived an expression for the global posterior density and showed how this may be approximated to arrive at a representation of the multitarget state in which each target is represented by an existence probability and a posterior density. Future work will entail implementation of the proposed algorithm and comparison with existing techniques such as the multiple hypothesis tracking algorithm and joint integrated probabilistic data association.

6. REFERENCES


