Realization of 2-D FIR Filters using Generalized Polyphase Structure
Combined with Singular-Value Decomposition

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Abstract

In this paper, a realization scheme that combines the singular-value decomposition (SVD) and the generalized polyphase (GP) structure is proposed for 2-D linear-phase FIR filters. With a small number of extra additions, a high-order 2-D FIR filter is converted to several lower-order 2-D subfilters. These subfilters are then realized using the SVD, yielding a parallel implementation structure for each 2-D subfilter which consists of a number of 1-D short-tap FIR filters. Due to the energy compaction of the SVD and the frequency-selective property of the GP structure, the number of parallel branches in each 2-D subfilter is significantly reduced without introducing a large error. It is also shown that the various symmetries of 2-D filters, such as the quadrantal symmetry and the central symmetry, are well preserved in the proposed GP-SVD structure.

I. Introduction

Two-dimensional (2-D) linear-phase FIR filtering as one of the fundamental signal processing techniques has found very important applications in fields such as image processing where the phase of 2-D signals needs to be preserved. However, the design and realization of 2-D filters is much more complicated than that of their 1-D counterparts. Many methods have been proposed to reduce the complexity for the design and implementation of 2-D filters. More recently, the singular-value decomposition (SVD) based approach has attracted lots of intention [1]-[3]. Applying an SVD to the desired 2-D frequency-response or the impulse-response matrix of a designed filter, a 2-D filter can be designed and realized with a parallel connection of cascaded 1-D subfilters. This SVD-based approach has many advantages. First, a 2-D filter design problem can be simplified to the problem of designing 1-D subfilters. Second, the SVD-based structure is very suitable for a parallel and modular implementation, which speeds up the processing of 2-D signals as well as reduces the computational cost. Third, with a properly chosen SVD, only the first a few branches corresponding to the larger singular values need to be retained, yielding a significantly simplified SVD realization.

Recently, a generalized polyphase (GP) structure has been proposed by Mitra et al for the realization of 1-D filters [4]. By inserting a pair of matrices between the delay chain and the subfilters of conventional polyphase structure, the GP realization is able to convert a large-tap filter into a number of short-length filters. In this paper, we extend the GP structure proposed in [4] to 2-D filters and propose to employ the SVD for the realization of each 2-D subfilter in the GP structure. Although the idea of combining GP with SVD is motivated by reducing the computational cost of direct SVD of very high-order 2-D filters, the proposed structure prevails in the following aspects:

1. The various symmetry properties normally available in a 2-D impulse-response matrix can be exploited and elegantly realized in the corresponding 1-D constituent filters by using the SVD. This symmetry can be used to reduce computing storage and arithmetic operations for filter coefficients and succeeding processing.
2. The GP structure forms a frequency-selective subband for each constituent filter. This feature can be well combined with the energy compaction property of the SVD so as to further reduce the number of branches in the SVD realization.

II. Brief Review of GP Structure and SVD

A. 1-D Generalized Polyphase Structure

The generalized polyphase (GP) structure for FIR filters is described as

\[
H(z) = \begin{bmatrix}
1 & z^{-1} & \cdots & z^{-(L+1)}
\end{bmatrix} \begin{bmatrix}
H_0(z^L) \\
H_1(z^L) \\
\vdots \\
H_L(z^L)
\end{bmatrix} = \sum_{k=0}^{L-1} F_k(z) G_k(z^L) \tag{1}
\]

where the product of matrices \( P \) and \( Q \) is an identity matrix, \( H_k(z^L) \) represents the subfilters in conventional polyphase
structure, and \(F_k(z)\) and \(G_k(z)\), which are referred to as
interpolators and constituent filters, respectively, can be written
using the elements of \(P\) and \(Q\) as
\[
F_k(z) = \sum_{l=0}^{L-1} P(l,k)z^{-l}
\]
\[
G_k(z) = \sum_{l=0}^{L-1} Q(k,l)H(z)
\]
In [4], Mitra et al. have investigated the frequency-selective
property of the interpolators based on Hadamard transform and
have provided a general framework for the realization of the
subfilters.

B. SVD of a 2-D Impulse Response
Let \(A\) be an \(M \times N\) matrix whose elements represent the
impulse response of a 2-D filter, i.e., \(A = [h(m,n)]\). The SVD of
\(A\) can be written as
\[
A = USV^T
\]
where \(U\) and \(V\) are unitary matrices, and \(S\) is a diagonal matrix
whose diagonal elements represent the singular values of \(A\), i.e.,
\[
S = \text{diag}(\sigma_0, \sigma_1, \ldots, \sigma_{N-1})
\]
with \(\sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_{N-1} \geq 0\). The superscript \(C\) in (4) denotes
the complex conjugate transpose of the associated matrix. The
simplified realization with \(r\) sections is given by
\[
A = \sum_{i=0}^{r-1} \sigma_i u_i v_i^T = \sum_{i=0}^{r-1} \tilde{u}_i \tilde{v}_i^T
\]
where \(u_i\) and \(v_i\) represent the \(i\)th column vectors of \(U\) and \(V\),
respectively, and \(\tilde{u}_i = \sqrt{\sigma_i} u_i\), \(\tilde{v}_i = \sqrt{\sigma_i} v_i\).

III. Proposed Structure
A. 2-D Generalized Polyphase Structure
Consider a 2-D impulse response \(h(m,n)\) of size \(M \times N\) with
\(z\)-transform \(H(z_1, z_2)\). Assuming \(M\) to be a multiple of \(K\) and \(N\)
a multiple of \(L\), a 2-D GP structure can be given by
\[
H(z_1, z_2) = \begin{bmatrix} 1 & z_1^{-1} & \ldots & z_1^{-(K-1)} \end{bmatrix} \begin{bmatrix} 1 \\ z_1^{-1} \\ \vdots \\ z_1^{-(L-1)} \end{bmatrix} = \sum_{k=0}^{K-1} \sum_{l=0}^{L-1} E_k(z_1)G_k(z_1^{K}, z_1^L)F_j(z_2)
\]
where
\[
H(z_1^K, z_1^L) = \begin{bmatrix} H_{00}(z_1^{K}, z_1^L) & \cdots & H_{0(L-1)}(z_1^{K}, z_1^L) \\ \vdots & \ddots & \vdots \\ H_{(K-1)0}(z_1^{K}, z_1^L) & \cdots & H_{(K-1)(L-1)}(z_1^{K}, z_1^L) \end{bmatrix}
\]
property of the interpolators in the GP structure and the energy compaction feature of SVD, more 1-D subfilters/branches can be discarded without causing a large error in comparison with a direct SVD realization of the original FIR filter. In [3], some SVD expressions for the impulse response matrix of 2-D linear-phase filters have been obtained by exploiting various symmetry properties and appropriate matrix transforms. Those SVDs are capable of producing 1-D linear-phase subfilters with similar symmetries. By applying the matrix transforms proposed in [3] to the 2-D constituent filters in the GP structure, one can easily verify that the linear-phase and symmetry features of the original 2-D FIR filter are well preserved in the resulting 1-D subfilters in the proposed GP-SVD realization.

### IV. Simulation Results

A few examples are presented in this section to demonstrate the performance of the proposed realization technique. The 2-D filters to be realized are designed by the least-squares design method [6]. The realization of these filters with the proposed GP-SVD structure contains in general many branches with insignificant coefficients. Neglecting these branches will reduce the implementation cost while causing a small approximation error. The maximum error resulting from omission of some sections is computed.

**Example 1: A Quadrantally Symmetric Fan-Shaped Filter:**
A 16×16 quadrantally symmetric fan-shaped filter is designed. The magnitude specification is depicted in Fig.2, with 1 indicating the passband and 0 the stopband. A 2×2 GP decomposition is carried out based on the Hadamard matrix given by

$$R_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad R_2^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

(13)

In order to show the efficiency of the proposed structure, the realization errors caused by neglecting the smallest branches are compared with that of a direct SVD decomposition of the original 2-D filter. The comparison is made at the same computational complexity, i.e., the two versions of SVD realization have the same number of 1-D filter coefficients. For example, if one branch is dropped in the direct SVD structure, then two branches are omitted in the proposed structure in order to keep the same complexity of the 1-D subfilters. The maximum errors in the amplitude frequency response with respect to different numbers of branches used in the structure are listed in Table 1. As the designed filter has a quadrantal symmetry, there are only a total of 8 non-zero singular values. The frequency responses with respect to two realization structures are depicted in Fig.3 and Fig.4, respectively, where 4 branches are used in each realization.

**Example 2: A Centro-Symmetric Fan-Shaped Filter**
A 15×15 centro-symmetric fan filter is designed. A 3×3 GP decomposition is performed using the following transform matrix

$$R_z = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} \quad \text{with} \quad R_z^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

(14)

The maximum amplitude errors caused by dropping insignificant branches for the proposed GP-SVD structure as well as a direct SVD realization of the designed 2-D filter are given in Table 2. Compared with a direct SVD implementation with the same computational complexity, the proposed method gives a smaller distortion.

### V. Conclusions

A parallel realization technique using the generalized polyphase structure in conjunction with SVD has been proposed for 2-D FIR filters. Owing to the frequency-selective property of the GP decomposition and the energy compaction of the SVD, a high-order 2-D filter can be simplified as a number of short-tap 1-D FIR filters. It has been shown through a few examples that the proposed GP-SVD structure is superior to the direct SVD realization of a 2-D FIR filter in terms of the total number of coefficients of the 1-D constituent filters. It has also been shown that the linear-phase characteristic as well as other symmetry properties that the original 2-D filter has are retained in the resulting 1-D subfilters.

### References:


Table 1. Maximum amplitude errors of two realizations in Example 1

<table>
<thead>
<tr>
<th>Number of Branches Used</th>
<th>Proposed GP-SVD Structure</th>
<th>Direct SVD Implementation</th>
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<tbody>
<tr>
<td>4</td>
<td>0.2190</td>
<td>0.2433</td>
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<tr>
<td>5</td>
<td>0.1638</td>
<td>0.1527</td>
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<td>6</td>
<td>0.1027</td>
<td>0.1168</td>
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<tr>
<td>7</td>
<td>0.0580</td>
<td>0.0833</td>
</tr>
</tbody>
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Table 2. Maximum amplitude errors of two realizations in Example 2

<table>
<thead>
<tr>
<th>Number of Branches Used</th>
<th>Proposed GP-SVD Structure</th>
<th>Direct SVD Implementation</th>
</tr>
</thead>
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<tr>
<td>6</td>
<td>0.1994</td>
<td>0.2318</td>
</tr>
<tr>
<td>7</td>
<td>0.1446</td>
<td>0.2264</td>
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<td>8</td>
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<td>9</td>
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<td>12</td>
<td>0.0965</td>
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<td>13</td>
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<td>0.1008</td>
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<td>14</td>
<td>0.0241</td>
<td>0.0691</td>
</tr>
</tbody>
</table>

Fig. 1. A $3 \times 2$ GP structure for 2-D filters

Fig. 2. Design specification in Ex. 1

Fig. 3. Frequency response of the quadrantally symmetric fan filter realized with 4-branch GP-SVD structure

Fig. 4. Frequency response of the quadrantally symmetric fan filter realized with a direct SVD of 4 branches

Fig. 5. Design specification in Ex. 2