Coded Random CDMA with Partitioned Spreading
Lukasz Krzymien, Dmitri Truhachev, and Christian Schlegel

Abstract—A Code Division Multiple Access (CDMA) system is considered, where a number of concurrent users, distinguished by random spreading waveforms, access the common additive white Gaussian noise (AWGN) channel. All users employ a regular low-density parity-check (LDPC) error control code (ECC). Additionally, a method, called Partitioned Spreading (PS) is used. Each user’s spreading waveform is divided into \( M \) partitions which are interleaved and spread in time. Asymptotic performance of the proposed system is studied for two different decoding schedules. In the first schedule, called two-stage schedule, iterative multiple access (MA) detection and LDPC decoding are executed separately. The second schedule, named full decoding allows to exchange information between MA and LDPC sides. Supportable system loads are derived as functions of the users’ signal-to-noise ratio (SNR). It is shown that significantly higher system loads can be achieved compared to the traditional LDPC coded random CDMA.

I. INTRODUCTION

The recent invention of turbo codes [1] and the rediscovery of LDPC codes [2] have caused a revolution in error control coding. Moreover, new approaches involving iterative information processing have also been sketched in several related fields. Iterative detection and decoding of CDMA signals is a flagship example of it [3]. The conjunction of CDMA with an ECC naturally imposes a serial concatenation. It can be successfully decoded using iterative principle, where a particular ECC plays the role of the inner code. However, the choice of the ECC remains somewhat an open question. In the situations when some form of received power control is possible the use of complex powerful codes results in good performance. Nevertheless, for equal power users, or more generally for uncontrolled power levels, simple codes such as repetition or convolutional codes outperform even strong turbo codes in terms of achievable spectral efficiencies [4] [5].

Partitioned spreading CDMA, introduced in [6], is based on division of the waveform which is spreading users’ bit into a number of partitions. These partitions are interleaved over time frame which is much longer than the original spreading waveform duration. The detection can be accomplished in an iterative manner where different partitions are delivering uncorrelated signal estimates. Moreover, PS CDMA has a great potential for concatenation with strong iteratively decodable codes since in this way a simple repetition coding is realized, which provides an interface between the CDMA channel and a complex powerful ECC.

PS CDMA can be seen as a generalization of interleaved CDMA (IDMA), proposed in [7]. IDMA relies on interfering chips instead of group of chips, i.e. \( M = N \). The implementation of this system in digital circuits is difficult for large \( N \) since extremely accurate timing is required. We show that asymptotic performance loss is small if moderate finite number of partitions \( M \), chosen independently of \( N \), is used.

Two iterative decoding detection methods for the considered system arise naturally. The first method, called two-stage schedule, separates the iterative detection of the PS CDMA and iterative decoding of the users’ LDPC codes. The second method, called full decoding, allows for exchange of the soft information between the PS detector and the LDPC decoders. Thus, even higher system loads can be supported in the latter case. However, the decoding complexity requirements are also increased.

The paper is organized as follows. In Section II we describe the system model. Section III treats the two-stage decoding schedule. First, we make connection to results in [9] derived from statistical mechanics. We show that for equal power and equal rate users iterative joint detection of PS CDMA performs identically to exponentially complex individually optimal APP detector for a large range of system loads. Moreover, we show that PS detector performance (achievable load) is lower bounded by that of minimum mean square error (MMSE) detector. For the coded system the dependence of the system load on the users’ signal-to-noise ratio (SNR) is derived. Additionally, unequal user power profiles with minimum average power are studied and it is proved that system load constantly improves with increase of the number of users’ power groups.

In Section IV we focus on description and performance evaluation of the second processing scheme - full decoding. It is demonstrated that performance of LDPC coded PS CDMA is superior to that of conventional CDMA with iterative MMSE filtering and decoding of users’ LDPC codes. Finally, Section V provides the conclusions.

II. PARTITIONED SPREADING CDMA

We consider a random CDMA system with \( K \) concurrent users communicating over an AWGN channel. All
$K$ users transmit information with the same rate, and employ length $L$ binary regular $d_v, d_c$ LDPC code. Each symbol $v_{k,l} \in \{-1,1\}$ from codeword $v_k = [v_{k,1}, v_{k,2}, \ldots, v_{k,L}]$ of user $k \in \{1, 2, \ldots, K\}$ is spread by $N$ chips $s_{k,l,1}, s_{k,l,2}, \ldots, s_{k,l,N}$ chosen randomly and uniformly from the set $\{-1/\sqrt{N}, 1/\sqrt{N}\}$. The duration of a symbol is $T$ and $p(t)$ is the unit-energy chip pulse. The chip duration is $T_c = T/N$, and we allow the system to be asynchronous with $\tau_k < T_c$.

The $N$ chips spreading one code bit $v_{k,l}$ are divided into $M$ equal size sections which are randomly interleaved over one or more frames using interleave function $\pi_{k,l}(m), m = 1, 2, \ldots, M$. Denoting user’s power by $P_k$ we can express the $k$-th user transmit signal as

$$ x_k(t) = \sum_{l=1}^{L} \sqrt{P_k} v_{k,l} \cdot \sum_{m=1}^{M} c_{k,l,m} \left( t - (l-1)T - \tau_k - \pi_{k,l}(m) \frac{T}{M} \right) $$

where

$$ c_{k,l,m}(t) = \sum_{n=1}^{N/M} s_{k,l,n-1+m/\pi_{k,l}T} \cdot p(t - (n-1)T_c) $$

is the $m$-th section of the spreading waveform for bit $v_{k,l}$. The combined signal from all users is transmitted over an additive white Gaussian channel, giving the received signal

$$ y(t) = \sum_{k=1}^{K} x_k(t) + n(t), \quad (3) $$

where $n(t)$ is the Gaussian noise process with double-sided power spectral density $\sigma^2 = N_0/2$.

### III. Two-Stage Decoding Schedule

The received signal is passed through a bank of filters, matched to the normalized partition signals $\sqrt{M} c_{k,l,m}(t), m = 1, 2, \ldots, M$ of users $k = 1, 2, \ldots, K$. In general more complex, but efficient filters can also be used, however, we consider matched filters for simplicity. The soft channel values are then forwarded to the iterative detector. The received value corresponding to $m$-th partition of bit $v_{k,l}$ after matched filtering is

$$ z_{k,l,m} = \sqrt{\frac{P_k}{M}} v_{k,l} + I_{k,l,m} + \eta_{k,l,m} \quad \text{(4)} $$

where $I_{k,l,m}$ denotes interference and $\eta_{k,l,m}$ the noise value. The log-likelihood ratio (LLR) of $v_{k,l}$ from the observations

$$ \ln \left( \frac{P(v_{k,l} = 1|z_{k,l,1}, \ldots, z_{k,l,M})}{P(v_{k,l} = -1|z_{k,l,1}, \ldots, z_{k,l,M})} \right) = \frac{2}{\sigma_0^2} \sum_{m=1}^{M} z_{k,l,m} \quad \text{(5)} $$

where $\sigma_0^2$ is the variance of $I_{k,l,m} + \eta_{k,l,m}$, which is shown in [10] to be Gaussian. With this assumption soft bit estimates created from signal of all $M$ partitions are

$$ \tilde{v}_{k,l,m} = \tanh \left( \frac{1}{\sigma_0^2} \sum_{m' \neq m} z_{k,l,m'} \right). \quad \text{(6)} $$

Values $\tilde{v}_{k,l,m}$ and $z_{k,l,m}$ can be updated iteratively.

The two-stage schedule separates the operations of PS CDMA detection and the decoding of the LDPC code into two independent processes. The convergence of the LDPC decoder is determined by the output SNR of the detection stage. This schedule operates according to the following algorithm.

In the first stage

- perform $i = 1, \ldots, I_1$ iterations with PS detector:
  - equality node: calculate soft bit estimates using
    $$ \tilde{v}_{k,l,m}(i-1) = \tanh \left( \frac{1}{\sigma_0^2} \sum_{m' \neq m} (z_{k,l,m'}(i-1)) \right) $$
    for $k = 1, \ldots, K, \ l = 1, \ldots, L, \ m = 1, \ldots, M$.
  - partition node: calculate partially canceled signals, $k = 1, \ldots, K, \ l = 1, \ldots, L, \ m = 1, \ldots, M$ as
    $$ y^{(i)}_k(t) = y(t) - \sum_{k' \neq k} \tilde{x}_{k'}(t) = x_{k}(t) + \sum_{k' \neq k} \sqrt{P_{k'}} \sum_{l=1}^{L} (v_{k',l} - \tilde{v}_{k',l,m}(i)) \cdot c_{k',l,m} \left( t - (l-1)T - \tau_{k'} - \pi_{k',l}(m) \frac{T}{M} \right) + n(t) $$
  - equality node: form the input LLRs to the LDPC decoder as
    $$ x_{k,l,m}(I_1) = \frac{2}{\sigma_1^2} \sum_{m=1}^{M} z_{k,l,m}(I_1) $$. 
    for $k = 1, \ldots, K, \ l = 1, \ldots, L$.
The two-stage decoding procedure for a particular user \(k\) is explained graphically in Figure 1. The PS part is connected to the LDPC code portion of the factor graph by equality nodes indicated by squares. These nodes act as an interface and illustrated by grey circles. The PS part is connected to the MA channel and the LDPC code. During the first, detection, stage the LLR messages are exchanged in iterative manner between the partition nodes and the equality nodes. After that the input LLRs \(\lambda_{k,l}^{\text{in}}\) are passed to the LDPC decoder block which outputs the final estimates \(\lambda_{k,l}^{\text{out}}\) after \(I_2\) decoding iterations.

Combined noise and interference variance in (4) from all the partitions at iteration \(i\) of the first stage is given by

\[
\sigma_i^2 = \frac{1}{N} \sum_{k=1}^{K} P_k \sigma_{2,i,k}^2 + \sigma^2
\]

for sufficiently large \(K\) and \(N\). The quantity \(\sigma_{2,i,k}^2 = \mathbb{E}[ (v_{k,l} - \tilde{v}_{k,l,m}^{(i)})^2] = \mathbb{E}[ (v_{k,l} - \hat{v}_{k,l,m}^{(i)})^2]\) is the mean squared estimation error of the coded symbol \(v_{k,l}\) called soft bit variance. The expectation generally does not depend on particular \(m\) and \(l\), which can be omitted. The soft bit estimate itself is

\[
\tilde{v}_{k,l,m}^{(i)} = \text{tanh} \left( \frac{1}{\sigma_i^2} \sum_{m' \neq m} z_{k,l,m'} (i-1) \right)
\]

\[
\hat{v}_{k,l,m}^{(i)} = \text{tanh} \left( \frac{(M-1)P_k}{M\sigma_i^{-1}} + \frac{\xi}{\sqrt{\frac{(M-1)P_k}{M\sigma_i^{-1}}}} \right)
\]

where \(\xi \sim \mathcal{N}(0,1)\) is standard Gaussian. Without loss of generality consider \(v_{k,l} = 1\). It is known \[12\] that

\[
\mathbb{E}[ (1 - \text{tanh}(x + \xi \sqrt{\sigma}))^2] \leq g(x) = \begin{cases} \frac{1}{x^2}, & x < 1 \\ \pi Q(\sqrt{\sigma}), & x \geq 1 \end{cases}
\]

and the upper bound is very tight. On the other hand, the closed form expression for the expectation is not known. Setting \(x = \frac{(M-1)P_k}{M\sigma_i^{-1}}\) and using (9) instead of \(\sigma_{2,i,k}^2\) we get

\[
\sigma_i^2 = \frac{1}{N} \sum_{k=1}^{K} P_k g \left( \frac{(M-1)P_k}{M\sigma_i^{-1}} \right) + \sigma^2.
\]

The variance \(\sigma_i^2\) decreases monotonically with iterations \(i\) and has a fixed point \(\sigma_\infty^2\). With the assumption that system is equal power (\(P = P_k = 1\)), equal rate users and the number of partitions \(M\) as well as spreading gain \(N\) tend to infinity the equation (10) becomes

\[
\sigma_\infty^2 = \alpha g \left( \frac{1}{\sigma_\infty^2} \right) + \sigma^2.
\]

After small modifications (11) can be expressed in terms of output signal to noise plus interference ratio (SINR) value \(\gamma = 1/\sigma_\infty^2\) produced by the PS detector as

\[
\gamma = \left[ \sigma^2 + \alpha \mathbb{E}[ 1 - \text{tanh}(\gamma + \sqrt{\gamma} \xi) ] \right]^{-1}
\]

where \(\alpha = \frac{K}{N}\) is system load. For system loads \(\alpha < \alpha_s \approx 1.49\) equation (12) describes the signal-to-noise ratio of individually optimal APP detector derived using statistical mechanics approach \[9\]. Similar equations appear in \[11\](Eq.(2,3,18)) where authors show a link between optimal APP detection and belief propagation\[2\]. Therefore, for a range of loads PS CDMA is optimal in equal power, equal rate configurations and moreover it is of considerably lower complexity (simple soft information exchange) than the APP detector.

The variance \(\sigma_i^2\) is monotonically decreasing with \(i\) \[12\]. For user \(k\) the decoding in the second stage is successful (asymptotically) if and only if

\[
P_k \geq 2R\gamma_{\text{code}}
\]

where \(\gamma_{\text{code}}\) is the code’s convergence threshold signal-to-noise ratio (in \(E_b/N_0\)), \(R\) is the code rate. If the optimal,
“Shannon code” is used instead of the LDPC, the condition for the asymptotically error free communication is

$$C_B \left( \frac{P_k}{\sigma_i^2} \right) > R$$

(14)

where \(C_B(\cdot)\) is the capacity of binary input AWGN channel as a function of signal-to-noise ratio. Figure 2 presents the maximum achievable system loads \(\alpha\) as a function of employed code rate \(R\). Equal power users are considered \(P = P_k\). The asymptotic two-stage decoding performance for LDPC coded PS CDMA is given by the dashed line. We assume that regular (3, x) LDPC codes are used. The dash-dotted line indicates the performance of regular LDPC coded CDMA with MMSE detector. Solid line indicates performance of full decoding schedule, discussed in Section IV. It can be observed that both, the two-stage schedule and the full decoding outperform the traditional system for code rates \(R \geq 0.5\). For \(R < 0.5\) the PS detector (with infinite number of iterations) behaves almost identically to the MMSE filter, since for \(\frac{(M-1)P}{\sigma_i^2} < 1\) the first equality in (9) is biting. The resulting variance formula (for \(I_1 = \infty\)) (using \(g(x)\) upper bound) is

$$\sigma_i^2 = \sigma_{\text{MMSE}}^2 = \frac{\alpha P}{1 + \frac{(M-1)P}{\sigma_i^2}} + \sigma^2$$

(15)

i.e. asymptotically \((M \to \infty)\) identical to that of the linear MMSE filter [13]. However the actual PS performance is slightly better since \(g(x)\) is an upper bound. It can be concluded that simple Partitioned Spreading detection is at least as effective as far more complex (especially for large number of users \(K\)) MMSE filtering. The dotted lines correspond to the PS CDMA with two-stage decoding and the regular CDMA both concatenated with optimal codes. Additionally, it is seen that the full decoding schedule achieves the best overall performance, which is not of surprise. What is interesting however, is that the two PS CDMA decoding schedules perform similarly for high rate codes i.e. for \(R > 0.75\). This key observation works in favor of the two-stage schedule, which is of lower complexity than the full decoding approach.

From (7) and (9), for \(M \to \infty\), follows

$$\sigma_i^2 = \alpha \sigma_{d,i,k}^2 + \sigma^2$$

$$\frac{P_k}{g^{-1}(\sigma_{d,i,k}^2)} \leq \sigma_i^2$$

(16)

for the detection iterations \(i = 1, 2, \ldots\), where \(\alpha = K/N\) is the system load. Without loss of generality, we set \(P_k = 1\). The LDPC decoder (second stage) convergence requirement is given by (13). It implies that \(\sigma_i^2 < \sigma_{i-1}^2\) should be satisfied for \(i = 1, 2, \ldots\) until finally \(\sigma_\infty^2 \leq 1/(2R\gamma_{\text{code}})\) is reached. Let us denote the soft bit variance \(\sigma_{d,i,k}^2\) by \(x\). This convergence condition is equivalent to

$$\alpha x + \sigma^2 < \frac{1}{g^{-1}(x)}$$

(17)

for \(x \in [g(2R\gamma_{\text{code}}), 1]\). The soft-bit variance will decrease from 1 to \(g(2R\gamma_{\text{code}})\) during detection iterations. The minimum receive signal-to-noise ratio \(E_b/N_0 = 1/(2R\sigma^2)\) required for the two-stage schedule to accommodate load \(\alpha\) is given by Lemma 1. The following auxiliary definitions
will be required. Let us denote $\alpha$ such that
\[ \alpha x = \frac{1}{g^{-1}(x)} \] (18)
has exactly one non-zero root, by $\alpha_{th}$. Let the corresponding root be $x_{th}$. Notice that it is equivalent to the condition
\[ \alpha_{th} = \min_{x \in [0,1]} \frac{1}{xg^{-1}(x)} = \max_{y \in (0, \infty)} yg(y) \] (19)
Numerically calculated $\alpha_{th} \approx 1.92$, $x_{th} \approx 0.37$, $y_{th} = g^{-1}(x_{th}) \approx 1.41$.

Lemma 1: For codes such that $R_{\gamma_{\text{code}}} \leq g^{-1}(x_{th})/2 \approx 0.7$ (low rate codes, $R \leq 0.5$ for $(3,x)$ family) the two-stage converges for $\alpha \leq 1/(2R_{\gamma_{\text{code}}}g(2R_{\gamma_{\text{code}}}))$. The smallest SNR required for convergence is
\[ \text{snr}_{\min} = \frac{\gamma_{\text{code}}}{1 - 2\alpha_{\gamma_{\text{code}}}R_{\gamma}(2R_{\gamma_{\text{code}}})} . \] (20)
For codes such that $R_{\gamma_{\text{code}}} > g^{-1}(x_{th})/2 \approx 0.7$ (high rate codes, $R > 0.5$ for $(3,x)$ family) the two-stage converges for $\alpha \leq \alpha_{th} \approx 1.92$. Now $\text{snr}_{\min}$ is the infimum over snr such that
\[ \frac{1}{g^{-1}(x)} = \alpha x + \frac{1}{2R\text{snr}} \] (21)
has exactly one positive root.

Proof: Straightforward from graphical analysis of variance transfer using (17).

A. Unequal Power Distributions

The analysis in the previous section and the corresponding derivations were obtained for the case of equal user received powers. It is clear that such power control requires high accuracy, may be complex and is often undesirable. In this section we will study optimal power profile for our LDPC coded Partitioned Spreading system if two-stage decoding is used. It is known that for the traditional LDPC coded CDMA system the asymptotically optimal power profile is exponential [12], i.e. $\frac{\lambda_{k,i}}{R_{\gamma_{\text{code}}}} = \text{const.}$ for $k \in \{2, \ldots, K\}$. The exponential distribution is very difficult for practical use due to the large power differences between the users. Nevertheless, for the coded PS CDMA the optimal profile is different. We assume that users form a number of power level groups and optimize the group number as well as the group powers in similar manner as proposed for a CDMA system with convolutional codes [15]. The results of numerical optimization are presented in Figure 3. They indicate that the optimum (requiring the least average amount of power) user power distribution is changing with the system load. We study the distributions for the case of $(3,4)$ rate $R = 0.25$ and $(3,30)$ rate $R = 0.9$ LDPC codes.

In the low load region the uniform power distribution is optimal. There is only one user power group. As the load increases it switches to bimodal and later the number of required power groups increases further. The distribution switching points are upper bounded by the asymptotes shown in the figure which are characterized by particular load values. These values describe the asymptotic performance of the system for a given number of groups which in order to be improved needs to accommodate more users and have more power. The description of boundary asymptotes, given by the following lemmas, depends on the LDPC code employed. We study PS CDMA with $M \to \infty$ and concatenated with LDPC codes of rate $R$ with threshold $\gamma_{\text{code}}$.

Lemma 2: For $R$ and $\gamma_{\text{code}}$ such that $R_{\gamma_{\text{code}}} > \frac{\alpha_{\gamma_{\text{code}}}}{2}$ (high rate codes, $R > 0.5$) the maximum load achievable with $J$ user power groups and two-stage decoding is upper bounded by
\[ \alpha_{\max} = \frac{J}{y_{th}g(y_{th})} = J\alpha_{th} \approx 1.92J . \] (22)
Moreover, for any $\varepsilon > 0$ load $\alpha_{\max} - \varepsilon$ can be achieved.

Lemma 3: For $R$ and $\gamma_{\text{code}}$ such that $R_{\gamma_{\text{code}}} \leq \frac{\alpha_{\gamma_{\text{code}}}}{2}$ (low rate codes, $R \leq 0.5$ for $(3,x)$ family) the maximum load achievable with $J$ user power groups and two-stage decoding is upper bounded by
\[ \alpha_{\max} \leq \frac{J}{2R_{\gamma_{\text{code}}}g(2R_{\gamma_{\text{code}}})} . \] (23)

Proof: See Appendix.

From these lemmas we can conclude that the capabilities of the two-stage schedule grow linearly with number of power groups $J$. The scaling factor is determined by the values $\alpha_{\max}$ and is equal to 1.92 for $(3,4)$ code and 2.57 for $(3,30)$code. Thus, increasing the number of groups is the necessary strategy for achieving very high system loads. However, it is done at the expense of rapid increase of power.

IV. FULL DECODING SCHEDULE

The full schedule involves message exchange between all the nodes in the system’s factor graph. On the other hand the full decoding is a repetition of the two-stage processing.

The full decoding consists of the following steps graphically illustrated in the Figure 4.

- Process $i = 1, 2, \ldots, I$ iterations
  - perform $I_1$ iterations in the PS detector with the a priori “information” $\lambda^{\text{out}}_{k,l}(i-1)$
  - pass the resulting LLRs as the input $\lambda^{\text{in}}_{k,l}(i)$ to the LDPC decoders, $k = 1, \ldots, K$
  - perform $I_2$ LDPC decoding iterations
  - pass the output $\lambda^{\text{out}}_{k,l}(i)$ to the PS detector, go to the first step
- output final LLRs $\lambda^{\text{out}}_{k,l}(I)$, for $k = 1, \ldots, K$, $l = 1, \ldots, L$ and make the hard decisions
  \[ \tilde{v}^{\text{full}}_{k,l} = \text{sign} (\lambda^{\text{out}}_{k,l}(I)) . \]
Fig. 3. Average signal-to-noise ratio required to achieve given load $\alpha$ using two-stage schedule with unequal user power levels.

In case of full decoding with equal received power levels the soft bit expression at iteration $i$ becomes

$$\sigma^2_{d,i,k} = E [(v_k - \tilde{v}_k,i)^2] \leq g \left( \frac{M - 1}{M} \frac{P_k}{\sigma^2_{i-1}} + \frac{d_k}{2} m_{c,i} \left( \frac{P_k}{\sigma^2_{i-1}} \right) \right)$$

where $m_{c,i}$ is the mean of the messages output by LDPC check nodes on the final $I_2$-th LDPC decoding iteration and $i$-th overall iteration.

Table I shows the minimum required SNR for both schedules to differ in performance by small constant $\epsilon$. For example for $R = 0.8$ signal to noise ratio $E_b/N_0 \geq 6.5$dB guarantees that the two-stage schedule performance is in $\epsilon$ neighborhood of more complex full decoding scheme.

Finally, Figure 5 demonstrates loads achievable by different CDMA systems concatenated with rate $R = 1/2$ (4, 8) LDPC code and for different decoding schedules. The solid curves show performance of the full iterative exchange between decoder and detector. The dashed curves show the performance of the two-stage schedule. Each group of curves contains coded PS CDMA with $M = \infty$, PS CDMA with $M = 20$ and MMSE$^3$ detected LDPC decoded CDMA. It can

3Here, for the full schedule, we assume that iterative realization of MMSE filter using first order stationary method is used [3], [17]. Therefore the filter can accept soft values coming from the LDPC code.

Table I

<table>
<thead>
<tr>
<th>Rate $R$</th>
<th>Minimum $E_b/N_0$ [dB]</th>
<th>$\epsilon$</th>
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<tr>
<td>0.75</td>
<td>9.0</td>
<td>0.1</td>
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<tr>
<td>0.80</td>
<td>6.5</td>
<td>0.1</td>
</tr>
<tr>
<td>0.85</td>
<td>5.75</td>
<td>0.1</td>
</tr>
<tr>
<td>0.90</td>
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<td>0.1</td>
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</table>
be seen that PS CDMA outperforms conventional CDMA. Moreover, $M = 20$ gives performance, sufficiently close to the infinite case.

V. CONCLUSIONS

LDPC coded Partitioned Spreading CDMA system is analyzed and shown to be significantly more efficient than conventional coded CDMA. Two schedules of iterative multiuser detection and decoding were considered: the two-stage schedule where the iterative detection of Partitioned Spreading signals precedes the LDPC iterative decoding and the full schedule when the detector and decoder exchange soft information throughout the iterations. For both methods the asymptotically supportable system loads are derived as functions of the signal-to-noise ratio for the case of equal power users. It is shown that the system can accommodate loads which are significantly larger than those supported by conventional coded spread spectrum systems even if the best possible linear filter is used for iterative detection and the code is chosen in the optimal way. These gains are especially visible when high rate LDPC codes are applied. It is important that despite the complexity difference of the two considered schedules their performance for certain conditions (employing higher rate regular LDPC codes) is nearly identical. The advantage of unequal received power level was also observed. For the two-stage decoding multi modal power distribution allows system to achieve higher supportable loads and spectral efficiencies.

Appendix

Proof of Lemma 2 and Lemma 3

Upper Bound on the achievable load

We will prove the upper bound by contradiction. Assume that $\alpha$ greater that stated in (22) and respectively (23) can be accommodated. We consider $\sigma^2 = 0$, this may only increase the load. There are $J$ groups of users with loads $\alpha_1, \alpha_2, \ldots, \alpha_J$ and powers $P_1 \leq P_2 \leq \ldots \leq P_J$. Variance evolution during the detection evolution is given by

$$
\begin{align*}
\sigma_0^2 &= \sum_{j=1}^{J} \alpha_j \sigma_j^2 \\
\sigma_1^2 &= \sum_{j=1}^{J} \alpha_j \sigma_j g \left( \frac{P_j}{\sigma_0^2} \right).
\end{align*}
$$

(25)

Convergence condition for two-stage schedule $\sigma_1^2 \leq P_1/(2\gamma_{\text{code}})$ together with (25) can be equivalently written as

$$
x > \sum_{j=1}^{J} \alpha_j P_j g \left( \frac{P_j}{x} \right) \quad \text{for } x \in \left[ \frac{P_1}{2\gamma_{\text{code}}}, \sum_{j=1}^{J} \alpha_j P_j \right].
$$

(26)

We observe that the interval on the right hand side is nonempty for $\alpha$’s we consider. Let us define $j = \arg \max_j \alpha_j$, $y = P_j/x$. We notice that $\alpha_j \geq \alpha/J$. From (26) follows

$$
1 > \sum_{j=1}^{J} \alpha_j P_j x g \left( \frac{P_j}{x} \right) \geq \frac{\alpha}{J} y g(y)
$$

for $y \in \left[ \frac{P_j}{\sum_{j=1}^{J} \alpha_j P_j}, 2\gamma_{\text{code}} \frac{P_j}{\sigma_1^2} \right].
$$

(27)

A necessary condition for the convergence would be

$$
\alpha < \frac{J}{y} \max_{y \in \left[ \frac{P_j}{\sum_{j=1}^{J} \alpha_j P_j}, 2\gamma_{\text{code}} \frac{P_j}{\sigma_1^2} \right]} y g(y)
$$

$$
= \left\{ \begin{array}{ll}
\frac{J}{y_{\text{th}} g(y_{\text{th}})} & \text{for } y_{\text{th}} < 2\gamma_{\text{code}} \\
2\gamma_{\text{code}} y_{\text{th}} & \text{for } y_{\text{th}} \geq 2\gamma_{\text{code}}.
\end{array} \right.
$$

(28)

In the first case $y_{\text{th}}$, the point of maximum for $yg(y)$, belongs to the maximization interval in (28). In the second case maximum is reached on the boundary $y = 2\gamma_{\text{code}} P_j/\sigma_1^2$ and is minimized (i.e. $\alpha$ is maximized) for $j = 1$. Thus $\alpha$ satisfies (28) and the initial assumption that $\alpha$ might be greater is invalid.

Achievable load

Consider equal group loads $\alpha_j = \alpha/J$, $j = 1, 2, \ldots, J$ and an exponential power distribution $P_j = Pa^{j-1}$, $j = 1, 2, \ldots, J$, $a > 1$. Then for $\sigma^2 \neq 0$ the convergence condition from (26) can be represented as

$$
\alpha < J \left( 1 - \frac{\sigma^2}{x} \right) \left( \sum_{j=1}^{J} \frac{P_j}{x} g \left( \frac{P_j}{x} \right) \right) \quad \text{for } x \in \left[ \frac{P_1}{2\gamma_{\text{code}}}, \sum_{j=1}^{J} \alpha_j P_j + \sigma^2 \right].
$$

(29)

We are given arbitrary $\varepsilon > 0$. Consider additionally $\delta_1, \delta_2 > 0$.

Since $\lim_{y \to 0} yg(y) = \lim_{y \to \infty} yg(y) = 0$ for any $\delta_1 > 0$ we can find $b_1(\delta_1)$ and $b_2(\delta_1)$ such that $yg(y) < \delta_1$ for $y \in [0, b_1(\delta_1)] \cup [b_2(\delta_1), \infty)$.

Choose $a = (1/\sqrt{b_1(\delta_1)})^2$, $b_2(\delta_1)$ and define $x_j = Pa^{j-1/2}$, $j = 0, 1, \ldots, J$. Then for $x \in [x_{j-1}, x_j]$, $j = 1, 2, \ldots, J$

$$
\frac{P_j}{x} g \left( \frac{P_j}{x} \right) < \left\{ \begin{array}{ll}
\delta_1 & \text{for } j' \neq j \quad \text{and } y_{\text{th}} g(y_{\text{th}}) \text{ for } j' = j
\end{array} \right.
$$

(30)

For $x < x_0$ or $x > x_J$

$$
\frac{P_j}{x} g \left( \frac{P_j}{x} \right) < \left\{ (J-1)\delta_1 + y_{\text{th}} g(y_{\text{th}}) \right. \quad \text{for } x < x_0
$$

Therefore for any $x \in [0, \infty)$

$$
\sum_{j=1}^{J} \frac{P_j}{x} g \left( \frac{P_j}{x} \right) < (J-1)\delta_1 + y_{\text{th}} g(y_{\text{th}})
$$

(32)
Choose \( P > 2R_{\text{code}} \sigma^2 / \delta_2 \), then \( \sigma^2 / x < \delta_2 \) for \( x \) the in the interval given in (29). Together with (32) this leads to

\[
\frac{J(1 - \delta_2)}{(J - 1) \delta_1 + y_\text{th} \delta(y_\text{th})} < J \left( 1 - \frac{\sigma^2}{x} \right) \left( \sum_{j=1}^{J} P_j x - \frac{P_j}{x} \right) \]

for \( x \in \left[ \frac{P}{2R_{\text{code}}} - \sum_{j=1}^{J} \alpha_j P_j + \sigma^2 \right]. \)

(33)

Now we can choose \( \delta_1 \), \( \delta_2 \) (and related \( a \) and \( P \)) small enough to satisfy

\[
\alpha_{\text{max}} - \varepsilon < \frac{J(1 - \delta_2)}{(J - 1) \delta_1 + y_\text{th} \delta(y_\text{th})}.
\]

(34)

Therefore \( \alpha_{\text{max}} - \varepsilon \) is achievable. Note that the exponential distribution was used in the proof for simplification and is, in most cases, not optimal.

REFERENCES


