International Journal of Mathematics And its Applications
Volume 4, Issue 1–D (2016), 135–148.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

A New Family of Exponentiated Weibull-Generated Distributions

Research Article

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Abstract: Exponentiated Weibull distribution is a generalization of the most popular Weibull distribution which provides a more flexible model for analyzing observed data. We suggest an extended version of the Weibull-generated family of distributions which was previously defined in [1]. Properties of the exponentiated Weibull generated family are studied including; quantile function, order statistics, moments and mean deviation. Some special models in the exponentiated Weibull generated distributions are provided. Further, the derived properties of the generated family apply to these selected models. Maximum likelihood method is applied to obtain parameter estimates of the exponentiated Weibull generated family. The resilience and usefulness of the extended family is accomplished through an application to a real data set.

Keywords: Weibull generated distributions, Order statistics, Quantiles.

1. Introduction

Generated family of continuous distributions is a new development for creating and extending the well-known distributions. Well-known traditional distributions often do not provide a good fit in relation to most great interesting data sets. The generated family has been broadly studied in statistics as they yield more flexibility in application. The beta-family of distributions has been studied in [2]. A generated family based on gamma distribution has been suggested in [3]. Kumaraswamy generalized family provided in [4]. An alternative gamma generator for any continuous distribution has been proposed in [5]. The Kummer beta generated family of distributions has been proposed in [6]. In [7] a continuous family of distribution which is an extended part of exponentiated distributions has been proposed. In [8] another generated family has been proposed, which is defined as

$$F(x) = \int_0^{W(G(x))} r(t)dt, \tag{1}$$

where, $r(t)$ be the probability density function (pdf) of a random variable $T \in [c, d]$ for $-\infty \le c < d \le \infty$. $W(G(x))$ be a function of the cumulative distribution function (cdf) of any random variable X so that $W(G(x))$ satisfies;

(1). $W(G(x)) \in [c, d]$.

(2). $W(G(x))$ is differentiable and monotonically non-decreasing.

(3). $W(G(x)) \to c$ as $x \to -\infty$, and $W(G(x)) \to d$ as $x \to \infty$.

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In the same trend, the Weibull-G family of distributions has been presented (see [1]). They considered a continuous distribution $G(\cdot)$ with density function $g(\cdot)$ where $W(G(x)) = G(x)/\ 1 - G(x)$, and the cumulative distribution function of Weibull is $R(t) = 1 - e^{-\alpha\, t^{\beta}}$ (for $t > 0$) with positive parameters α and β. The $W - G$ family is given by

$$F(x;\alpha,\beta,\xi) = \int_0^{\frac{G(x;\xi)}{1-G(x;\xi)}} \alpha\beta t^{\beta-1} e^{-\alpha t^{\beta}}\, dt \ = 1 - e^{-\alpha[\frac{G(x;\xi)}{1-G(x;\xi)}]^{\beta}}, \quad x \in D \subseteq R,\ \alpha,\beta > 0,$$

where; $G(x,\xi)$ is a baseline cdf, which depends on a parameter vector ξ. This paper aims to introduce an extended version of the Weibull-G $(W - G)$ distributions called exponentiated Weibull generated family of distributions. The organization of the current paper is as follows. In the next section, the exponentiated Weibull-generated family is defined and subsequent properties are provided. In Section 3, some general mathematical properties of the family are examined. In Section 4, some new special models of the generated family are introduced. In Section 5, estimation of the parameters of the family is performed through maximum likelihood method.An illustrative purpose on the basis of real data is investigated, in Section 6. Finally, concluding remarks are offered in Section 7.

2. The New Family of Generated Distributions

In this section, the new generated distributions, namely, exponentiated Weibull generated $(EW - G)$ distributions is introduced. The exponentiated Weibull distribution was proposed in [9],which has the following cdf and pdf respectively

$$R(t) = [1 - e^{-\alpha\, t^{\beta}}]^{a}, \ \text{ and } \ r(t) = a\alpha\beta t^{\beta-1} e^{-\alpha\, t^{\beta}}[1 - e^{-\alpha\, t^{\beta}}]^{a-1}, \tag{2}$$

where, $t > 0$, $\alpha > 0$ is the scale parameter, $a, \beta > 0$ are the shape parameters. Replacing; (i) the generator $r(t)$ defined in (1) by the pdf generator $r(t)$ given in (2) and (ii) $W(G(x))$ in (1) with $G(x,\xi)/1 -\ G(x,\xi)$, we obtain the following new family

$$F(x) = \int_0^{\frac{G(x;\xi)}{1-G(x;\xi)}} a\alpha\beta t^{\beta-1} e^{-\alpha\, t^{\beta}} [1 - e^{-\alpha\, t^{\beta}}]^{a-1} dt = [1 - e^{-\alpha[\frac{G(x;\xi)}{1-G(x;\xi)}]^{\beta}}]^{a},\ x > 0\ ,\ a,\ \alpha,\ \beta,\ \xi > 0, \tag{3}$$

where $G(x,\xi)$ is a baseline cdf, which depends on a parameter vector ξ. The distribution function (3) provides a broadly exponentiated Weibull generated distributions. Therefore, the pdf of the exponentiated Weibull generated family is as follows

$$f(x) = \frac{a\alpha\beta(G(x,\xi))^{\beta-1} g(x,\xi)}{(1 - G(x,\xi))^{\beta+1}} e^{-\alpha[\frac{G(x,\xi)}{1-G(x,\xi)}]^{\beta}} [1 - e^{-\alpha[\frac{G(x,\xi)}{1-G(x,\xi)}]^{\beta}}]^{a-1},\ x > 0,\ a,\ \alpha,\ \beta > 0. \tag{4}$$

Hereafter,a random variable X with pdf (4) is denoted by $X \sim EW - G$. We observe that; for $a = b = 1$ the $EW - G$ family reduces to the $W - G$ distributions which is previously given in [1]. The survival, hazard, reversed hazard and cumulative hazard rate functions are, respectively, given by

$$\begin{aligned}
\overline{F}(x) &= 1 - [1 - e^{-\alpha[\frac{G(x,\xi)}{1-G(x,\xi)}]^{\beta}}]^{a},\ x > 0\ ,\ a, \alpha\,, \beta > 0,\\
h(x) &= \frac{a\alpha\beta G(x,\xi)^{\beta-1} g(x,\xi) e^{-\alpha[\frac{G(x,\xi)}{1-G(x,\xi)}]^{\beta}} [1 - e^{-\alpha[\frac{G(x,\xi)}{1-G(x,\xi)}]^{\beta}}]^{a-1}}{(1 - G(x,\xi))^{\beta+1} [1 - (1 - e^{-\alpha[\frac{G(x,\xi)}{1-G(x,\xi)}]^{\beta}})^{a}]},\\
\tau(x) &= \frac{a\alpha\beta G(x,\xi)^{\beta-1} g(x,\xi) e^{-\alpha[\frac{G(x,\xi)}{1-G(x,\xi)}]^{\beta}}}{(1 - G(x,\xi))^{\beta+1} [1 - e^{-\alpha[\frac{G(x,\xi)}{1-G(x,\xi)}]^{\beta}}]}, \ \text{ and}\\
H(x) &= -\ln|1 - F(x)| = -\ln\left|[1 - (1 - e^{-\alpha[\frac{G(x,\xi)}{1-G(x,\xi)}]^{\beta}})^{a}]\right|.
\end{aligned}$$

3. Statistical Properties

Here, we provide some statistical properties of exponentiated Weibull generated family of distributions.

3.1. Quantile and Median

The quantile function, denoted by, $Q(u) = F^{-1}(u)$ of X is derived as follows

$$u = \left[1 - e^{-\alpha[\frac{G(Q(u))}{1-G(Q(u))}]^{\beta}}\right]^{a}.$$

After simplification, the quantile function takes the following form

$$Q(u) = G^{-1}\left[\frac{\{\ln[1-u^{\frac{1}{a}}]^{\frac{-1}{\alpha}}\}^{\frac{1}{\beta}}}{1+\{\ln[1-u^{\frac{1}{a}}]^{\frac{-1}{\alpha}}\}^{\frac{1}{\beta}}}\right], \tag{5}$$

where, u is a uniform distribution on the interval (0, 1) and $G^{-1}(\cdot)$ is the inverse function of $G(\cdot)$. In particular, $Q(0.5)$ is the median of the family and defined by

$$Median = G^{-1}\left[\frac{\{\ln[1-0.5^{\frac{1}{a}}]^{\frac{-1}{\alpha}}\}^{\frac{1}{\beta}}}{1+\{\ln[1-0.5^{\frac{1}{a}}]^{\frac{-1}{\alpha}}\}^{\frac{1}{\beta}}}\right].$$

3.2. Useful Representation

In this subsection, a useful expansion of the probability density and distribution functions for exponentiated Weibull family of distributions is covered. Firstly, we obtain an expansion for pdf defined in (4) as follows: Since the generalized binomial series is

$$(1-z)^{\beta-1} = \sum_{i=0}^{\infty}(-1)^{i}\begin{pmatrix}\beta-1\\ i\end{pmatrix}z^{i}, \tag{6}$$

for $|z| < 1$ and β is a positive real non-integer. Then, by applying the binomial theorem (6) to $[1-e^{-\alpha[\frac{G(x,\xi)}{1-G(x,\xi)}]^{\beta}}]^{a-1}$ in (4), where a is real non integer, the density function of $EW-G$ distribution becomes

$$f(x) = \frac{a\alpha\beta\, G(x,\xi)^{\beta-1}g(x,\xi)}{(1-G(x,\xi))^{\beta+1}}\sum_{i=0}^{\infty}(-1)^{i}\begin{pmatrix}a-1\\ i\end{pmatrix}e^{-\alpha(i+1)[\frac{G(x,\xi)}{1-G(x,\xi)}]^{\beta}}. \tag{7}$$

But,

$$e^{-\alpha(i+1)[\frac{G(x,\xi)}{1-G(x,\xi)}]^{\beta}} = \sum_{j=0}^{\infty}\frac{(-1)^{j}\alpha^{j}(i+1)^{j}}{j!}[\frac{G(x,\xi)}{1-G(x,\xi)}]^{j\beta}. \tag{8}$$

So that, (7) becomes

$$f(x) = a\alpha\beta g(x,\xi)\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}\alpha^{j}(i+1)^{j}}{j!}\begin{pmatrix}a-1\\ i\end{pmatrix}[G(x,\xi)]^{\beta(j+1)-1}\times[1-G(x,\xi)]^{-[\beta(j+1)+1]}. \tag{9}$$

Now,using the generalized binomial series, we can write

$$[1-G(x,\xi)]^{-[\beta(j+1)+1]} = \sum_{k=0}^{\infty}\begin{pmatrix}\beta(j+1)+k\\ k\end{pmatrix}(G(x,\xi))^{k}. \tag{10}$$

Inserting (10) in (9), the pdf of $EW-G$ can be defined as an infinite linear combination of pdf of exponentiated generated i.e.

$$f(x)=\sum_{i,j,k=0}^{\infty}\eta_{i,j,k}g(x,\xi)G(x,\xi)^{\beta(j+1)+k-1}, \tag{11}$$

then,

$$f(x)=\sum_{i,j,k=0}^{\infty}W_{i,j,k}h_{\beta(j+1)+k}(x), \tag{12}$$

where,

$$W_{i,j,k}=\frac{a\alpha^{j+1}(-1)^{i+j}(i+1)^{j}}{[\beta(j+1)+k]j!}\begin{pmatrix}a-1\\ i\end{pmatrix}\begin{pmatrix}\beta\,(j+1)+k\\ k\end{pmatrix},$$

and,

$$W_{i,j,k}=\frac{\eta_{i,j,k}}{[\beta(j+1)+k]},\qquad h_a(x)=a\,g(x,\xi)G(x,\xi)^{a-1}.$$

If a and β are integers the index i stops at $a-1$ and k stops at $\beta(j+1)+k$. When β is non-integer, a more general form can be derived from (10) by adding and subtracting 1 to $G(x,\xi)^{\beta(j+1)+k-1}$ as follows

$$f(x)=\sum_{i,j,k=0}^{\infty}\eta_{i,j,k}(\beta\,(j+1)+k)g(x)[1-[1-G(x)]]^{\beta(j+1)+k-1}.$$

By applying binomial series (6) to $[1-[1-G(x)]]^{\beta(j+1)+k-1}$ in the previous pdf yields:

$$f(x)=\sum_{i,j,k,u=0}^{\infty}(-1)^{u}\eta_{i,j,k}\begin{pmatrix}\beta(j+1)+k-1\\ u\end{pmatrix}g(x,\xi)[1-G(x,\xi)]^{u}.$$

Using binomial expansion one more time yields:

$$f(x)=\sum_{i,j,k,u=0}^{\infty}\sum_{m=0}^{u}(-1)^{u+m}\eta_{i,j,k}\begin{pmatrix}\beta(j+1)+k-1\\ u\end{pmatrix}\begin{pmatrix}u\\ m\end{pmatrix}g(x,\xi)(G(x,\xi))^{m}.$$

So, the the final expansion form of pdf is as follows

$$f(x)=\sum_{i,j,k,u=0}^{\infty}\sum_{m=0}^{u}\eta_{i,j,k,u,m}g(x,\xi)(G(x,\xi))^{m}, \tag{13}$$

where,

$$\eta_{i,j,k,u,m}=(-1)^{u+m}\eta_{i,j,k}\begin{pmatrix}\beta(j+1)+k-1\\ u\end{pmatrix}\begin{pmatrix}u\\ m\end{pmatrix}.$$

Secondly, an expansion for the $[F(x)]^h$ is obtained as following: Again, the binomial expansion is worked out for $[F(x)]^h$, with h is integer and a is real non-integer.

$$[F(x)]^{h}=\sum_{q=0}^{\infty}(-1)^{q}\begin{pmatrix}ah\\ q\end{pmatrix}e^{-\alpha q\left[\frac{G(x,\xi)}{1-G(x,\xi)}\right]^{\beta}}.$$

But,

$$e^{-\alpha q\left[\frac{G(x,\xi)}{1-G(x,\xi)}\right]^{\beta}}=\sum_{t=0}^{\infty}\frac{(\alpha q)^{t}}{t!}\left[\frac{G(x,\xi)}{1-G(x,\xi)}\right]^{\beta t}$$

then $[F(x)]^h$ takes the following form

$$[F(x)]^h = \sum_{q=0}^{\infty}\sum_{t=0}^{\infty} \frac{(-1)^{q+t}(\alpha q)^t}{t!} \binom{ah}{q} \left[\frac{G(x,\xi)}{1-G(x,\xi)}\right]^{\beta t}.$$

by using the relation (10) in the preceding equation where β is real non-integer, we obtain

$$[F(x)]^h = \sum_{q=0}^{\infty}\sum_{t,u=0}^{\infty} \frac{(-1)^{q+t+u}(\alpha q)^t}{t!} \binom{ah}{q} \binom{\beta t}{u} G(x,\xi)^{\beta t+u}.$$

Again, the binomial expansion is applied to $G(x,\xi)^{\beta t+u}$ by adding and subtracting 1, then $[F(x)]^h$ will be as follows

$$[F(x)]^h = \sum_{q=0}^{\infty}\sum_{t,u,l=0}^{\infty} \frac{(-1)^{q+t+u+l}(\alpha q)^t}{t!} \binom{ah}{q} \binom{\beta t}{u} \binom{\beta t+u}{l} [1-G(x,\xi)]^l.$$

For l is a real, then $[F(x)]^h$ is as follows

$$[F(x)]^h = \sum_{q,t,u,l=0}^{\infty}\sum_{z=0}^{l} \frac{(-1)^{q+t+u+l+z}(\alpha q)^t}{t!} \binom{ah}{q} \binom{\beta t}{u} \binom{\beta t+u}{l} \binom{l}{z} G(x,\xi)^z.$$

Replacing $\sum_{l=0}^{\infty}\sum_{z=0}^{l}$ with $\sum_{l=z}^{\infty}\sum_{z=0}^{\infty}$ yielding

$$[F(x)]^h = \sum_{q,t,u=0}^{\infty}\sum_{l=z}^{\infty}\sum_{z=0}^{\infty} \frac{(-1)^{q+t+u+l+z}(\alpha q)^t}{t!} \binom{ah}{q} \binom{\beta t}{u} \binom{\beta t+u}{l} \binom{l}{z} G(x,\xi)^z.$$

Finally,

$$[F(x)]^h = \sum_{z=0}^{\infty} s_z G(x,\xi)^z, \tag{14}$$

where,

$$s_z = \sum_{q,t,u=0}^{\infty}\sum_{l=z}^{\infty} \frac{(-1)^{q+t+u+l+z}(\alpha q)^t}{t!} \binom{ah}{q} \binom{\beta t}{u} \binom{\beta t+u}{l} \binom{l}{z}.$$

3.3. The Probability Weighted Moments

For a random variable X, the probability weighted moments can be calculated by the following relation

$$\tau_{r,s} = E[X^r F(x)^s] = \int_{-\infty}^{\infty} x^r f(x)(F(x))^s dx. \tag{15}$$

First, when β is an integer, by substituting (11) and (14) into (15), replacing h with s, leads to:

$$\tau_{r,s} = \int_{-\infty}^{\infty} \sum_{i,j,k=0}^{\infty}\sum_{z=0}^{\infty} s_z \eta_{i,j,k} x^r g(x,\xi)(G(x,\xi))^{z+\beta(j+1)+k-1} dx.$$

Then,

$$\tau_{r,s} = \sum_{i,j,k=0}^{\infty}\sum_{z=0}^{\infty} s_z \eta_{i,j,k} \tau_{r,z+\beta(j+1)+k-1},$$

where,

$$\tau_{r,z+\beta(j+1)+k-1} = \int_{-\infty}^{\infty} x^r g(x,\xi)(G(x,\xi))^{z+\beta(j+1)+k-1} dx.$$

Second, when β is real non-integer, by substituting (13) and (14) into (15), replacing h with s, leads to:

$$\tau_{r,s} = \int_{-\infty}^{\infty} \sum_{i,j,k,u=0}^{\infty} \sum_{m=0}^{u} \sum_{z=0}^{\infty} s_z \eta_{i,j,k,u,m} x^r g(x,\xi) G(x,\xi)^{m+z} dx.$$

Then

$$\tau_{r,s} = \sum_{i,j,k,u=0}^{\infty} \sum_{m=0}^{u} \sum_{z=0}^{\infty} s_z \eta_{i,j,k,u,m} \tau_{r,z+m},$$

where,

$$\tau_{r,z+m} = \int_{-\infty}^{\infty} x^r g(x,\xi) G(x,\xi)^{m+z} dx.$$

Additionally; two different forms will be yielded by using quantile function. The first form is derived for β an integer as follows

$$\tau_{r,s} = \int_{0}^{1} \sum_{i,j,k=0}^{\infty} \sum_{z=0}^{\infty} s_z \eta_{i,j,k} (Q_G(u))^r u^{z+\beta(j+1)+k-1} du.$$

While, the second form is obtained when β real non-integer

$$\tau_{r,s} = \int_{0}^{1} \sum_{i,j,k,u=0}^{\infty} \sum_{m=0}^{u} \sum_{z=0}^{\infty} s_z \eta_{i,j,k,u,m} (Q_G(u))^r u^{z+m} du.$$

3.4. Moments

If X has the pdf (13), then r^{th} moment is obtained as follows

$$\mu'_r = \int_{-\infty}^{\infty} x^r f(x) dx\,. \tag{16}$$

First, when β is an integer, then substituting (11) into (16) yields:

$$\mu'_r = \int_{-\infty}^{\infty} \sum_{i,j,k=0}^{\infty} \eta_{i,j,k} x^r g(x,\xi)(G(x,\xi))^{\beta(j+1)+k-1} dx.$$

Then,

$$\mu'_r = \sum_{i,j,k=0}^{\infty} \eta_{i,j,k} \tau_{r,\beta(j+1)+k-1},$$

where, $\tau_{r,\beta(j+1)+k-1}$ is the probability weighted moments of the $G(x,\xi)$ distribution. Second, when β is real non-integer, then substituting (13) into (16) yields:

$$\mu'_r = \int_{-\infty}^{\infty} \sum_{i,j,k,u=0}^{\infty} \sum_{m=0}^{u} \eta_{i,j,k,u,m} x^r g(x,\xi)(G(x,\xi))^m dx.$$

Then, $\mu'_r = \sum_{i,j,k,u=0}^{\infty} \sum_{m=0}^{u} \eta_{i,j,k,u,m} \tau_{r,m}$, where, $\tau_{r,m} = \int_{-\infty}^{\infty} x^r g(x,\xi)(G(x,\xi))^m dx$. Based on the parent quantile function, two different forms can be derived. The first form can be deduced for β an integer;

$$\mu'_r = \sum_{i,j,k=0}^{\infty} \eta_{i,j,k} \int_{0}^{1} (Q_G(u))^r u^{\beta(j+1)+k-1} du\,.$$

While the second structure, when β is real non-integer yields

$$\mu'_r = \sum_{i,j,k,u=0}^{\infty} \sum_{m=0}^{u} \eta_{i,j,k,u,m} \int_{0}^{1} (Q_G(u))^r u^m du.$$

3.5. Moment Generating Function

For a random variable X, it is known that, the moment generating function is defined as

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r.$$

Two different forms will be produced, the first form for β is an integer; while the second form for β is real non-integer, they are as follows;

$$M_X(t) = \sum_{r=0}^{\infty} \sum_{i,j,k=0}^{\infty} \frac{t^r}{r!} \eta_{i,j,k} \tau_{r,\beta(j+1)+k-1},$$

$$M_X(t) = \sum_{r=0}^{\infty} \sum_{i,j,k,u=0}^{\infty} \sum_{m=0}^{u} \frac{t^r}{r!} \eta_{i,j,k,u,m} \tau_{r,m}.$$

Additionally; two different forms will be yielded by using quantile function, the first form for β is an integer; while the second form for β is real non-integer, they are as follows;

$$M_u(t) = \sum_{i,j,k=0}^{\infty} \eta_{i,j,k} \int_0^1 e^{(tQ_G(u))} u^{\beta(j+1)+k-1} du.$$

$$M_u(t) = \sum_{i,j,k,u=0}^{\infty} \sum_{m=0}^{u} \eta_{i,j,k,u,m} \int_0^1 e^{(tQ_G(u))} u^m du.$$

3.6. The Mean Deviation

Generally, the mean deviation can be calculated by the following relation $\delta_1(X) = 2\mu F(\mu) - 2T(\mu)$ and $\delta_2(X) = \mu - 2T(M)$, where, $T(q) = \int_{-\infty}^{q} x f(x) dx$ which is the first incomplete moment. Depending on the parent quantile function, two additional forms are obtained. Firstly, when β is an integer

$$T(q) = \sum_{i,j,k=0}^{\infty} \eta_{i,j,k} \int_0^q Q_G(u) u^{\beta(j+1)+k-1} du.$$

While the second form is derived when β is real non-integer

$$T(q) = \sum_{i,j,k,u=0}^{\infty} \sum_{m=0}^{u} \eta_{i,j,k,u,m} \int_0^q Q_G(u) u^m du.$$

3.7. Order Statistics

Let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ be an ordered random sample from a population of size n following the exponentiated Weibull generated family, then as mentioned in [10] the pdf of the k^{th} order statistic, is calculated by

$$f_{X_{(k)}}(x_{(k)}) = \frac{f(x_{(k)})}{B(k, n-k+1)} \sum_{v=0}^{n-k} (-1)^v \begin{pmatrix} n-k \\ v \end{pmatrix} F(x_{(k)})^{v+k-1}. \tag{17}$$

First, when β is an integer substituting (11) and (14) in (17), replacing h with $v + k - 1$, leads to

$$f_{X_{(k)}}(x_{(k)}) = \frac{g(x_{(k)}, \xi)}{B(k, n-k+1)} \sum_{v=0}^{n-k} \sum_{z=0}^{\infty} \sum_{i,j,k=0}^{\infty} \eta_{i,j,k} p_{z,v} G(x_{(k)}, \xi)^{z+\beta(j+1)+k-1}, \tag{18}$$

where $p_{z,v} = (-1)^v \begin{pmatrix} n-k \\ v \end{pmatrix} s_z$. Second, when β is real non-integer substituting (13) and (14) in (17), replacing h with $v+k-1$ leads to

$$f_{X_{(k)}}(x_{(k)}) = \frac{g(x_{(k)})}{B(k,n-k+1)} \sum_{v=0}^{n-k} \sum_{z=0}^{\infty} \sum_{i,j,k,u=0}^{\infty} \sum_{m=0}^{u} \eta_{i,j,k,u,m} p_{z,v} G(x_{(k)},\xi)^{m+z}, \tag{19}$$

where $g(\cdot)$ and $G(\cdot)$ are the density and cumulative distribution functions of the $EW-G$ distribution, respectively. Moments of order statistics is defined by:

$$E({X_{(k)}}^r) = \int_{-\infty}^{\infty} {x_{(k)}}^r f(x_{(k)}) dx_{(k)} \,. \tag{20}$$

First, when β is an integer substituting (18) in (20), leads to

$$E({X_{(k)}}^r) = \frac{1}{B(k,n-k+1)} \sum_{v=0}^{n-k} \sum_{z=0}^{\infty} \sum_{i,j,k=0}^{\infty} \eta_{i,j,k} p_{z,v} \int_{-\infty}^{\infty} {x_{(k)}}^r g(x_{(k)},\xi) G(x_{(k)},\xi)^{z+\beta(j+1)+k-1} dx_{(k)}.$$

Then,

$$E({X_{(k)}}^r) = \frac{1}{B(k,n-k+1)} \sum_{v=0}^{n-k} \sum_{z=0}^{\infty} \sum_{i,j,k=0}^{\infty} \eta_{i,j,k} p_{z,v} \tau_{r,z+\beta(j+1)+k-1}.$$

Second, when β is real non integer substituting (19) in (20), leads to

$$E({X_{(k)}}^r) = \frac{1}{B(k,n-k+1)} \int_{-\infty}^{\infty} \sum_{v=0}^{n-k} \sum_{z=0}^{\infty} \sum_{i,j,k,u=0}^{\infty} \sum_{m=0}^{u} \eta_{i,j,k,u,m} p_{z,v} {x_{(k)}}^r g(x_{(k)},\xi) G(x_{(k)},\xi)^{m+z} dx_{(k)}.$$

Then,

$$E({X_{(k)}}^r) = \frac{1}{B(k,n-k+1)} \sum_{v=0}^{n-k} \sum_{z=0}^{\infty} \sum_{i,j,k,u=0}^{\infty} \sum_{m=0}^{u} \eta_{i,j,k,u,m} p_{z,v} \tau_{r,z+m}.$$

4. Special Distributions

In this section, we give some selected special distributions from $EW-G$ family. The selected models are EW-uniform, EW-Burr XII, EW-Weibull and EW-quasi Lindley.

4.1. EW-uniform Distribution

As a first example, suppose that the parent distribution is uniform in the interval $0 < x < \theta$ and $g(x,\theta) = \frac{1}{\theta}$. Therefore, the EW uniform distribution, denoted by $EWU\,(a,\,\alpha,\,\beta,\,\theta)$, has the following cdf and pdf by direct substituting $G(x,\theta) = \frac{x}{\theta}$, in (3) and (4) as follows $F(x) = [1-e^{-\alpha(\frac{x}{\theta-x})^\beta}]^a$, $f\,(x) = \frac{a\alpha\beta\theta x^{\beta-1}}{(\theta-x)^{\beta+1}} e^{-\alpha(\frac{x}{\theta-x})^\beta}[1-e^{-\alpha(\frac{x}{\theta-x})^\beta}]^{a-1}$, $a,\,\alpha,\,\beta>0,\ 0<x<\theta$. Furthermore, the survival and the hazard rate functions are given, respectively, as follows $\overline{F}(x) = 1-[1-e^{-\alpha(\frac{x}{\theta-x})^\beta}]^a$, and

$$h(x) = \frac{a\alpha\beta\theta x^{\beta-1} e^{-\alpha(\frac{x}{\theta-x})^\beta}[1-e^{-\alpha(\frac{x}{\theta-x})^\beta}]^{a-1}}{(\theta-x)^{\beta+1}[1-[1-e^{-\alpha(\frac{x}{\theta-x})^\beta}]^a]}.$$

4.2. EW-Burr XII Distribution

Let us consider the Burr XII distribution with probability density and distribution functions given, respectively, by

$$g(x,c,\mu,\,\sigma) = c\sigma\mu^{-c}x^{c-1}[1+(\frac{x}{\mu})^c]^{-\sigma-1}, c,\mu,\,\sigma>0,$$

and, $G(x,c,\mu,\sigma) = 1-[1+(\frac{x}{\mu})^c]^{-\sigma}$. Then the $EW-BurrXII$ distribution, denoted by, $EWBurrXII\,(a,b,\alpha,\beta,c,\mu,\sigma)$ has the following cdf, pdf, survival and hazard rate functions $F(x) = [1-e^{-\alpha[(1+(\frac{x}{\mu})^c)^\sigma-1]^\beta}]^a$, $a,\ \alpha,\ \beta,\ c,\mu,\ \sigma > 0,\ x>0,$

$$f(x) = \frac{ac\alpha\beta\sigma\mu^{-c}x^{c-1}e^{-\alpha[(1+(\frac{x}{\mu})^c)^\sigma-1]^\beta}[(1+(\frac{x}{\mu})^c)^\sigma-1]^{\beta-1}[1-e^{-\alpha[(1+(\frac{x}{\mu})^c)^\sigma-1]^\beta}]^{a-1}}{(1+(\frac{x}{\mu})^c)^{1-\sigma}},$$

$\overline{F}(x) = 1-[1-e^{-\alpha[(1+(\frac{x}{\mu})^c)^\sigma-1]^\beta}]^a$, and

$$h(x) = \frac{a\,c\alpha\beta\sigma\mu^{-c}x^{c-1}e^{-\alpha[(1+(\frac{x}{\mu})^c)^\sigma-1]^\beta}[(1+(\frac{x}{\mu})^c)^\sigma-1]^{\beta-1}[1-e^{-\alpha[(1+(\frac{x}{\mu})^c)^\sigma-1]^\beta}]^{a-1}}{(1+(\frac{x}{\mu})^c)^{1-\sigma}[1-[1-e^{-\alpha[(1+(\frac{x}{\mu})^c)^\sigma-1]^\beta}]^a]}.$$

4.3. EW-Weibull Distribution

Let us consider, the Weibull distribution with distribution function given by $G(x,\lambda,\gamma) = 1-e^{-\lambda x^\gamma}$ for $x>0$ where, $\lambda>0$ is scale parameter and $\gamma>0$ is the shape parameter. The cdf, pdf, survival and the hazard rate functions of the EW-Weibull distribution, denoted by $EWW\,(a,\ \alpha,\ \beta,\ \lambda,\ \gamma)$, take the following forms respectively,

$$F(x) = [1-e^{-\alpha(e^{\lambda x^\gamma}-1)^\beta}]^a, \quad a,\ \alpha,\ \beta,\ \lambda,\ \gamma>0, x>0,$$

$$f(x) = a\alpha\beta\lambda\gamma x^{\gamma-1}[e^{\lambda x^\gamma}-1]^{\beta-1}e^{-\{\alpha(e^{\lambda x^\gamma}-1)^\beta-\lambda x^\gamma\}}(1-e^{-\alpha(e^{\lambda x^\gamma}-1)^\beta})^{a-1},$$

$$\overline{F}(x) = 1-[1-e^{-\alpha(e^{\lambda x^\gamma}-1)^\beta}]^a,$$

and

$$h(x) = \frac{a\alpha\beta\lambda\gamma x^{\gamma-1}[e^{\lambda x^\gamma}-1]^{\beta-1}e^{-\{\alpha(e^{\lambda x^\gamma}-1)^\beta-\lambda x^\gamma\}}(1-e^{-\alpha(e^{\lambda x^\gamma}-1)^\beta})^{a-1}}{1-(1-e^{-\alpha(e^{\lambda x^\gamma}-1)^\beta})^a},$$

when $\gamma=1$, we get EW-exponential distribution.

4.4. EW-quasi Lindley Distribution

The quasi Lindley distribution, denoted by $QL(\theta,\ p)$, distribution has been studied by [11]. The probability density and distribution functions of $QL(\theta,\ p)$, are defined as follows;

$$G(x,\theta) = 1-e^{-\theta x}\left[1+\frac{\theta x}{p+1}\right],$$

$$g(x,\theta) = \frac{\theta}{p+1}(p+\theta x)\,e^{-\theta x}.$$

The cdf, pdf, survival and the hazard rate functions for EW-quasi Lindley distribution (EWQL) are obtained from (3) and (4), respectively as

$$F(x) = [1-e^{-\alpha[e^{\theta x}(1+\frac{\theta x}{p+1})^{-1}-1]^\beta}]^a, \qquad a,\ \alpha,\ \beta,\ \theta>0\ ,\ p>-1\,,\ x>0,$$

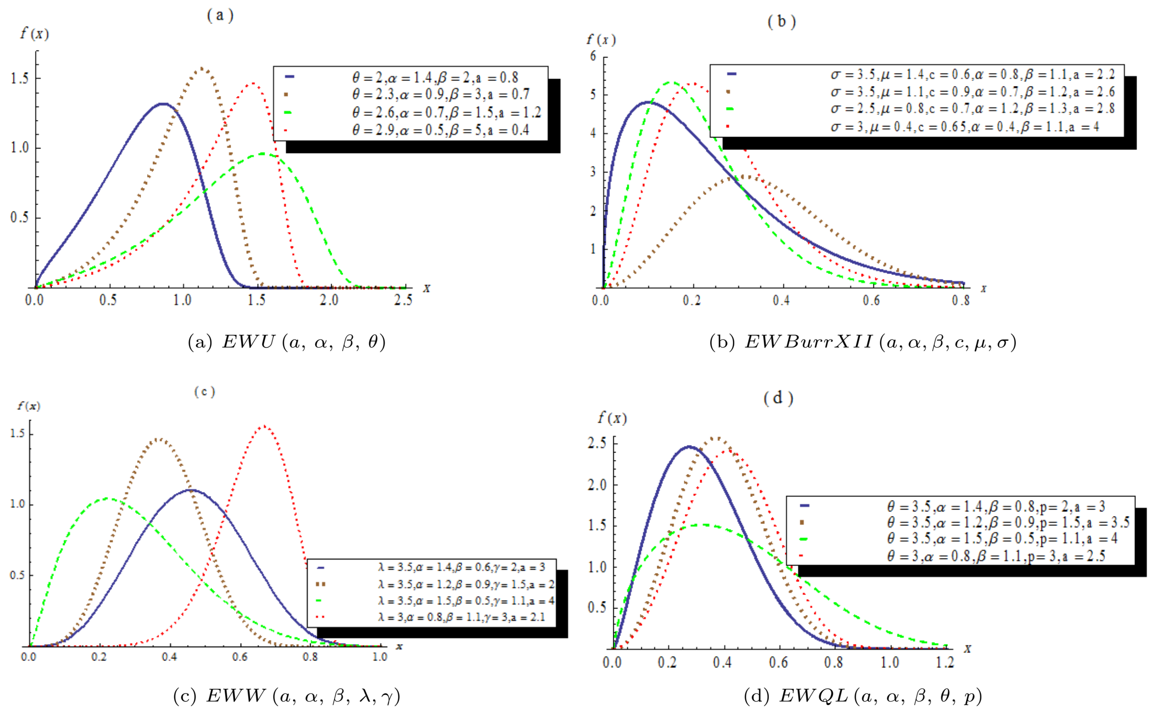
$$f(x) = \frac{a\alpha\beta\theta(p+\theta x)e^{-[\alpha[e^{\theta x}(1+\frac{\theta x}{p+1})^{-1}-1]^\beta-\theta x]}[e^{\theta x}(1+\frac{\theta x}{p+1})^{-1}-1]^{\beta-1}(1-e^{-\alpha[e^{\theta x}(1+\frac{\theta x}{p+1})^{-1}-1]^\beta})^{a-1}}{(p+1)(1+\frac{\theta x}{p+1})^2},$$

$$\overline{F}(x) = 1-[1-e^{-\alpha[e^{\theta x}(1+\frac{\theta x}{p+1})^{-1}-1]^\beta}]^a,$$

and,

$$h(x) = \frac{a\alpha\beta\theta(p+\theta x)[e^{\theta x}(1+\frac{\theta x}{p+1})^{-1}-1]^{\beta-1}(1-e^{-\alpha[e^{\theta x}(1+\frac{\theta x}{p+1})^{-1}-1]^\beta})^{a-1}\ e^{-[\alpha[e^{\theta x}(1+\frac{\theta x}{p+1})^{-1}-1]^\beta-\theta x]}}{(p+1)(1+\frac{\theta x}{p+1})^2[1-(1-e^{-\alpha[e^{\theta x}(1+\frac{\theta x}{p+1})^{-1}-1]^\beta})^a]}.$$

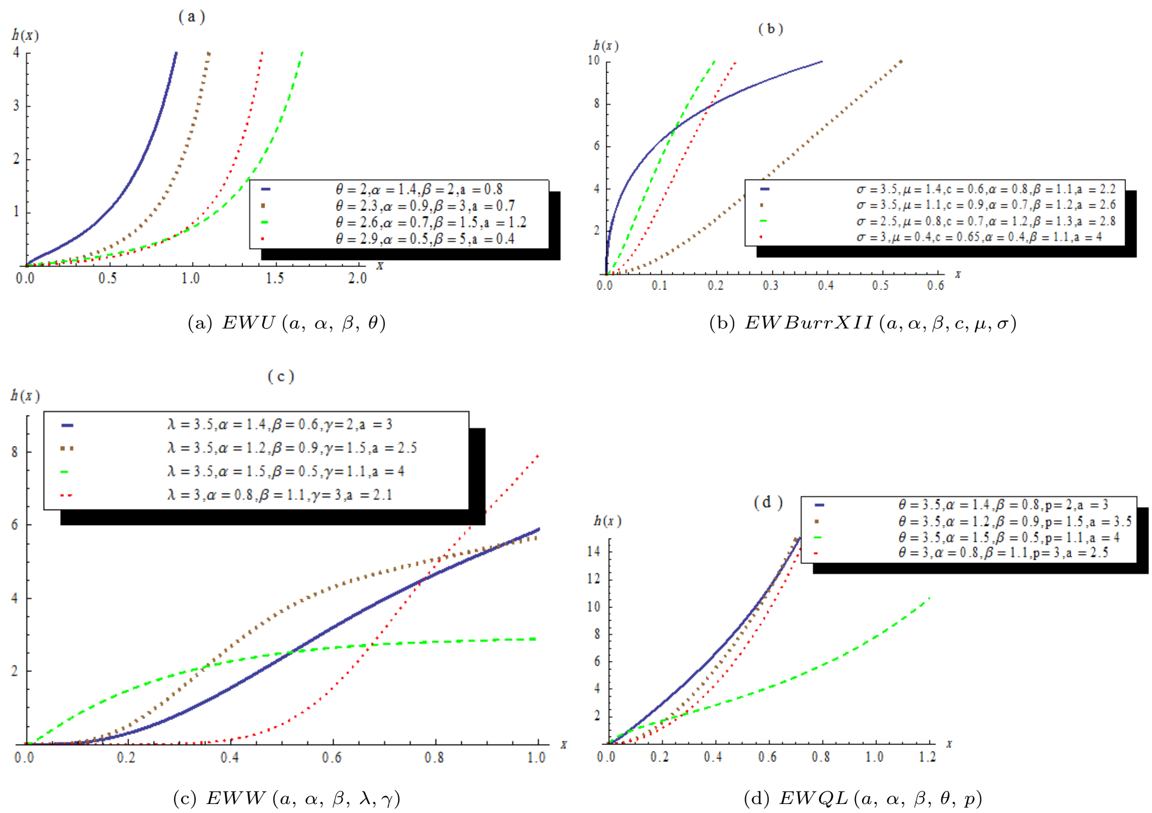
For $p=\theta$ the EW-Lindley distribution is obtained. Plots of pdf and hazard rate function for the selected distributions are showed in Figures 1 and 2 respectively.



(a) $EWU\,(a,\ \alpha,\ \beta,\ \theta)$ (b) $EWBurrXII\,(a, \alpha, \beta, c, \mu, \sigma)$

(c) $EWW\,(a,\ \alpha,\ \beta,\ \lambda, \gamma)$ (d) $EWQL\,(a,\ \alpha,\ \beta,\ \theta,\ p)$

Figure 1. **Plots of pdfs for some selected values of parameters**



(a) $EWU\,(a,\ \alpha,\ \beta,\ \theta)$ (b) $EWBurrXII\,(a, \alpha, \beta, c, \mu, \sigma)$

(c) $EWW\,(a,\ \alpha,\ \beta,\ \lambda, \gamma)$ (d) $EWQL\,(a,\ \alpha,\ \beta,\ \theta,\ p)$

Figure 2. **Plots of hazard rate functions for some selected values of parameters**

5. Maximum Likelihood Method

This section deals with the maximum likelihood estimators of the unknown parameters for the $EW-G$ family of distributions based on complete samples. Let $X_1, ..., X_n$ be observed values from the $EW-G$ family with set of parameter $\Theta =$

$(a,\, \alpha,\, \beta,\, \zeta)^T$. The log-likelihood function for parameter vector $\Theta = (a,\, \alpha,\, \beta,\, \zeta)^T$ is obtained as follows

$$\ln L(\Theta) = n\ln a + n\ln\alpha + n\ln\beta + (\beta-1)\sum_{i=1}^{n}\ln\left[G(x_i,\zeta)\right] - (\beta+1)\sum_{i=1}^{n}\ln\left[1-G(x_i,\zeta)\right]$$
$$+\sum_{i=1}^{n}\ln\left[g(x_i,\zeta)\right] - \alpha\sum_{i=1}^{n}\left[H(x_i,\zeta)\right]^{\beta} + (a-1)\sum_{i=1}^{n}\ln[1-e^{-\alpha[H(x_i,\zeta)]^{\beta}}],$$

where $H(x_i,\zeta) = \frac{G(x_i,\zeta)}{1-G(x_i,\zeta)}$. The elements of the score function $U(\Theta) = (U_a,\, U_\alpha,\, U_\beta,\, U_\zeta)$ are given by

$$U_a = \frac{n}{a} + \sum_{i=1}^{n}\ln[1-e^{-\alpha[H(x_i,\zeta)]^{\beta}}],$$
$$U_\alpha = \frac{n}{\alpha} - \sum_{i=1}^{n}[H(x_i,\zeta)]^{\beta} + (a-1)\sum_{i=1}^{n}\frac{[H(x_i,\zeta)]^{\beta}}{e^{\alpha[H(x_i,\zeta)]^{\beta}}-1},$$
$$U_\beta = \frac{n}{\beta} + \sum_{i=1}^{n}\ln[H(x_i,\zeta)] - \alpha\sum_{i=1}^{n}[H(x_i,\zeta)]^{\beta}\ln[H(x_i,\zeta)] + \alpha(a-1)\sum_{i=1}^{n}\frac{[H(x_i,\zeta)]^{\beta}\ln[H(x_i,\zeta)]}{e^{\alpha[H(x_i,\zeta)]^{\beta}}-1},$$

and

$$U_{\zeta_k} = (\beta-1)\sum_{i=1}^{n}\frac{\partial G(x_i,\zeta)/\partial\zeta_k}{G(x_i,\zeta)} + (\beta+1)\sum_{i=1}^{n}\frac{\partial G(x_i,\zeta)/\partial\zeta_k}{1-G(x_i,\zeta)} + \sum_{i=1}^{n}\frac{\partial g(x_i,\zeta)/\partial\zeta_k}{g(x_i,\zeta)}$$
$$-\alpha\beta\sum_{i=1}^{n}[H(x_i,\zeta)]^{\beta-1}\partial H(x_i,\zeta)/\partial\zeta_k + \alpha\beta(a-1)\sum_{i=1}^{n}\frac{[H(x_i,\zeta)]^{\beta-1}e^{-\alpha[H(x_i,\zeta)]^{\beta}}/\partial H(x_i,\zeta)/\partial\zeta_k}{1-e^{-\alpha[H(x_i,\zeta)]^{\beta}}}.$$

Setting U_a, U_α, U_β and U_ζ equal to zero and solving the equations simultaneously yields the maximum likelihood estimate (MLE)$\widehat{\Theta} = (\widehat{a}, \widehat{\alpha}, \widehat{\beta}, \widehat{\zeta})$ of $\Theta = (a,\, \alpha,\, \beta,\, \zeta)^T$. These equations cannot be solved analytically and statistical software can be used to solve them numerically using iterative methods.

6. Application

In this section, the flexibility of some special models of EW generated distributions are examined using a real data set from [12]. We illustrate the superiority of new selected distribution as compared with some of their sub-models. The selected submodels are exponentiated Weibull exponential (EWE) distribution, Weibull exponential (WE) distribution and exponential exponential (EE) distribution. The corresponding densities of the selected sub-models are:

$$f_{EWE}(x) = a\alpha\beta\lambda[e^{\lambda x}-1]^{\beta-1}e^{-\{\alpha(e^{\lambda x}-1)^{\beta}-\lambda x\}}(1-e^{-\alpha(e^{\lambda x}-1)^{\beta}})^{a-1} \quad a,\, \alpha,\, \beta,\, \lambda > 0,\;\; x > 0,$$
$$f_{WE}(x) = \alpha\beta\lambda[e^{\lambda x}-1]^{\beta-1}e^{-\{\alpha(e^{\lambda x}-1)^{\beta}-\lambda x\}} \quad \alpha,\, \beta,\, \lambda > 0,\;\; x > 0,\;\; \text{and}$$
$$f_{EE}(x) = \alpha\lambda e^{-\{\alpha(e^{\lambda x}-1)-\lambda x\}} \quad \alpha,\, \lambda > 0,\;\; x > 0.$$

The following data gives the observed 72 survival times data (in days) of infected guinea pigs. Its infected with virulent tubercle bacilli. The data is recorded as follows:

0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

On the basis of maximum likelihood method, the parameters of EWE, WE and EE distributions are estimated. Some selected measuresas; $-2\ln L$, Akaike information criterion (AIC), Bayesian information criterion (BIC), the correct Akaike information criterion (CAIC), Hannan- Quinn information criterion (HQIC), the Kolmogorov-Smirnov $(K-S)$ and $p-value$ statistics are obtained to compare the fitted models (as seen in Table 1). The mathematical form of these measures is as follows:

$$AIC = 2k - 2\ln L,$$
$$CAIC = AIC + \frac{2k(k+1)}{n-k-1},$$
$$BIC = k\ln(n) - 2\ln L,$$
$$HQIC = 2k\ln[\ln(n)] - 2\ln L,$$

where k is the number of models parameter, n is the sample size and $\ln L$ is the maximized value of the log- likelihood function under the fitted models. Also, $k-s = \sup_y [F_n(y) - F(y)]$ where $F_n(y) = \frac{1}{n}$ (number of observation $\leq y$), and $F(y)$ denotes the cdf. Additionally, plots of estimated cumulative and estimated densities of the fitted EWE, WE and EE models for the data set are achieved (as seen in Figures 3 and 4)

	Distributions		
	EWE	WE	EE
Parameter Estimates	$\hat{a} = 10.973$	-	-
	$\hat{\alpha} = 32.229$	$\hat{\alpha} = 163.738$	$\hat{\alpha} = 73.753$
	$\widehat{\beta} = 0.601$	$\widehat{\beta} = 1.584$	-
	$\hat{\lambda} = 0.011$	$\hat{\lambda} = 0.02$	$\hat{\lambda} = 0.0074$
$-2\ln L$	283.242	292.659	379.023
AIC	291.242	298.659	383.023
$CAIC$	292.842	299.582	383.197
BIC	290.671	298.036	382.738
$HQIC$	294.867	301.378	384.836
$K-S$	0.111	0.1394	0.286
$p-value$	0.338	0.122	0.00002

Table 1. **Values of MLE and information measures**

It is obvious from the measure values in Table 1,that the EWE is appropriate model than the two other.

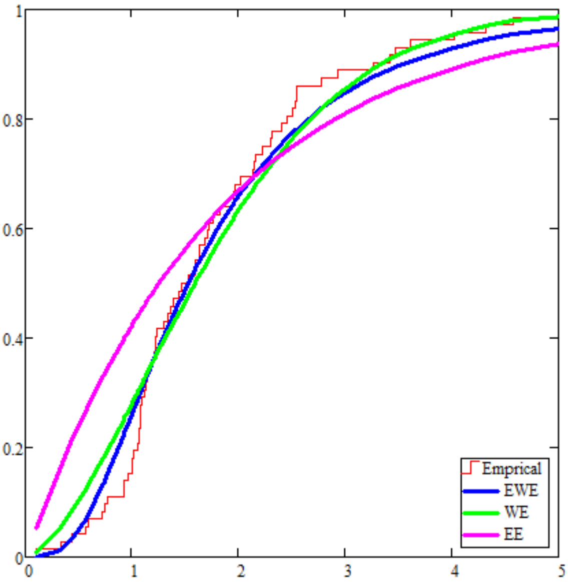


Figure 3. **Estimated cumulative distribution functions for the data set**

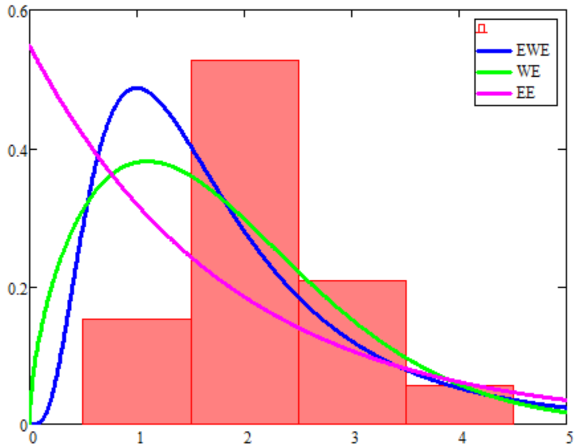


Figure 4. **Estimated densities of models for the data set**

It is clear that EWE model provides the overall best fit and therefore could be chosen as the more adequate model for explaining data.

7. Conclusion

In this article, we introduced the new generated family of distributions which is extended the Weibull generated family. More specifically, the exponentiated Weibull generated family covers several new distributions. We wish a broadly statistical application in some area for this new generalization. Properties of the $EW-G$ were discussed, such as, expressions for the density function, moments, mean deviation, quantile function and order statistics. The maximum likelihood method is employed for estimating the model parameters. Four special models namely; exponentiated Weibull uniform, exponentiated WeibullBurrXII, exponentiated WeibullWeibull and exponentiated Weibull quasi Lindley are provided.. Further, the derived properties of the generated family are valid to these selected models. We fit some $EW-G$ distributions to one real data to demonstrate the potentiality of this family.

Acknowledgement

The authors would like to express their thanks to the editor for important remarks, careful reading and support.

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