Improving homotopy analysis method for system of nonlinear algebraic equations

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Abstract. In this paper, a new iterative method for solving the system of nonlinear equations is suggested based on Newton-Raphson method and homotopy analysis method. Comparison of the result obtained by present method with Newton-Raphson method and homotopy perturbation method reveals that the accuracy and fast convergence of the proposed method.

Keywords: Homotopy analysis method; System of nonlinear equations; Newton-Raphson method.

Mathematics Subject Classification 2010: 65H10.

1 Introduction

One of the oldest and most basic problems in mathematics is that of solving an nonlinear system of equations \( \phi(X) = 0 \). This problem has motivated many theoretical developments including the fact that solution formulas do not in general exist. Thus, the development of algorithms for finding solutions has historically been an important enterprise. Newton-Raphson method [8] is the most popular technique for solving nonlinear system of equations. Many topics related to Newton's method still attract attention from researchers. As is well-known, a disadvantage of the method is that the initial approximation \( X_0 \), must be chosen sufficiently close to a true solution in order to guarantee their convergence. Finding a criterion for choosing \( X_0 \) is quite difficult and therefore effective and globally convergent algorithms are needed [32]. In recent years, several iterative methods have been developed to solve the nonlinear system of equations \( \phi(X) = 0 \), by using Adomian’s decomposition method [1,7,13,14], homotopy perturbation method [16,17], quadrature formulas [3,6,9,10,12,15] and other techniques [11,31].

In 1992, Liao [29] employed the basic ideas of homotopy to propose a general method

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for nonlinear problems, namely the homotopy analysis method. This method and its modifications and extensions have been successfully applied to solve many types of nonlinear problems [2,5,18-28,30]. In 2007, Abbasbandy et al. [2], employed homotopy analysis method to solve some nonlinear algebraic equations. Furthermore, in [2], an efficient algorithm named Newton-homotopy analysis method was presented based on Newton-Raphson method and homotopy analysis method and applied to solve nonlinear algebraic equations. In this work, homotopy analysis method and Newton-homotopy analysis method are generalized to the system of nonlinear equations \( \varphi(X) = 0 \). Several examples are presented and compared to other methods, showing the accuracy and fast convergence of the proposed methods.

2 The homotopy analysis method

Consider the system of nonlinear equations:

\[
\varphi(X) = \begin{cases} 
  f(x,y) = 0, \\
  g(x,y) = 0, 
\end{cases} \quad X = (x,y) \in \mathbb{R}^2 \tag{2.1}
\]

where \( f, g : \mathbb{R}^2 \to \mathbb{R} \) and \( \varphi : \mathbb{R}^2 \to \mathbb{R}^2 \). Here, homotopy analysis method is proposed to solve Eq.(2.1). Let \( q \in [0,1] \) denotes an embedding parameter, and \( h, k \) nonzero auxiliary parameters. We construct a homotopy \( H : \mathbb{R}^2 \to \mathbb{R} \) which satisfies

\[
H(\bar{X}(q); q) = \begin{cases} 
  (1-q) \left( f(\bar{X}(q)) - f(X_0) \right) = qhf(\bar{X}(q)), \\
  (1-q) \left( g(\bar{X}(q)) - g(X_0) \right) = qhg(\bar{X}(q)), 
\end{cases} \quad \bar{X}(q) \in \mathbb{R}^2, q \in [0,1] \tag{2.2}
\]

where \( X_0 = (x_0, y_0) \) is an initial approximation of Eq. (2.1). Since \( h, k \neq 0 \) it is obvious that

\[
H(\bar{X}(0); 0) = \varphi(\bar{X}(0)) - \varphi(X_0) = 0 \tag{2.3}
\]

and

\[
H(\bar{X}(1); 1) = \varphi(\bar{X}(1)) = 0. \tag{2.4}
\]

When \( q \) increases from 0 to 1 as (2.3) is continuously deformed to original problem \( H(\bar{X}(1); 1) = \varphi(\bar{X}(1)) = 0 \). The homotopy analysis method uses Taylor’s theorem to expanding \( \bar{X}(q) \) with respect to the \( q \) one has

\[
\bar{X}(q) = \begin{cases} 
  \bar{x}(q) = x_0 + \sum_{m=1}^{+\infty} x_m q^m, \\
  \bar{y}(q) = y_0 + \sum_{m=1}^{+\infty} y_m q^m,
\end{cases} \tag{2.5}
\]

where

\[
\begin{align*}
  x_m &= \frac{1}{m!} \left. \frac{d^m \bar{x}(q)}{dq^m} \right|_{q=0} \\
  y_m &= \frac{1}{m!} \left. \frac{d^m \bar{y}(q)}{dq^m} \right|_{q=0}
\end{align*} \tag{2.6}
\]
If the initial guess, and the auxiliary parameters \( h, k \) are so properly chosen that the series (2.5) converges at \( q = 1 \), one has

\[
X = \lim_{q \to 1} \bar{X}(q) = \begin{cases} 
\lim_{q \to 1} \bar{x}(q) = x_0 + \sum_{m=1}^{+\infty} x_m, \\
\lim_{q \to 1} \bar{y}(q) = y_0 + \sum_{m=1}^{+\infty} y_m,
\end{cases} \tag{2.7}
\]

which must be one of the solutions of Eq.(2.1). Taking the 1st-order derivative with respect to the \( q \) on both sides of equation (2.2) and setting \( q = 0 \), we have

\[
\begin{cases} 
x_1 f_x(X_0) + y_1 f_y(X_0) = hf(X_0) \\
x_1 g_x(X_0) + y_1 g_y(X_0) = kg(X_0)
\end{cases}
\]

whose solution is

\[
\begin{bmatrix}
x_1 \\
y_1
\end{bmatrix} = \begin{bmatrix}
f_x(X_0) & f_y(X_0) \\
g_x(X_0) & g_y(X_0)
\end{bmatrix}^{-1} \begin{bmatrix}
hf(X_0) \\
kg(X_0)
\end{bmatrix}
\]

Taking the second order derivative with respect to the \( q \) on both sides of equation (2.2) and then dividing them by \( 2! \) and finally setting \( q = 0 \), we have

\[
\begin{cases} 
x_2 f_x(X_0) + y_2 f_y(X_0) = -\frac{1}{2} \left[ x_1^2 f_{xx}(X_0) + 2x_1 y_1 f_{xy}(X_0) + y_1^2 f_{yy}(X_0) - A \right] \\
x_2 g_x(X_0) + y_2 g_y(X_0) = -\frac{1}{2} \left[ x_1^2 g_{xx}(X_0) + 2x_1 y_1 g_{xy}(X_0) + y_1^2 g_{yy}(X_0) - B \right] \tag{2.8}
\end{cases}
\]

where

\[
A = 2 (h + 1) \left( x_1 f_x(X_0) + y_1 f_y(X_0) \right)
\]

and

\[
B = 2 (k + 1) \left( x_1 g_x(X_0) + y_1 g_y(X_0) \right) \tag{2.9}
\]

and whose solution (2.8) is

\[
\begin{bmatrix}
x_2 \\
y_2
\end{bmatrix} = -\begin{bmatrix}
f_x(X_0) & f_y(X_0) \\
g_x(X_0) & g_y(X_0)
\end{bmatrix}^{-1} \begin{bmatrix}
\frac{1}{2} \left[ x_1^2 f_{xx}(X_0) + 2x_1 y_1 f_{xy}(X_0) + y_1^2 f_{yy}(X_0) - A \right] \\
\frac{1}{2} \left[ x_1^2 g_{xx}(X_0) + 2x_1 y_1 g_{xy}(X_0) + y_1^2 g_{yy}(X_0) - B \right]
\end{bmatrix}
\]

Substituting all the above terms into (2.7), we can obtain the solution of Eq.(2.1) as follows:

\[
X = \begin{bmatrix}
x_0 \\
y_0
\end{bmatrix} + \begin{bmatrix}
f_x(X_0) & f_y(X_0) \\
g_x(X_0) & g_y(X_0)
\end{bmatrix}^{-1} \begin{bmatrix}
hf(X_0) \\
kg(X_0)
\end{bmatrix} - \begin{bmatrix}
f_x(X_0) & f_y(X_0) \\
g_x(X_0) & g_y(X_0)
\end{bmatrix}^{-1} \begin{bmatrix}
\frac{1}{2} \left[ x_1^2 f_{xx}(X_0) + 2x_1 y_1 f_{xy}(X_0) + y_1^2 f_{yy}(X_0) - A \right] \\
\frac{1}{2} \left[ x_1^2 g_{xx}(X_0) + 2x_1 y_1 g_{xy}(X_0) + y_1^2 g_{yy}(X_0) - B \right]
\end{bmatrix} + \ldots
\]

The above formulations allow us to suggest the following iterative methods for solving system of nonlinear equations (2.1).
Algorithm 1. \( A.1(h) \). Put \( h = h = k \) and for a given \( X_0 \), compute the approximate solution \( X_{i+1} \) by the iterative scheme

\[
\begin{bmatrix}
x_{i+1} \\
y_{i+1}
\end{bmatrix} = \begin{bmatrix}
x_i \\
y_i
\end{bmatrix} + \begin{bmatrix}
f_x(X_i) & f_y(X_i) \\
g_x(X_i) & g_y(X_i)
\end{bmatrix}^{-1} \begin{bmatrix}
hf(X_i) \\
kg(X_i)
\end{bmatrix}
\]

(2.10)

\[
\begin{bmatrix}
x_{i+1} \\
y_{i+1}
\end{bmatrix} = \begin{bmatrix}
x_i \\
y_i
\end{bmatrix} + h \begin{bmatrix}
f_x(X_i) & f_y(X_i) \\
g_x(X_i) & g_y(X_i)
\end{bmatrix}^{-1} \begin{bmatrix}
f(X_i) \\
g(X_i)
\end{bmatrix}
\]

(2.11)

Obviously, Newton-Raphson method [8] is special case of (2.11) when \( h = -1 \). The values of \( h \) can be determined by plotting the so-called \( h \)-curves, as suggested by Liao [27].

Algorithm 2. \( A.2(h) \). Put \( h = h = k \) and for a given \( X_0 \), compute the approximate solution \( X_{i+1} \) by the iterative scheme

\[
\begin{bmatrix}
l_i \\
m_i
\end{bmatrix} = h \begin{bmatrix}
 f_x(X_i) & f_y(X_i) \\
g_x(X_i) & g_y(X_i)
\end{bmatrix}^{-1} \begin{bmatrix}
f(X_i) \\
g(X_i)
\end{bmatrix},
\]

\[
\begin{bmatrix}
x_{i+1} \\
y_{i+1}
\end{bmatrix} = \begin{bmatrix}
x_i \\
y_i
\end{bmatrix} + \begin{bmatrix}
l_i \\
m_i
\end{bmatrix} - \begin{bmatrix}
f_x(X_i) & f_y(X_i) \\
g_x(X_i) & g_y(X_i)
\end{bmatrix}^{-1} \left( \frac{1}{2} \left[ f_{xx}(X_0) + 2l_i m_i f_{xy}(X_0) + m_i^2 f_{yy}(X_0) \right] \right) - 2(h + 1) \left[ \frac{l_i f_x(X_0) + m_i f_y(X_0)}{m_i f_x(X_0) + m_i f_y(X_0)} \right]
\]

Note that the above method for \( h = -1 \) is the same as modified homotopy perturbation method which was presented in [16]. The values of \( h \) can be determined with by plotting the so-called \( h \)-curves.

3 Newton-homotopy analysis method

In \( A.1(h) \) and \( A.2(h) \), we put \( h = h = k \) as a fixed constant and it can be determined by \( h \)-curves. However, in Newton-homotopy analysis method, we determine \( h, k \) by Newton-Raphson scheme in each step as follows. Let \( X_0 = (x_0, y_0) \) and \( h_0 = (h_0, k_0) \) are the initial values for \( X = (x, y) \) and \( h = (h, k) \), respectively. From (2.10), we have

\[
\begin{align*}
x_{i+1} &= x_i + ha_i + kb_i \\
y_{i+1} &= y_i + h c_i + kd_i
\end{align*}
\]

where

\[
\begin{align*}
a_i &= \frac{g_y(X_i) f(X_i)}{f_x(X_i) g_y(X_i) - f_y(X_i) g_x(X_i)}, & b_i &= -\frac{f_y(X_i) g(X_i)}{f_x(X_i) g_y(X_i) - f_y(X_i) g_x(X_i)}, \\
c_i &= -\frac{g_x(X_i) f(X_i)}{f_x(X_i) g_y(X_i) - f_y(X_i) g_x(X_i)}, & d_i &= \frac{f_x(X_i) g(X_i)}{f_x(X_i) g_y(X_i) - f_y(X_i) g_x(X_i)}.
\end{align*}
\]
After computing $X_{i+1}$, we want to renew $h_i, k_i$ by Newton-Rafson scheme on

$$
\psi_i(h) = \begin{cases} 
F(h) = f(x_{i+1}, y_{i+1}) = 0, \\
G(h) = g(x_{i+1}, y_{i+1}) = 0,
\end{cases} \quad h = (h, k) \in \mathbb{R}^2.
$$

Hence

$$
h_{i+1} = h_i - \psi'_i(h_i)^{-1}\psi_i(h_i), \quad i = 0, 1, 2, \ldots,
$$

where $\psi'_i(h_i)$ is the Jacobian matrix at the point $h_i = (h_i, k_i)$ and

$$
h_0 = -\psi'_0(0)^{-1}\psi_0(0),
$$

Therefore, this method does not need to choose $h = (h, k)$ and has better numerical behavior. Some results are listed in Tables 2-6.

4 Numerical examples

In this section, we present some numerical results to illustrate the efficiency of the new methods proposed in this paper. We present some numerical test results for various well-known schemes in Tables 2-5. Compared were modified homotopy perturbation method (MHPM) [16], Newton-Raphson method (N-R) [8] and the Newton-Homotopy analysis method (N-HAM) which introduced in the present contribution. All computations are done by Maple 12 with 64 digits precision. We accept an approximate solution rather than the exact root, depending on the precision ($\varepsilon$) of the computer. The following stopping criteria is used for computer programmes:

$$(i)\|X_{i+1} - X_i\|_\infty < \varepsilon, \quad \text{and} \quad (ii)\|\varphi(X_i)\|_\infty < \varepsilon.$$ 

For every method, we analyze the number of iterations needed to converge to the solution. Furthermore, the computational order of convergence $p$ is approximated by means of

$$p \approx \ln \left( \frac{\|X_{i+1} - X_i\|_\infty / \|X_{i+1} - X_i\|_\infty}{\ln (\|X_i - X_{i-1}\|_\infty / \|X_{i-1} - X_{i-2}\|_\infty)} \right).$$

Example 4.1. System of nonlinear equations

\[
\begin{align*}
\sin^2(x) + \cos(y) &= 1, \\
3e^x \cos(y) + \sin^2(x) &= 1
\end{align*}
\]

Starting with the same initial value $X_0 = (\frac{\pi}{6}, \frac{\pi}{3})^t$ and using Algorithm 2, A.2(h), we get a number of roots in Table 1. Here stopping criteria is $\|X_{i+1} - X_i\|_\infty < 10^{-6}$. Note that, using Newton’s method, after 25 iterations we get approximate solution $X_{25} = (26.7035, 67.5442)^t$ with $\|\varphi(X_{25})\|_\infty < 7.31 \times 10^{-13}$. 

Example 4.2. Consider the following system of nonlinear equations \[1\]:
\[
\begin{align*}
    x^2 + y^2 - 2 &= 0, \\
    e^{x-1} + y^3 - 2 &= 0,
\end{align*}
\]
with exact solution \((1, 1)^t\) and the same initial value \(X_0 = (2.25, 2.5)^t\), Table 2.

Example 4.3. Consider the following system of nonlinear equations:
\[
\begin{align*}
    x^4y - xy + 2x - y - 1 &= 0, \\
    ye^{-x} + x - y - e^{-1} &= 0,
\end{align*}
\]
with exact solution \((1, 1)^t\) and the same initial value \(X_0 = (5.4, 4.2)^t\), Table 3.

Example 4.4. Consider the following system of nonlinear equations \[3\]:
\[
\begin{align*}
    x^2 - 2x - y + 0.5 &= 0, \\
    x^2 + 4y^2 - 4 &= 0,
\end{align*}
\]
with exact solution \((1.90067672636706, 0.311218565419294)^t\) and the same initial value \(X_0 = (4, 6)^t\), Table 4.

Example 4.5. Consider the following system of nonlinear equations:
\[
\begin{align*}
    x^2 + 3\log(x) - y^2 &= 0, \\
    2x^2 - xy - 2x + 1 &= 0,
\end{align*}
\]
with exact solution \((1.94463168830610, 2.40349957157572)^t\) and the same initial value \(X_0 = (8, 8)^t\), Table 5.
Table 2, for example2.

<table>
<thead>
<tr>
<th>Method</th>
<th>IN</th>
<th>$p$</th>
<th>$| \varphi(X_{IN}) |_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>N-R</td>
<td>7</td>
<td>2.01</td>
<td>$2.92 \times 10^{-16}$</td>
</tr>
<tr>
<td>MHPM</td>
<td>5</td>
<td>2.99</td>
<td>$6.18 \times 10^{-24}$</td>
</tr>
<tr>
<td>N-HAM</td>
<td>4</td>
<td>3.87</td>
<td>$3.34 \times 10^{-15}$</td>
</tr>
</tbody>
</table>

Table 3, for example3.

<table>
<thead>
<tr>
<th>Method</th>
<th>IN</th>
<th>$p$</th>
<th>$| \varphi(X_{IN}) |_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>N-R</td>
<td>12</td>
<td>2.00</td>
<td>$2.35 \times 10^{-15}$</td>
</tr>
<tr>
<td>MHPM</td>
<td>9</td>
<td>3.00</td>
<td>$8.00 \times 10^{-34}$</td>
</tr>
<tr>
<td>N-HAM</td>
<td>5</td>
<td>4.05</td>
<td>$4.96 \times 10^{-21}$</td>
</tr>
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</table>

Table 4, for example4.

<table>
<thead>
<tr>
<th>Method</th>
<th>IN</th>
<th>$p$</th>
<th>$| \varphi(X_{IN}) |_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>N-R</td>
<td>8</td>
<td>1.94</td>
<td>$1.77 \times 10^{-22}$</td>
</tr>
<tr>
<td>MHPM</td>
<td>6</td>
<td>2.97</td>
<td>$4.42 \times 10^{-32}$</td>
</tr>
<tr>
<td>N-HAM</td>
<td>4</td>
<td>3.91</td>
<td>$7.79 \times 10^{-24}$</td>
</tr>
</tbody>
</table>

Table 4, for example5.

<table>
<thead>
<tr>
<th>Method</th>
<th>IN</th>
<th>$p$</th>
<th>$| \varphi(X_{IN}) |_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>N-R</td>
<td>8</td>
<td>2.00</td>
<td>$5.48 \times 10^{-19}$</td>
</tr>
<tr>
<td>MHPM</td>
<td>6</td>
<td>3.00</td>
<td>$7.46 \times 10^{-34}$</td>
</tr>
<tr>
<td>N-HAM</td>
<td>4</td>
<td>5.01</td>
<td>$7.38 \times 10^{-19}$</td>
</tr>
</tbody>
</table>

Numerical results for Examples 4.2-4.5 and comparison

Example 4.6. Consider the following system of nonlinear equations[4]:

\[
\begin{align*}
3x - \cos(yz) - 0.5 &= 0, \\
x^2 - 81(y + 0.1)^2 + \sin(z) + 1.06 &= 0, \\
e^{-xy} + 20z + \frac{10\pi-3}{3} &= 0
\end{align*}
\]

with exact solution $(0, 0.5, -0.5235987756)^t$ and the same initial value $(5, 5, 3)^t$. Note that, using Newton-Raphson method, after 10 iterations we get approximate solution $(0, 0.5, -0.5235987756)^t$ with $\|\varphi(X_{10})\|_\infty < 4.33 \times 10^{-36}$.

Table 3. Numerical results given by N-HAM for 4.6

<table>
<thead>
<tr>
<th>IN</th>
<th>$| \varphi(X_{IN}) |_\infty$</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>2080.60</td>
</tr>
<tr>
<td>1</td>
<td>123.91</td>
</tr>
<tr>
<td>2</td>
<td>1.54</td>
</tr>
<tr>
<td>3</td>
<td>0.08</td>
</tr>
<tr>
<td>4</td>
<td>$0.04 \times 10^{-3}$</td>
</tr>
<tr>
<td>5</td>
<td>$2.88 \times 10^{-14}$</td>
</tr>
<tr>
<td>6</td>
<td>$4.33 \times 10^{-36}$</td>
</tr>
</tbody>
</table>

In Tables 2 and 3, IN denotes the number of iteration and denotes the computational order of convergence, defined in [15].

5 Conclusion

In this paper, homotopy analysis method and Newton-homotopy analysis method have been applied to solve the system of nonlinear equations. The Newton-homotopy analysis method was compared in performance to the homotopy perturbation method and Newton-Raphson method. The proposed method is both effective and convenient. Furthermore, it has been shown that the homotopy perturbation method for solving nonlinear system of equations is exactly especial case of that given by homotopy analysis method.
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References