## THE GENERAL QUASILINEAR ULTRAHYPERBOLIC SCHRÖDINGER EQUATION

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#### 1. INTRODUCTION

In this article we consider nonlinear Schrödinger equations of the form

(1.1) 
$$\begin{cases} \partial_t u = -i \,\partial_{x_j}(a_{jk}(x,t,u,\bar{u},\nabla u,\nabla \bar{u})\partial_{x_k}u) \\ + \vec{b}_1(x,t,u,\bar{u},\nabla u,\nabla \bar{u})\cdot\nabla u + \vec{b}_2(x,t,u,\bar{u},\nabla u,\nabla \bar{u})\cdot\nabla \bar{u} \\ + c_1(x,t,u,\bar{u})u + c_2(x,t,u,\bar{u})\bar{u} + f(x,t), \end{cases}$$

where  $x \in \mathbb{R}^n$ , t > 0, and  $A = (a_{jk}(\cdot))_{j,k=1,\dots,n}$  is a real, symmetric matrix.

Our aim is to study the existence, uniqueness and regularity of local solutions to the initial value problem (IVP) associated to the equation (1.1).

In the case where  $A = (a_{jk}(\cdot))_{j,k=1,..,n}$  is assumed to be elliptic the local solvability of the IVP associated to (1.1) was recently established in [20]. Hence, in this work we should be concerned with the case where  $(a_{jk}(\cdot))_{j,k=1,..,n}$  is just a non-degenerate matrix.

Equations of the form described in (1.1) with  $A = (a_{jk}(\cdot))_{j,k=1,..,n}$  merely invertible arise in water wave problems, and as higher dimensions completely integrable models, see for example [1], [7], [8], [15], [27], and [30].

There are significant differences in the arguments required for the local solvability in the case where A is a non-degenerate matrix in comparison with the elliptic case treated in [20]. To illustrate them as well as to review some of the previous related results we consider first the semi-linear equation with constant coefficients (for more details and further references and comments see [19], [20], [21], and references therein)

(1.2) 
$$\partial_t u = -i(\partial_{x_1}^2 + ... + \partial_{x_k}^2 - \partial_{x_{k+1}}^2 - ... - \partial_{x_n}^2)u + P(u, \bar{u}, \nabla_x u, \nabla_x \bar{u}),$$

where  $P(\cdot)$  is the non-linearity (by simplicity a polynomial in its variables without constant or linear terms).

In [18] based on the smoothing effects (homogeneous and inhomogeneous, see [16], [22], [4], [25], [29], [17]) associated to the group  $\{e^{-it(\partial_{x_1}^2+..+\partial_{x_k}^2-\partial_{x_{k+1}}^2-..-\partial_{x_n}^2): t \in \mathbb{R}\}$  the local wellposedness for "small" data for the IVP associated to (1.2) was deduced. In [12] for the one dimensional case (n = 1), Hayashi and Ozawa eliminated the size restriction on the data in [18]. Their argument was based on a change of variable which transforms the equation into a new system without the term  $\partial_x u$ , so that the standard energy estimate yields the desired local result. In [5] Chihara, for the elliptic case (i.e. k = n in (1.2)), removed the size restriction on the data in any dimension n. Roughly speaking, the argument there first uses the ellipticity to diagonalize the system for  $(u, \bar{u})$ , and then introduced an operator K so that the commutator  $i[K; \Delta]$  "controls" the term  $K\vec{b}(x) \cdot \nabla_x$ . This is achieved by combining some result of Doi [9] concerning the local smoothing effects in the solution with the sharp version of Garding inequality.

If instead of "controlling" it one asks for the operator K to verify that

(1.3) 
$$-i[K;\Delta] + K\vec{b}(x) \cdot \nabla_x = 0 + \text{ order zero}$$

one finds that K has symbol

(1.4) 
$$k(x,\xi) = exp(-\int_0^\infty \vec{b}(x+2s\xi)\cdot\xi ds),$$

which is in the non-standard class studied by Craig-Kappeler-Strauss [6]. In particular, it satisfies that

$$|\partial_x^{\alpha}\partial_{\xi}^{\beta}k(x,\xi)| \le c_{\alpha\beta}\langle x \rangle^{|\beta|} \langle \xi \rangle^{-|\beta|}, \qquad \alpha, \, \beta \in (\mathbb{Z}^+)^n.$$

However, in the non-elliptic case with coefficients depending just on the space variable x, the geometric assumptions in [6] (Chapter 3.1, section 3.1) does not hold for the relevant symbols.

The local wellposedness of the IVP associated to the equation in (1.2)  $(1 < k \leq n)$  was established in [19]. The method of proof there, among other arguments, utilizes the symbol class  $S_{0,0}^0$  of Calderón-Vaillancourt [3]. However, this approach does not seem to extend to the variable coefficients case.

Next, we consider the linear IVP

(1.5) 
$$\begin{cases} \partial_t u = -i\partial_{x_k}a_{jk}(x)\partial_{x_j}u + \vec{b}_1(x)\cdot\nabla u + \vec{b}_2(x)\cdot\nabla\overline{u} + f(x,t), \\ u(x,0) = u_0(x). \end{cases}$$

We recall the notion of the bicharacteristic flow associated to the symbol of the principal part of the operator  $-\partial_{x_k}a_{jk}(x)\partial_{x_j}$ , i.e.  $h(x,\xi) = a_{jk}(x)\xi_k\xi_j$ .

Let  $(X(s; x_0, \xi_0), \Xi(s; x_0, \xi_0))$  denote the solution of the Hamiltonian system

(1.6) 
$$\begin{cases} \frac{d}{ds} X_j(s; x_0, \xi_0) = 2a_{jk}(X(s; x_0, \xi_0)) \Xi_k(s; x_0, \xi_0) = \partial_{\xi_j} h, \\ \frac{d}{ds} \Xi_j(s; x_0, \xi_0) = -\partial_{x_j} a_{lk}(X(s; x_0, \xi_0)) \Xi_k \Xi_l(s; x_0, \xi_0) = -\partial_{x_j} h, \end{cases}$$

for j = 1, ..., n, with data

$$(X(0; x_0, \xi_0), \Xi(0; x_0, \xi_0)) = (x_0, \xi_0).$$

Under mild regularity assumptions on the coefficients  $a_{jk}(x)$ 's the bicharacteristic flow  $(X(s; x_0, \xi_0), \Xi(s; x_0, \xi_0))$  is defined in the interval  $s \in (-\delta, \delta)$  with  $\delta = \delta(x_0, \xi_0) > 0$  depending continuously on  $(x_0, \xi_0)$ .

If the operator  $-a_{jk}(x)\partial_{x_kx_j}^2$  is elliptic, i.e.  $(a_{jk}(x))$  is positive definite, using that the flow preserves h, i.e.

$$H_h h = \sum_{j=1}^n \left( \partial_{\xi_j} h \partial_{x_j} p - \partial_{x_j} h \partial_{\xi_j} p \right) = \{h, p\} = 0,$$

one has that

$$\nu^{-2}|\xi_0|^2 \le |\Xi(s;x_0,\xi_0)|^2 \le \nu^2|\xi_0|^2.$$

Hence  $\delta = \infty$ , i.e. the bicharacteristic flow is globally defined. In the non-elliptic case one needs to prove it.

In [14] Ichinose established that for the  $L^2$ -local wellposedness of IVP (1.5), with  $-\partial_{x_k} a_{jk}(x) \partial_{x_j}$  elliptic and  $b_2(x) \equiv 0$ , it is necessary that

(1.7) 
$$\sup_{x \in \mathbb{R}^n, \omega \in S^{n-1}, R > 0} \left| Im \int_0^R \vec{b}_1(X(s; x, \omega)) \cdot \Xi(s; x, \omega) ds \right| < \infty.$$

This extends previous results of Mizohata [23] and Takeuchi [28] for the constant coefficient case, where  $(X(s; x_0, \xi_0), \Xi(s; x_0, \xi_0)) = (x_0 + 2s\xi_0, \xi_0)$ . Notice that in this variable coefficient case the "integrating factor" in (1.4) reads

(1.8) 
$$k(x,\xi) = \exp\left(-\int_0^\infty \vec{b}_1(X(s;x,\xi)) \cdot \Xi(s;x,\xi)ds\right).$$

The condition (1.7) justifies the following "non-trapping" assumption. The bicharacteristic flow (1.6) is non-trapped if the set

$$\{X(s; x_0, \xi_0) : s \in \mathbb{R}^+\}$$

is unbounded in  $\mathbb{R}^n$  for each  $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n - \{0\}$ .

As it was already mentioned the IVP for the equation (1.1) with  $(a_{jk}(\cdot))_{j,k=1,\dots,n}$  elliptic was studied in [20]. There the local solvability was obtained under regularity and decay assumptions on the coefficients. Also a non-trapping character of the bicharacteristic flow associated to the principal symbol of the elliptic operator

$$-\partial_{x_j}(a_{jk}(x,t,u,\bar{u},\nabla_x u,\nabla_x \bar{u})\partial_{x_k}$$

when evaluated at the data  $u(x, 0) = u_0(x)$  and at a time t = 0 was assumed, i.e. the bicharacteristic flow associated to the symbol

(1.9) 
$$h(u_0) = h_{u_0}(x,\xi) = a_{jk}(x,0,u_0,\bar{u}_0,\nabla_x u_0,\nabla_x \bar{u}_0)\xi_j\xi_k.$$

The proofs of the semi-linear results in [18], [19] follow a fixed point theorem (via contraction principle). This approach does not extend to the quasi-linear case, thus in [20] the proof used the so called "artificial viscosity" method.

We recall that one of the advantages of the contraction principle approach is that it also provides the continuity (in fact, the analyticity) of the solution upon the data.

Returning to the non-degenerate case, in [21] the following semi-linear IVP was studied

(1.10) 
$$\begin{cases} \partial_t u = -i\partial_{x_k}a_{jk}(x)\partial_{x_j}u + \vec{b}_1(x)\cdot\nabla u + \vec{b}_2(x)\cdot\nabla\overline{u} \\ + c_1(x)u + c_2(x)\overline{u} + P(u,\nabla u,\overline{u},\nabla\overline{u}), \\ u(x,0) = u_0(x), \end{cases}$$

where the non-linearity P is given by a polynomial without linear or constant terms. Under the following assumptions:

(a) (Non-degeneracy) There exists  $\nu \in (0, 1)$ 

$$\nu|\xi| \le |A(x)\xi| \le \nu^{-1}|\xi|, \qquad x, \xi \in \mathbb{R}^n,$$

where  $A(x) = (a_{jk}(x))_{j,k=1,..,n}$ .

(b) (Asymptotic Flatness) There exist  $c_0 > 0$  and  $N, \tilde{M} \in \mathbb{Z}^+$  large enough such that

(1.11) 
$$|\partial_x^{\alpha}(a_{jk}(x) - a_{jk}^0)| \le \frac{c_0}{\langle x \rangle^N}, \quad |\alpha| \le \tilde{M}, \quad j, k = 1, .., n,$$

where

(1.12) 
$$A_h = (a_{jk}^0)_{j,k=1,\dots,n} = \begin{pmatrix} I_{k\times k} & 0\\ 0 & -I_{(n-k)\times(n-k)} \end{pmatrix}$$

(c) (Non-trapping condition) The initial data  $u_0$  satisfies that the bicharacteristic flow associated to (1.9) is non-trapping.

(d) (Growth condition of the first order coefficients) There exist  $c_0 > 0$  and  $N, \tilde{M} \in \mathbb{Z}^+$  large enough such that

(1.13) 
$$|\partial_x^{\alpha} b_{j_k}(x)| \le \frac{c_0}{\langle x \rangle^N}, \quad |\alpha| \le \tilde{M}, \quad j = 1, 2, \quad k = 1, .., n.$$

(e) Regularity of the coefficients For  $J \in \mathbb{Z}^+$  sufficiently large

$$a_{jk}, b_{m_j}, c_m \in C_b^J(\mathbb{R}^n : \mathbb{C}), \quad j, k = 1, ..., n, \ m = 1, 2.$$

the following result was established in [21].

**Theorem 1.1.** [21] There exist s, N > 0, s > N, and  $\tilde{N} > 0$ , depending only on n, such that for  $u_0 \in H^s(\mathbb{R}^n) \cap L^2(\langle x \rangle^N dx)$  there exists  $T = T(||u_0||_{s,2}, ||\langle x \rangle^{N/2} u_0||_2) > 0$  such that the IVP (1.10) has a unique solution u defined in the time interval [0, T] satisfying

$$u \in C([0,T]: H^s(\mathbb{R}^n) \cap L^2(\langle x \rangle^N dx)) \equiv X_T^{s,N}.$$

Moreover,

$$\int_0^T \int |J^{s+1/2}u(x,t)|^2 \langle x \rangle^{-\tilde{N}} dx dt < \infty,$$

#### SCHRÖDINGER EQUATION

and for every  $u_0 \in H^s(\mathbb{R}^n) \cap L^2(\langle x \rangle^N dx)$  there exists a neighborhood  $\mathfrak{U}$  of  $u_0$  and a T' > 0 such that the map data  $\rightarrow$  solution of is continuous from  $\mathfrak{U}$  into  $X_{T'}^{s,N}$ .

Here  $\|\cdot\|_{s,2}$  denotes the norm in the Sobolev spaces  $H^s(\mathbb{R}^n)$ .

As in the previous semi-linear cases the proof in [21] was based on the contraction principle.

In this non-degenerate case the operator describing the "integrating factor" in (1.8) has not been shown to be an  $L^2$ -bounded operator. However, thanks to the local smoothing effects it suffices to solve (1.3) up to "small" first order term. This is achieved in [21] by introducing a new class of symbols.

**Definition 1.2.** (i) It will be said that  $a \in S(\mathbb{R}^n : S_{1,0}^m)$  (where  $S_{1,0}^m$  the class of  $\psi$ .d.o's of classical symbols of order m) if  $a \in C^{\infty}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$  and a satisfies

(1.14) 
$$|\langle z \rangle^{\mu} \partial_s^{\alpha} \partial_{\xi}^{\beta} \partial_{\xi}^{\gamma} a(s; x, \xi)| \le c_{\mu\alpha\beta\gamma} \langle \xi \rangle^{m-|\gamma|}, \quad \forall z, x, \xi \in (\mathbb{Z}^+)^n, \quad \forall \mu, \alpha, \beta, \gamma \in (\mathbb{Z}^+)^n.$$

(ii) For  $a \in S(\mathbb{R}^n : S_{1,0}^m)$  let

(1.15) 
$$b(x,\xi) = \chi(|\xi|)a(P(x,A_h\xi);x,\xi),$$

where  $P(y,z) = y - (y \cdot z)z/|z|^2$  for  $y, z \in \mathbb{R}^n$ ,  $z \neq 0$ , is the projection of y onto the hyperplane perpendicular to z,  $A_h$  as in (1.12), and  $\chi \in C^{\infty}(\mathbb{R}^n)$  with  $\chi(l) = 0$  for |l| < 1 and  $\chi(l) = 1$  for |l| > 2.

In fact, we showed in [21] that it suffices to have (1.14) for sufficiently large  $\mu$ ,  $\alpha$ ,  $\beta$ ,  $\gamma \in (\mathbb{Z})^n$ .

We observe that if  $a \in S(\mathbb{R}^n : S_{1,0}^m)$ , then

$$\partial_{\xi}^{\alpha}(\xi^{\beta}a(\cdot)) \in \mathbb{S}(\mathbb{R}^{n}: S_{1,0}^{k}), \ k = m + |\beta| - |\alpha|,$$

i.e. this class is closed with respect to differentiation and multiplication in the  $\xi$ -variable. We shall show below that this class is also closed with respect to differentiation in the *x*-variable.

Roughly speaking, for  $r \in \mathbb{Z}^+$  large enough

$$\langle x \rangle^{-r} \chi(|\xi|) a(P(x, A_h \xi); x, \xi) = \langle x \rangle^{-r} b(x, \xi),$$

is a symbol in the class  $S_{1,0}^m$ .

In [21], we deduce several properties of operators with symbol in our class. These include their continuity from  $H^m(\mathbb{R}^n)$  to  $L^2$  and their composition rules with classical differential operators  $P(x, \partial_x)$  with decaying coefficients, i.e. with  $P(x, \partial_x) = \phi_{\alpha}(x)\partial_x^{\alpha}$  with  $|x|^l \phi_{\alpha}$  bounded for  $l \in \mathbb{Z}$  sufficiently large.

The proof of the nonlinear results concerning the IVP (1.10) relies on two key linear estimates. The first one is concerned with the smoothing effect described for solutions of

the IVP (1.5) with  $(a_{ik}(x))$  being just an invertible matrix

(1.16) 
$$\int_{0}^{T} \int_{\mathbb{R}^{n}} |J^{s+1/2}u(x,t)|^{2} \langle x \rangle^{-\tilde{N}} dx dt$$
$$\leq c(1+T) \sup_{0 \leq t \leq T} \|u(t)\|_{s,2}^{2} + c \int_{0}^{T} \int_{\mathbb{R}^{n}} |f(x,t)|^{2} dx dt$$

or

(1.17) 
$$\int_{0}^{T} \int_{\mathbb{R}^{n}} |J^{s+1/2}u(x,t)|^{2} \langle x \rangle^{-\tilde{N}} dx dt \leq c(1+T) \sup_{0 \leq t \leq T} ||u(t)||_{s,2}^{2} + \int_{0}^{T} \int_{\mathbb{R}^{n}} |J^{s-1/2}f(x,t)|^{2} \langle x \rangle^{\tilde{N}} dx dt$$

for  $\tilde{N} > 1$ .

The second is related with the local wellposedness in  $L^2$  (and in  $H^s$ ) of the IVP (1.5). To establish it we followed an indirect approach. First, we truncated at infinity the operator  $\mathcal{L}(x) = -\partial_{x_k} a_{jk}(x) \partial_{x_j}$  using  $\theta \in C_0^{\infty}(\mathbb{R}^n)$  with  $\theta(x) = 1$ ,  $|x| \leq 1$ , and  $\theta(x) = 0$ ,  $|x| \geq 2$ . For R > 0 we define

(1.18) 
$$\mathcal{L}^{R}(x) = \theta(x/R)\mathcal{L}(x) + (1 - \theta(x/R))\mathcal{L}^{0},$$

where  $\mathcal{L}^0 = -a_{jk}^0 \partial_{x_j x_k}^2$ ,  $A_h = (a_{jk}^0)_{j,k=1,\dots,n}$  as in (1.12), with the decay assumption  $a_{jk}(x) - a_{jk}^0 \in \mathcal{S}(\mathbb{R}^n)$ ,  $j,k = 1,\dots,n$ , (the same proof worked if one just assumed that the corresponding estimate held for a sufficiently large number of semi-norms of  $\mathcal{S}(\mathbb{R}^n)$ ). Thus,

$$\mathcal{L}(x) = \mathcal{L}^R(x) + \mathcal{E}^R(x).$$

For R large enough we considered the bicharacteristic flow  $(X^R(s; x, \xi), \Xi^R(s; x, \xi))$  associated to the operator  $\mathcal{L}^R(x)$  and the corresponding integrating factor  $K^R$ , i.e. the operator with symbol as in (1.8) but evaluated in the bicharacteristic flow  $(X^R(s; x, \xi), \Xi^R(s; x, \xi))$ . We deduced several estimates concerning the operator  $K^R$ . In particular, we showed in [21] that there exists  $N_0 \in \mathbb{Z}^+$  (depending only on the dimension n) such that for any  $M \in \mathbb{Z}^+$  there exist  $N_1 = N_1(M) \in \mathbb{Z}^+$  and  $R_0$  sufficiently large such that for  $R \ge R_0$  it follows that

(1.19) 
$$\sup_{0 \le t \le T} \|K^R u(t)\|_2^2 \le c_0 R^{N_0} \|u(0)\|_2^2 + R^{-M} \int_0^T \int_{\mathbb{R}^n} |J^{1/2} u|^2 \langle x \rangle^{-\tilde{N}} dx dt + c_0 T R^{N_0 + N_1(M)} \sup_{0 \le t \le T} \|u(t)\|_2^2$$

for  $\tilde{N}$  large enough depending only on the dimension n. In [21] the coefficients in (1.10) were taken in the Schwartz class  $S(\mathbb{R}^n)$ . However, it is clear from the proof there that the same argument works if one just assumes that a fixed large number of semi-norms of  $S(\mathbb{R}^n)$  of the coefficients are bounded. In this case, M will be chosen depending only

on the decay of the coefficients. More precisely, one chooses M = M(N),  $M(N) \uparrow \infty$  as  $N \uparrow \infty$ , with N as in (1.11), (1.13).

To complete the estimate one needs to consider the operator  $E^R = I - \tilde{K}^R (K^R)^*$ , where the symbol of  $\tilde{K}^R$  differs from that of  $K^R$  only in the sign of the exponent, and  $(K^R)^*$  is the adjoint of  $K^R$ . It was established that  $E^R u(t)$  satisfies an estimate similar to that in (1.16). Combining these results we get that

(1.20)  
$$\begin{aligned} \sup_{0 \le t \le T} \|u(t)\|_{2}^{2} \le c_{0} R^{N_{0}} \|u(0)\|_{2}^{2} \\ &+ R^{-M} \int_{0}^{T} \int_{\mathbb{R}^{n}} |J^{1/2}u|^{2} \langle x \rangle^{-\tilde{N}} dx dt + c_{0} T R^{N_{0}+N_{1}(M)} \sup_{0 \le t \le T} \|u(t)\|_{2}^{2} \\ &+ c_{0} R^{N_{0}} \int_{0}^{T} \int_{\mathbb{R}^{n}} |f(x,t)|^{2} dx dt. \end{aligned}$$

From (1.16) and (1.20) fixing T small enough one gets that

(1.21) 
$$\sup_{0 \le t \le T} \|u(t)\|_2^2 \le c_0 R^{N_0} \|u(0)\|_2^2 + c_0 R^{N_0} \int_0^T \int_{\mathbb{R}^n} |f(x,t)|^2 dx dt.$$

This allows to use the contraction principle to obtain in [21] the desired nonlinear result. Returning to the IVP (1.1) we shall assume that the coefficients satisfy the following hypotheses:

(H1) <u>Non-degeneracy</u>. Given  $r_0 > 0$  there exists  $\gamma_{r_0} \in (0, 1)$  such that for any

(1.22) 
$$(x,t,\vec{z}) \in \mathbb{R}^n \times \mathbb{R} \times (\overline{(B_{r_0}(0))})^{2n+2} \equiv D_{r_0},$$
$$\gamma_{r_0}|\xi| \le |a_{jk}(x,t,\vec{z})\xi| \le \gamma_{r_0}^{-1}|\xi|, \quad \forall \xi \in \mathbb{R}^n.$$

where  $\overline{B_{r_0}(0)} = \{z \in \mathbb{C} : |z| \le r_0\}$ , and with

(1.23) 
$$A(x,0,\vec{0}) = (a_{jk}(x,0,\vec{0}))_{j,k=1,\dots,n}$$
$$= A_0(x) + A_h = (a_{0,jk}(x))_{j,k=1,\dots,n} + (a_{jk}^0)_{j,k=1,\dots,n}$$

where for some  $N, \ \tilde{M} \in \mathbb{Z}^+$  large enough

(1.24) 
$$|\partial_x^{\alpha} a_{0,jk}(x)| \le \frac{c_0}{\langle x \rangle^N}, \quad |\alpha| \le \tilde{M} \quad j,k=1,..,n,$$

and  $A_h$  as in (1.12).

(H2) <u>Asymptotic flatness.</u> There exists c > 0 such that for any  $(x,t) \in \mathbb{R}^n \times \mathbb{R}$  and  $0 < |\alpha| \le \tilde{M}, \ 0 \le |\alpha'| \le \tilde{M}$  it follows that

(1.25) 
$$|\partial_x^{\alpha} a_{jk}(x,t,\vec{0})| + |\partial_t \partial_x^{\alpha'} a_{jk}(x,t,\vec{0})| \le \frac{c}{\langle x \rangle^N},$$

for k, j = 1, ..., n with  $N, \tilde{M}$  as in (1.24).

(H3) <u>Growth of the first order coefficients.</u> There exist  $c, c_0 > 0$  such that for any  $x \in \mathbb{R}^n$  and any  $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ 

(1.26) 
$$\begin{cases} |\partial_x^{\alpha} b_{m_j}(x,0,\vec{0})| \le \frac{c_0}{\langle x \rangle^N}, \ |\alpha| \le \tilde{M}, \ m = 1, 2, \ j = 1, ..., n, \\ |\partial_t \partial_x^{\alpha} b_{m_j}(x,t,\vec{0})| \le \frac{c}{\langle x \rangle^N}, \ |\alpha| \le \tilde{M}, \ m = 1, 2, \ j = 1, ..., n, \end{cases}$$

where  $\vec{b}_l = (b_{l_1}, .., b_{l_n}), \ l = 1, 2$  with  $N, \ \tilde{M}$  as in (1.24). (H4) <u>Regularity.</u> For any  $J \in \mathbb{Z}^+$  and  $r_0 > 0$  the coefficients

(1.27) 
$$a_{jk}, b_{1_j}, b_{2_j} \in C_b^J(\mathbb{R}^n \times \mathbb{R} \times (\overline{(B_{r_0}(0))})^{2n+2}),$$

and

(1.28) 
$$c_1, c_2 \in C_b^J(\mathbb{R}^n \times \mathbb{R} \times (\overline{(B_{r_0}(0))})^2)$$

Our main result is the following.

**Theorem 1.3.** Under the hypotheses (H1)-(H4) there exists  $N = N(n) \in \mathbb{Z}^+$  such that given any

(1.29) 
$$u_0 \in H^s(\mathbb{R}^n) \quad with \quad \langle x \rangle^N \partial_x^\alpha u_0 \in L^2(\mathbb{R}^n), \ |\alpha| \le s_1,$$

and

(1.30) 
$$f \in L^1(\mathbb{R}^+ : H^s(\mathbb{R}^n))$$
 with  $\langle x \rangle^N \partial_x^\alpha f \in L^1(\mathbb{R}^+ : L^2(\mathbb{R}^n)), \ |\alpha| \le s_1,$ 

where  $s, s_1 \in \mathbb{Z}^+$ , sufficiently large with  $s > s_1 + 4$ , for which the Hamiltonian flow  $H_{h(u_0)}$ associated to the symbol

(1.31) 
$$h(u_0) = h_{u_0}(x,\xi) = \sum_{j,k=1}^n a_{jk}(x,0,u_0,\bar{u}_0,\nabla u_0,\nabla \bar{u}_0)\xi_j\xi_k,$$

is non-trapping, there exist  $T_0 > 0$ , depending on

$$\begin{aligned} \lambda &= \|u_0\|_{s,2} + \sum_{|\alpha_1| \le s_1} \|\langle x \rangle^N \partial_x^{\alpha} u_0\|_2 \\ &+ \int_0^\infty \|f(t)\|_{s,2} dt + \sum_{|\alpha_1| \le s_1} \int_0^\infty \|\langle x \rangle^N \partial_x^{\alpha} f(t)\|_2 dt, \end{aligned}$$

the constants in (H1)-(H4) and on the non-trapping condition (H5), and a unique solution u = u(x,t) of the equation in (1.1) with initial data  $u(x,0) = u_0(x)$  on the time interval  $[0,T_0]$  satisfying

(1.32) 
$$u \in C([0,T_0]:H^{s-1}) \cap L^{\infty}([0,T_0]:H^s) \cap C^1((0,T_0):H^{s-3}), \\ \langle x \rangle^N \partial_x^{\alpha} u \in C([0,T_0]:L^2), \quad |\alpha| \le s_1.$$

Moreover, if  $(u_0, f) \in H^{s'}(\mathbb{R}^n) \times L^1(\mathbb{R}^+ : H^{s'}(\mathbb{R}^n))$  with s' > s then (1.32) holds with s' instead of s in the same time interval  $[0, T_0]$ .

<u>Remarks</u> Here, we are not concerned with the problem of estimating the optimal values of s,  $s_1$  or N in Theorem 1.3.

Also we shall not attempt to obtain the sharp form of the persistence property of the solution, (i.e.  $u \in C([0, T_0] : H^s(\mathbb{R}^n))$ ) as well as the continuous dependence of the solution upon the data. These can be established by combining the argument in [2] with our key *a priori* estimates in Lemma 2.1.

Similarly, from our arguments it is easy to deduce that the local solution possesses the local smoothing effect, i.e. if  $u_0 \in H^s(\mathbb{R}^n)$ , then  $J^{s+1/2}u \in L^2(\mathbb{R}^n \times [0, T_0] : \langle x \rangle^{-\tilde{N}} dx dt)$ .

The use of the weights in Theorem 1.3 comes from two sources. First, in order to convert  $L^1$  conditions, such as (1.7), into  $L^2$  conditions. Secondly, one needs them in order to maintain the asymptotic flatness condition (H2), when  $\vec{0}$  is replaced by  $(u, \bar{u}, \nabla_x u, \nabla_x \bar{u})$ , for a solution u.

As it was mentioned quasilinear results as those in Theorem 1.3 cannot be obtained by using just a fixed point argument. Instead, as in [20], we shall rely on the artificial viscosity method. First, we consider the linear problem

(1.33) 
$$\begin{cases} \partial_t u = -\epsilon \Delta^2 u - i \,\partial_{x_j} (a_{jk}(x,t)\partial_{x_k} u) + \vec{b}_1(x,t) \cdot \nabla u + \vec{b}_2(x,t) \cdot \nabla \bar{u} \\ + c_1(x,t)u + c_2(x,t)\bar{u} + f(x,t) \equiv -\epsilon \Delta^2 u + L(x,t)u + f(x,t), \\ u(x,0) = u_0(x), \end{cases}$$

with  $\epsilon \in (0, 1)$ . The main step is to obtain the following *a priori* estimate for solutions of the linear IVP (1.33) : there exits T > 0 such that

(1.34) 
$$\sup_{0 \le t \le T} \|u(t)\|_2^2 \le c_T(\|u_0\|_2^2 + \int_0^T \|f(t)\|_2^2 dt)$$

with T, c independent of  $\epsilon$  and depending on an appropriate manner on the coefficients in (1.33).

The inequality (1.34) will be proved under general hypotheses on the coefficients in (1.33). This will allow us to find a class of functions such that when the coefficients of the equation in (1.1) are evaluated in an element in this class they satisfy these general hypotheses.

We denote by  $u^{\epsilon}$  the solution of the following nonlinear IVP associated to equation in (1.1)

(1.35) 
$$\begin{cases} \partial_t u = -\epsilon \Delta^2 u - i \, \partial_{x_j} (a_{jk}(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) \partial_{x_{jk}} u) \\ + \vec{b}_1(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) \cdot \nabla u + \vec{b}_2(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) \cdot \nabla \bar{u} \\ + c_1(x, t, u, \bar{u})u + c_2(x, t, u, \bar{u})\bar{u} + f(x, t), \\ u(x, 0) = u_0(x). \end{cases}$$

The viscosity method provides the solution  $u^{\epsilon}$ , in the appropriate class, in the time interval  $[0, T_{\epsilon}]$  with  $T_{\epsilon} = O(\epsilon)$ . Evaluating the coefficients in (1.35) in this solution we get a linear problem as that in (1.33) for which the estimate (1.34) holds. This *a priori* estimate allows us to extend the solution  $u^{\epsilon}$ , in the same class, to a time interval  $[0, T_0]$ with  $T_0$  independent of  $\epsilon \in (0, 1)$ .

Once the estimate (1.34) is available the proof of Theorem 1.3 follows an argument quite similar to that explained in detail in [20]. So we shall concentrate in the proof of the inequality (1.1). This will be given in section 2, Lemma 2.1.

Finally, we point out another difference between the elliptic quasi-linear case and the non-degenerate one.

In [20] the following general class of quasilinear equation was considered

$$(1.36) \qquad \begin{cases} \partial_t u = -i \,\partial_{x_j} (a_{jk}(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) \partial_{x_k} u + \partial_{x_j} (b_{jk}(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) \partial_{x_k} u \\ + \vec{b}_1(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) \cdot \nabla u + \vec{b}_2(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) \cdot \nabla \bar{u} \\ + c_1(x, t, u, \bar{u}) u + c_2(x, t, u, \bar{u}) \bar{u} + f(x, t), \end{cases}$$

where  $(b_{ik})$  is a symmetric complex valued matrix.

Under the ellipticity assumption : given  $r_0 > 0$  there exists  $\gamma_{r_0} > 0$  such that for any

$$(x, t, \vec{z}) \in \mathbb{R}^n \times \mathbb{R} \times \overline{((B_{r_0}(0)))}^{2n+2},$$

it follows that

$$\langle a_{jk}(x,t,\vec{z})\xi,\xi\rangle - |\langle b_{jk}(x,t,\vec{z})\xi,\xi\rangle| \ge \gamma_{r_0}|\xi|^2, \ \forall \xi \in \mathbb{R}^n,$$

with  $B_{r_0}(0)$  as in (1.22) and similar assumptions on the asymptotic flatness, the growth of first order coefficients, the regularity of the coefficients, and a non-trapping hypothesis it was established in [20] that the IVP associated to the equation (1.36) is locally well posed.

The non-trapping hypothesis in [20] was the following : Given  $u_0 \in H^r(\mathbb{R}^n)$ , r > n/2+2, define

$$\varpi_2(x,0,\xi) = -a_{jk}(x,0,u_0,\bar{u}_0,\nabla u_0,\nabla \bar{u}_0)\xi_k\xi_j, 
\varpi_3(x,0,\xi) = -b_{jk}(x,0,u_0,\bar{u}_0,\nabla u_0,\nabla \bar{u}_0)\xi_k\xi_j,$$

and

$$\kappa(x,\xi) = \sqrt{\varpi_2^2(x,0,\xi) - |\varpi_3(x,0,\xi)|^2}$$

It was said that  $u_0$  satisfies the non-trapping hypothesis if there exists  $0 < \eta < 1$  and functions  $a(x,\xi), a_1(x,\xi)$  such that

$$\kappa(x,\xi) = a(x,\xi) + \eta a_1(x,\xi),$$

(with  $a(x,\xi)$  real, homogeneous of degree 2, with  $|\partial_x^\beta a(x,\xi)| \in C^{1,1}(\mathbb{R}^n \times \mathbb{R}^n), |\beta| \leq N(n)$ and  $\theta(\xi) a(x,\xi) \in C^{N(n)}(\mathbb{R}^n \times \mathbb{R}^n)$ , where  $\theta \equiv 1$  for  $|\xi| > 1, \theta \equiv 0$  for  $|\xi| < 1/2, \theta \in C^{\infty}(\mathbb{R}^n)$ , with  $a_1$  verifying similar estimates), the Hamiltonian flow  $H_a$  associated to the symbol a is non-trapping, for more details see [20].

In the non-degenerate setting the equation in (1.36) does not allow for a more general one that that considered in (1.1). This is due to the following linear algebra result (whose proof follows by induction).

**Lemma 1.1.** Let A, B be two  $n \times n$  matrices, A be real symmetric non-positive and non-degenerate, and B a symmetric complex valued one such that

 $\langle A\xi, \xi \rangle = 0$  implies  $\langle B\xi, \xi \rangle = 0$ .

Then there exists  $\lambda \in \mathbb{C}$  such that  $B = \lambda A$ .

#### 2. The linear problem

In this section we shall consider the linear IVP

(2.1) 
$$\begin{cases} \partial_t u = -i \,\partial_{x_j} (a_{jk}(x,t)\partial_{x_k} u) + \vec{b}_1(x,t) \cdot \nabla u + \vec{b}_2(x,t) \cdot \nabla \bar{u} \\ + c_1(x,t)u + c_2(x,t)\bar{u} + f(x,t) \equiv L(x,t)u + f(x,t), \\ u(x,0) = u_0(x), \end{cases}$$

where  $x \in \mathbb{R}^n$ , n > 1,  $t \in [0, T]$ , with T > 0, and its associated  $\epsilon$ -viscosity version

(2.2) 
$$\begin{cases} \partial_t u = -\epsilon \Delta^2 u + L(x,t)u + f(x,t), & \epsilon \in (0,1), \\ u(x,0) = u_0(x), \end{cases}$$

under the following assumptions:

(H<sub>l</sub>1) <u>Non-degeneracy</u> :  $A(x,t) = (a_{jk}(x,t))_{j,k=1,\dots,n}$  is a real symmetric matrix and there exist  $\gamma, \gamma_0 \in (0,1)$  such that for any  $\xi \in \mathbb{R}^n, (x,t) \in \mathbb{R}^n \times [0,T]$ 

(2.3) 
$$\begin{cases} \gamma|\xi| \le |A(x,t)\xi| \le \gamma^{-1}|\xi|,\\ \gamma_0|\xi| \le |A(x,0)\xi| \le \gamma_0^{-1}|\xi|, \end{cases}$$

with

(2.4)  
$$A(x,0) = (a_{jk}(x,0))_{j,k=1,\dots,n}$$
$$= A_0(x) + A_h = (a_{0,jk}(x))_{j,k=1,\dots,n} + (a_{jk}^0)_{j,k=1,\dots,n}$$

where for some  $N, \ \tilde{M} \in \mathbb{Z}^+$  large enough

(2.5) 
$$|\partial_x^{\alpha} a_{0,jk}(x)| \le \frac{c_0}{\langle x \rangle^N}, \quad |\alpha| \le \tilde{M}, \quad j,k = 1,..,n,$$

and  $A_h$  as in (1.12).

(H<sub>l</sub>2) <u>Asymptotic Flatness</u> : There exists c > 0 such that for all  $\xi \in \mathbb{R}^n, (x,t) \in \mathbb{R}^n \times [0,T], j, k = 1, ..., n$ , and  $0 < |\alpha| \le \tilde{M}, 0 \le |\alpha'| \le \tilde{M}$ 

(2.6) 
$$\left|\partial_x^{\alpha} a_{jk}(x,t)\right| + \left|\partial_t \partial_x^{\alpha'} a_{jk}(x,t)\right| \le \frac{c}{\langle x \rangle^N}$$

with N, M as in  $H_l 1$ .

(H<sub>l</sub>3) <u>Growth Condition of the First Order Coefficients</u> : There exist  $c, c_0 > 0$  such that for all  $\xi \in \mathbb{R}^n, (x, t) \in \mathbb{R}^n \times [0, T]$ 

(2.7) 
$$|\partial_x^{\alpha} b_{m_j}(x,0)| \le \frac{c_0}{\langle x \rangle^N}, \quad |\alpha| \le \tilde{M}, \quad m = 1, 2, \quad j = 1, .., n,$$

and

(2.8) 
$$|\partial_x^{\alpha} b_{m_j}(x,t)| \le \frac{c}{\langle x \rangle^N}, \quad |\alpha| \le \tilde{M}, \quad m = 1, 2, \quad j = 1, .., n,$$

with  $N, \tilde{M}$  as in  $H_l 1$ .

 $(H_l 4)$  <u>Regularity of the Coefficients</u> :

(2.9) 
$$a_{jk}, b_{m_j}, c_m \in C_b^J(\mathbb{R}^n \times [0,T]), \quad j,k = 1,..,n, \ m = 1,2,$$

with  $J = J(n) \in \mathbb{Z}^+$  sufficiently large such that the proofs below involving  $\psi$ .d.o's can be carried out, with norm

(2.10) 
$$\|\partial_x^{\alpha}\partial_t^r d\|_{L^{\infty}(\mathbb{R}^n \times [0,T])} \le c_J, \quad |\alpha| + r \le J.$$

with  $d = a_{jk}$ ,  $b_{m_j}$  or  $c_m$ , j, k = 1, ..., n, m = 1, 2.

(H<sub>l</sub>5) <u>Non-trapping Condition</u> : The bicharacteristic flow associated to the symbol of A(x, 0) (see (2.4)), i.e.

(2.11) 
$$h_2^0(x,\xi) = \sum_{j,k=1}^n a_{jk}(x,0)\xi_j\xi_k$$

is non-trapping.

We shall use  $c_0$  to denote a generic constant which only depends on the coefficients evaluated at time t = 0.

The main ingredient in the proof of our main result Theorem 1.3 is the following estimate for the solution of the linear IVP (2.1).

**Lemma 2.1.** There exist  $\tilde{c}_0 = \tilde{c}_0(\gamma_0, c_0, n) > 0$  with  $\gamma_0$  as in (2.3),  $c_0$  as in (2.5) and (2.7), and the constant  $c_0$  in Lemma 2.4 below,  $\tilde{N} = \tilde{N}(n)$ , and  $K_0 > 0$ ,  $T_0 \in (0, T]$  depending on (2.3)-(2.10) such that for  $\tilde{T} \in (0, T_0)$  the solution of the IVP (2.2)  $u^{\epsilon}$  satisfies

(2.12)  
$$\sup_{0 \le t \le \tilde{T}} \|u^{\epsilon}(t)\|_{2} + \int_{0}^{T} \int |J^{1/2}u^{\epsilon}|^{2} \langle x \rangle^{-\tilde{N}} dx dt$$
$$\le \tilde{c}_{0} e^{K_{0}\tilde{T}} (\|u_{0}\|_{2} + \int_{0}^{\tilde{T}} \|f(\cdot, t)\|_{2} dt).$$

Moreover, (2.12) still holds if we replace its last term by

$$(\int_0^{\hat{T}} \int |J^{-1/2} f(x,t)|^2 \langle x \rangle^{-\tilde{N}} dx dt)^{1/2}.$$

We recall the class of  $\psi$ .d.o's introduced in [21]:

$$\Psi_a f(x) = \int e^{ix \cdot \xi} a(x,\xi) \hat{f}(\xi) d\xi,$$

with symbol

$$a(x,\xi) = \chi(|\xi|)a(P(x,A_h\xi);x,\xi),$$

with  $\chi$  as in (1.15),

$$P(x, A_h\xi) = x - \left(\frac{x \cdot A_h\xi}{|\xi|^2}\right)A_h\xi,$$

i.e.  $P(x, A_h\xi)$  is the projection of x into the hyperplane perpendicular to  $A_h\xi$ , and

$$a = a(s; x, \xi) \in \mathcal{S}(\mathbb{R}^n : S_{1,0}^m),$$

 $\mathbb{S}(\cdot)$  denoting the Schwartz class, and  $S^m_{1,0}$  the class of classical symbol of  $\psi.\text{d.o.'s}$  of order m.

In [21] (Theorem 3.2.1) we prove that

(2.13) 
$$\|\Psi_a f\|_2 \le c \|f\|_{m,2}.$$

We shall use that our class is closed under differentiation of the symbol in the x-variable.

Lemma 2.2. Let

$$a_{\alpha}(x,\xi) = \partial_x^{\alpha}(a(P(x,A_h\xi);x,\xi)\chi(|\xi|)),$$

with  $a(s; x, \xi)$  as above. Then  $a_{\alpha}(x, \xi)$  defines a symbol in our class. Moreover,

(2.14) 
$$\|\Psi_{a_{\alpha}}f\|_{2} \le c_{\alpha}\|f\|_{m,2}.$$

*Proof.* First we consider the case  $\alpha = (1, 0, .., 0)$ . So

$$\partial_{x_1} a(P(x, A_h \xi); x, \xi) = \frac{\partial a}{\partial x_1} (P(x, A_h \xi); x, \xi) + \sum_{j=1}^n \frac{\partial a}{\partial z_j} (P(x, A_h \xi); x, \xi) \frac{\partial P}{\partial x_1} (x, A_h \xi)$$

Since

$$\partial_{x_1}(P(x, A_h\xi))_j = \partial_{x_1}\left(x_j - \frac{x \cdot A_h\xi}{|\xi|^2}(A_h\xi)_j\right) = \delta_{1j} - \frac{(A_h\xi)_1(A_h\xi)_j}{|\xi|^2},$$

it follows that (see remark 3.1.3(b) in [21])

$$\partial_{x_1} a(P(x, A_h \xi); x, \xi) \chi(|\xi|) = b_1(P(x, A_h \xi); x, \xi) \chi(|\xi|)),$$

with  $b_1 = b_1(z; x, \xi) \in \mathfrak{S}(\mathbb{R}^n : S_{1,0}^m)$  which yields the result.

The proof of the general case combines the above argument and induction in  $|\alpha|$ .  $\Box$ 

Using the notation

(2.15) 
$$\mathcal{L} = \mathcal{L}(x,t) = -\partial_{x_j}(a_{jk}(x,t)\partial_{x_k}\cdot)$$

and taking complex conjugate in the equation (2.2) we obtain the system

(2.16) 
$$\begin{cases} \partial_t \vec{w} = -\epsilon \Delta^2 I \vec{w} + i H \vec{w} + B \vec{w} + C \vec{w} + \vec{F}, \\ \vec{w}(x,0) = \vec{w}_0(x), \end{cases}$$

where

(2.17) 
$$\vec{w} = \begin{pmatrix} u \\ \bar{u} \end{pmatrix}, \quad \vec{F} = \begin{pmatrix} f \\ \bar{f} \end{pmatrix},$$

(2.18) 
$$H = \begin{pmatrix} \mathcal{L} & 0\\ 0 & -\mathcal{L} \end{pmatrix}, \quad B = \begin{pmatrix} \underline{\Psi}_{b_1} & \vec{b}_2(x,t) \cdot \nabla\\ \overline{\vec{b}_2(x,t)} \cdot \nabla & \Psi_{\bar{b}_1} \end{pmatrix},$$

with  $b_1 \in S_{1,0}^1$  with odd symbol in  $\xi$ , and C a 2 × 2 matrix of  $\psi$ .d.o's of order zero.

At some point we will take derivative of the equation in (1.1), so the new coefficients of  $\nabla u$  will be a combination of the original ones and some derivatives of the  $a_{jk}$ 's. In fact, these coefficients depend on the order of the derivative just as a multiplicative constant. For this reason we consider a general  $b_1 \in S_{1,0}^1$  with odd symbol in  $\xi$  in (2.15), (2.18).

**Lemma 2.3.** There exists  $N \in \mathbb{Z}^+$  such that for any  $T_0 \in (0,T]$  and any  $\epsilon \in (0,1)$  the solution  $\vec{w}^{\epsilon} = \vec{w}$  of the IVP (2.16) satisfies

(2.19) 
$$\int_{0}^{T_{0}} \int |J^{1/2}\vec{w}|^{2} \langle x \rangle^{-\tilde{N}} dx dt \leq (c_{0} + cT_{0}) \sup_{0 \leq t \leq T_{0}} \|\vec{w}(t)\|_{2}^{2} + c_{0}\epsilon \int_{0}^{T_{0}} \|\Delta \vec{w}(t)\|_{2}^{2} dt + c_{0} \int_{0}^{T_{0}} \|\vec{F}(t)\|_{2}^{2} dt,$$

where  $c_0$  depends only on the coefficients evaluated at t = 0 and on Lemma 2.1 and c depends on the estimates in (2.3)-(2.10).

To prove Lemma 2.3 we will follow the argument in [21] (Lemma 5.1.3). First, we recall Lemma 5.1.1 in [21].

**Lemma 2.4.** Let A(x,0) be as in (2.4). Assume that the bicharacteristic flow is nontrapped, i.e.

$$\{X(s; x_0, \xi_0) : s \in \mathbb{R}^+\}\$$

is unbounded for any  $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n - \{0\}$ . Suppose  $\lambda \in L^1([0, \infty)) \cap C([0, \infty))$  is strictly positive and non-increasing. Then there exist  $c_0 > 0$  and a real symbol  $p \in S^0_{1,0}$ , both depending on  $h^0_2(x, \xi)$  in (2.11) and  $\lambda$  such that

$$H_{h_2^0}p = \{h_2^0, p\}(x,\xi) \ge \lambda(|x|) |\xi| - c_0, \quad \forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n.$$

We fix N as in (2.5), (2.6), (2.7), and (2.8), and choose

$$\lambda(\rho) = 1/(1+\rho^2)^{\tilde{N}/2} = \langle \rho \rangle^{\tilde{N}}.$$

to obtain the following time dependent version of Lemma 2.4.

**Lemma 2.5.** These exists  $T_0 > 0$  depending only on  $H_l$  such that for any  $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ 

(2.20) 
$$H_{h_2}p = \{h_2; p\}(x, t, \xi) \ge \frac{|\xi|}{2\langle x \rangle^{\tilde{N}}} - 2c_0,$$

where

(2.21) 
$$h_2 = h_2(x, t, \xi) = a_{jk}(x, t)\xi_k\xi_j$$

and  $p \in S_{1,0}^0$  is the operator in Lemma 2.4.

*Proof.* We recall the notation

$$h_2^0(x,\xi) = a_{jk}(x,0)\xi_k\xi_j.$$

By definition

$$H_{h_2}p = \partial_{\xi_j}h_2\partial_{x_j}p - \partial_{x_j}h_2\partial_{\xi_j}p$$
  
=  $2a_{jk}(x,t)\xi_k\partial_{x_j}p - \partial_{x_j}a_{jk}(x,t)\xi_k\xi_j\partial_{\xi_j}p.$ 

From our hypothesis  $H_l^2$  it follows that for any  $t \in (0, T_0]$ 

$$\begin{aligned} |\partial_{\xi_j} h_2(x,t,\xi) - \partial_{\xi_j} h_2^0(x,\xi)| \\ &= 2|(a_{jk}(x,t) - a_{jk}(x,0))\xi_k| \le c \frac{T_0}{\langle x \rangle^N} |\xi|, \end{aligned}$$

and

$$\begin{aligned} |\partial_{x_j} h_2(x,t,\xi) - \partial_{x_j} h_2^0(x,\xi)| \\ &= |(\partial_{x_j} a_{jk}(x,t) - \partial_{x_j} a_{jk}(x,0))\xi_k\xi_j| \le c \frac{T_0}{\langle x \rangle^N} |\xi|^2. \end{aligned}$$

Since  $p \in S_{1,0}^0$  choosing  $N \gg \tilde{N}$  one has that

$$|H_{h_2}p - H_{h_2^0}p| \le c \frac{T_0}{\langle x \rangle^{\tilde{N}}} |\xi|.$$

From Lemma 2.4 we know that

$$H_{h_2^0}p(x,\xi) \ge \frac{|\xi|}{\langle x \rangle^{\tilde{N}}} - c_0$$

so taking  $T_0$  sufficiently small we obtain the desired result (2.20).

Now we shall prove Lemma 2.3.

*Proof.* Let

(2.22) 
$$k(x,\xi) = \begin{pmatrix} exp(p(x,\xi)) & 0\\ 0 & -exp(p(x,\xi)) \end{pmatrix},$$

where  $p \in S_{1,0}^0$  is the symbol in Lemma 2.5 so  $K = \Psi_k$  is a diagonal matrix of  $\psi$ .d.o's of order zero. We calculate

(2.23)  
$$\partial_t \langle K\vec{w}, \vec{w} \rangle = \langle K\partial_t \vec{w}, \vec{w} \rangle + \langle K\vec{w}, \partial_t \vec{w} \rangle$$
$$= -\epsilon (\langle K\Delta^2 I\vec{w}, \vec{w} \rangle + \langle K\vec{w}, \Delta^2 I\vec{w} \rangle)$$
$$+ \langle (i[KH - HK] + KB + B^*K)\vec{w}, \vec{w} \rangle$$
$$\langle (KC + C^*K)\vec{w}, \vec{w} \rangle + \langle K\vec{F}, \vec{w} \rangle + \langle K\vec{w}, \vec{F} \rangle.$$

We disregard the symbols of order zero and consider first the symbol of  $i[KH - HK] + KB + B^*K$ , i.e.

$$\sigma(i[KH - HK] + KB + B^*K),$$

which is equal up to a symbol of order zero to

(2.24) 
$$-e^{p} \begin{pmatrix} H_{h_{2}}p & 0\\ 0 & H_{h_{2}}p \end{pmatrix} + e^{p} \begin{pmatrix} b_{1}(x,t,\xi) & 2i\vec{b}_{2}(x,t)\cdot\xi\\ -2i\vec{b}_{2}(x,t)\cdot\xi & \overline{b}_{1}(x,t,\xi) \end{pmatrix}$$

We write

$$\vec{b}_2(x,t) = \vec{b}_2(x,0) + t \frac{\vec{b}_2(x,t) - \vec{b}_2(x,0)}{t},$$

with a similar identity for  $b_1(x, t, \xi)$ . Also since  $K = \Psi_k$  has order zero it is easy to see that

$$\left|-\epsilon(\langle K\Delta^2 I\vec{w}, \vec{w}\rangle + \langle K\vec{w}, \Delta^2 I\vec{w}\rangle)\right| \le c_0 \epsilon \|\vec{w}(t)\|_{2,2}^2$$

This combined with the matrix version of the sharp Garding inequality, the hypothesis  $H_l3$ , and Lemma 2.5 yields, after integration by part, the desired result, i.e. inequality (2.19) (for details we refer to the proof of Lemma 5.1.3 in [21]).

Next, we shall recall some notations and definitions used in [21]. First, we define

(2.25) 
$$a_{jk}^{R}(x,0) = \theta\left(\frac{x}{R}\right)a_{jk}(x,0) + \left(1 - \theta\left(\frac{x}{R}\right)\right)a_{jk}^{0}$$

where  $\theta \in C_0^{\infty}(\mathbb{R}^n)$ ,  $\theta(x) = 1$  for |x| < 1, and  $\theta(x) = 0$  for |x| > 2,

(2.26) 
$$\mathcal{L}^{R}(x) = \partial_{x_{j}}(a_{jk}^{R}(x,0)\partial_{x_{k}}), \text{ and } \mathcal{L}(x,t) = \partial_{x_{j}}(a_{jk}(x,t)\partial_{x_{k}}).$$

We apply the operator  $(K^R)^*$ , which is independent of t, to the equation in (2.2) and use that

$$\mathcal{L}(x,t) = \mathcal{L}(x,t) - \mathcal{L}(x,0) + \mathcal{L}^{R}(x) + \mathcal{E}^{R}(x),$$

with

$$\mathcal{E}^{R}(x) = \partial_{x_{j}} \left( \left( 1 - \theta \left( \frac{x}{R} \right) \right) (a_{jk}(x, 0) - a_{jk}^{0}) \partial_{x_{k}} \right),$$

and

$$\vec{b}_j(x,t) \cdot \nabla = (\vec{b}_j(x,t) - \vec{b}_j(x,0)) \cdot \nabla + \vec{b}_j(x,0) \cdot \nabla, \quad j = 1, 2$$

We shall need the following symbols used in [21],

(2.27)  

$$b_{1}(x,\xi) = -b_{1}(x,0)i\xi,$$

$$p^{R}(x,\xi) = -\chi(|\xi|) \int_{-\infty}^{0} b_{1}(X^{R}(s;x,\xi),\Xi^{R}(s;x,\xi))ds,$$

$$p_{e}^{R}(x,\xi) = \frac{1}{2}(p^{R}(x,\xi) + p^{R}(x,-\xi)),$$

$$k^{R}(x,\xi) = \exp(p_{e}^{R}(x,\xi)),$$

$$k^{R}(x,\xi) = \exp(-p_{e}^{R}(x,\xi)).$$

where

$$(X^R(s;x,\xi),\Xi^R(s;x,\xi)),$$

denotes the bicharacteristic flow associated to the symbol of the truncated operator  $\mathcal{L}^{R}(x)$  defined in (2.26) and  $\chi$  as in (1.15).

We remark that after taking s-derivatives in the equation (1.1) and linearizing the resulting equation the new term  $b_1(x,\xi)$  obtained has the form

$$s \sum_{j,k,l} \partial_{x_j} a_{jk}^R(x,0) \xi_j \xi_k \xi_l \langle \xi \rangle^{-2} - b_1(x,0) i \xi.$$

From our hypotheses it is clear that this new  $b^R(x,\xi)$  satisfies similar estimates than that in (2.27).

Thus, from (2.2) we get

$$(2.28) \begin{aligned} \partial_t (K^R)^* u &= -\epsilon (K^R)^* \Delta^2 u - i[(K^R)^*; \mathcal{L}^R] u + i\mathcal{L}^R (K^R)^* u \\ &+ i[(K^R)^*; (\mathcal{L}(x,t) - \mathcal{L}(x,0))] u + i(\mathcal{L}(x,t) - \mathcal{L}(x,0)) (K^R)^* u \\ &+ i\mathcal{E}^R (K^R)^* u + i[\mathcal{E}^R; (K^R)^*] u \\ &+ (K^R)^* \vec{b}_1(x,0) \cdot \nabla u + (K^R)^* (\vec{b}_1(x,t) - \vec{b}_1(x,0)) \cdot \nabla u \\ &+ (K^R)^* \vec{b}_2(x,0) \cdot \nabla \bar{u} + (K^R)^* (\vec{b}_2(x,t) - \vec{b}_2(x,0)) \cdot \nabla \bar{u} \\ &+ (K^R)^* f + \text{terms of order zero in } u. \end{aligned}$$

Next, we shall estimate

(2.29) 
$$\partial_t \langle (K^R)^* u, (K^R)^* u \rangle = \frac{d}{dt} \int |(K^R)^* u|^2 (x, t) dx.$$

After using the equation (2.28) to estimate (2.29) we separate the terms obtained into four groups  $I_{K;j}$ , j = 1, ..., 4.

In  $I_{K;1}$  we set the terms with coefficients independent of t, i.e. those involving the operators

$$\mathcal{L}^{R}(x), \ \vec{b}_{1}(x,0) \cdot \nabla, \ \vec{b}_{2}(x,0) \cdot \nabla, \ \mathcal{E}^{R}(x),$$

in the equation.

In  $I_{K;2}$  one has the terms involving the difference of the coefficients at the time t and time 0, i.e. the terms containing the operators

$$(\mathcal{L}(x,t) - \mathcal{L}(x,0)), (\vec{b}_1(x,t) - \vec{b}_1(x,0)) \cdot \nabla, (\vec{b}_2(x,t) - \vec{b}_2(x,0)) \cdot \nabla.$$

 $I_{K;3}$  contains the terms coming from the  $\epsilon$  viscosity part of the equation, i.e.

$$-\epsilon \langle (K^R)^* \Delta^2 u, (K^R)^* u \rangle - \epsilon \langle (K^R)^*, (K^R)^* \Delta^2 u \rangle$$

In  $I_{K;4}$  we collect the terms of order zero and those coming from the inhomogeneous term f(x, t). From the  $L^2$ -continuity of  $(K^R)^*$  they are bounded by

(2.30) 
$$cR^{2N_0} \|u(t)\|_2^2 + cR^{2N_0} \|u(t)\|_2 \|f\|_2,$$

with  $N_0$  depending only on the dimension n.

Similarly, we shall estimate

(2.31) 
$$\partial_t \langle E^R u, E^R u \rangle = \frac{d}{dt} \int |E^R u|^2 (x, t) dx$$

where

$$I = E^R + \tilde{K}^R (K^R)^*,$$

see Lemma 5.2.6 in [21].

Thus,

(2.32) 
$$\partial_t E^R u = -\epsilon E^R \Delta^2 u + i E^R \mathcal{L}(x,t) u + E^R \vec{b}_1(x,t) \cdot \nabla u + E^R \vec{b}_2(x,t) \cdot \nabla \bar{u} + E^R c_1(x,t) u + E^R c_2(x,t) \bar{u} + E^R f.$$

Inserting the identity

(2.33) 
$$E^{R}\mathcal{L}(x,t) = E^{R}\mathcal{L}(x,0) + E^{R}(\mathcal{L}(x,t) - \mathcal{L}(x,0))$$
$$= \mathcal{L}(x,0)E^{R} + [E^{R};\mathcal{L}(x,0)] + (\mathcal{L}(x,t) - \mathcal{L}(x,0))E^{R}$$
$$+ [E^{R};\mathcal{L}(x,t) - \mathcal{L}(x,0)],$$

into the equation (2.32) and (2.31) we split the terms obtained into four groups  $I_{E;j}$ , j = 1, ..., 4.

In  $I_{E;1}$  we place the terms with coefficients independent of t coming from the expressions

$$\mathcal{L}^{R}(x), \quad \vec{b}_{1}(x,0) \cdot \nabla, \quad \vec{b}_{2}(x,0) \cdot \nabla, \quad \mathcal{E}^{R}(x),$$

in the equation.

 $I_{E;2}$  contains the terms involving the operator

$$(\mathcal{L}(x,t) - \mathcal{L}(x,0)),$$

(in this case the terms involving the operators  $E^R \Psi_{b_1}$ ,  $E^R \vec{b}_2(x,t) \cdot \nabla$  can be directly bounded by using Lemma 5.2.6 in [21]).

 $I_{E;3}$  contains the terms coming from the  $\epsilon$  viscosity part of the equation, i.e.

$$-\epsilon \langle E^R \Delta^2 u, E^R u \rangle - \epsilon \langle E^R, E^R \Delta^2 u \rangle.$$

In  $I_{E;4}$  we collect the remainder terms of order zero which using Lemma 5.2.6 in [21] are bounded by

(2.34) 
$$cR^{2N_0} \|u(t)\|_2^2 + cR^{2N_0} \|u(t)\|_2 \|f\|_2$$

with  $N_0$  as in (2.30).

The terms in  $I_{K;1}$  and  $I_{E;1}$  (all involving time independent coefficients) were already considered in Section 5 of [21], (i.e. their contribution to the equations of (2.29) and (2.31)). Thus, to bound them we shall use the following inequalities proved in Lemmas 4.2.4-5.2.6 in [21].

**Proposition 2.1.** There exist  $M = M(N) \in \mathbb{Z}^+$ , N denoting the decay of the coefficients in (2.5)-(2.8), with  $M(N) \uparrow \infty$  as  $N \uparrow \infty$ , and  $R_0$  large enough such that for  $R \ge R_0$  one has

(2.35) 
$$\Re e \, i \, \langle \mathcal{L}^R(x) (K^R)^* u, (K^R)^* u \rangle \equiv 0,$$

(2.36) 
$$\Re e \, i \, \langle \mathcal{L}(x,0) E^R u, E^R u \rangle \equiv 0,$$

(2.37) 
$$\begin{aligned} |\langle i[\mathcal{L}^{R}; (K^{R})^{*}]u + (K^{R})^{*}\dot{b}_{1}(x, 0) \cdot \nabla u, (K^{R})^{*}u\rangle| \\ + |\langle \mathcal{E}^{\mathcal{R}}(K^{R})^{*}u, (K^{R})^{*}u\rangle| + |\langle [\mathcal{E}^{R}; (K^{R})^{*}]u, (K^{R})^{*}u\rangle| \\ + |\langle (K^{R})^{*}\dot{b}_{2}(x, 0) \cdot \nabla \bar{u}, (K^{R})^{*}u\rangle| \end{aligned}$$

$$\leq c_0 R^{-M} \| (J^{1/2}u) \langle x \rangle^{-\tilde{N}} \|_2^2 + c_0 R^{N_0 + N_1(M)} \| u(t) \|_2^2 + c_0 \| f \|_2^2.$$

and

(2.38)  
$$\begin{aligned} |\langle [E^{R}; \mathcal{L}(x, 0)]u, E^{R}u \rangle| \\ + |\langle E^{R}\vec{b}_{1}(x, 0) \cdot \nabla u, E^{R}u \rangle| + |\langle E^{R}\vec{b}_{2}(x, 0) \cdot \nabla \bar{u}, E^{R}u \rangle \\ \leq c_{0}R^{N_{0}} ||u(t)||_{2}^{2} + c_{0} ||f||_{2}^{2}, \end{aligned}$$

with  $N_0$  as in (2.30),  $\tilde{N}$  depending only on n, with  $N \gg \tilde{N}$ , and  $N_1$  depending only on M.

To handle the contribution coming from the terms in  $I_{K;2}$  we shall establish the following inequalities.

**Proposition 2.2.** Take  $N_0$  as in (2.30), then

(2.39) 
$$\begin{aligned} |\langle [(\mathcal{L}(x,t) - \mathcal{L}(x,0)); (K^R)^*] u, (K^R)^* u \rangle | \\ &\leq cT R^{N_0} || (J^{1/2} u) \langle x \rangle^{-\tilde{N}} ||_2^2 + c R^{N_0} || u(t) ||_2^2, \end{aligned}$$

(2.40) 
$$\operatorname{\mathcal{R}e} i \left\langle (\mathcal{L}(x,t) - \mathcal{L}(x,0))(K^R)^* u, (K^R)^* u \right\rangle \equiv 0,$$

(2.41) 
$$\begin{aligned} |\langle (K^R)^* (\vec{b}_1(x,t) - \vec{b}_1(x,0)) \cdot \nabla u, (K^R)^* u \rangle | \\ &\leq cT R^{N_0} || (J^{1/2} u) \langle x \rangle^{-\tilde{N}} ||_2^2 + c R^{N_0} || u(t) ||_2^2, \end{aligned}$$

and

(2.42) 
$$\begin{aligned} |\langle (K^R)^* (\vec{b}_2(x,t) - \vec{b}_2(x,0)) \cdot \nabla \bar{u}, (K^R)^* \bar{u} \rangle| \\ &\leq cT R^{N_0} || (J^{1/2} u) \langle x \rangle^{-\tilde{N}} ||_2^2 + c R^{N_0} || u(t) ||_2^2, \end{aligned}$$

with  $\tilde{N}$  depending only on n and  $N \gg \tilde{N}$ .

Proof. We write

$$([(\mathcal{L}(x,t) - \mathcal{L}(x,0)); (K^{R})^{*}])^{*}$$
  
=  $(\mathcal{L}(x,t) - \mathcal{L}(x,0))^{*}K^{R} - K^{R}(\mathcal{L}(x,t) - \mathcal{L}(x,0))^{*}$ 

where

$$\begin{aligned} (\mathcal{L}(x,t) - \mathcal{L}(x,0))^* \\ &= (a_{jk}(x,t) - a_{jk}(x,0))\partial_{jk}^2 + \partial_j(a_{jk}(x,t) - a_{jk}(x,0))\partial_k \\ &= \beta_{jk}(x,t)\partial_{jk}^2 + \tilde{\beta}_k(x,t)\partial_k = \Theta_1 + \Theta_2, \end{aligned}$$

with the notation  $\partial_j$  instead of  $\partial_{x_j}$ . Notice that the coefficients  $\beta_{jk}(x,t)$  and  $\tilde{\beta}_k(x,t)$  and their derivatives up to order p (large enough) have strong decay at infinity uniformly in  $t \in [0, T]$ . In fact, a sufficient number of semi-norms in the Schwartz class are bounded by cT for all  $t \in [0, T]$ .

The term  $\Theta_2$  yields

$$\langle (\tilde{\beta}_k(x,t)\partial_k K^R - K^R \tilde{\beta}_k(x,t)\partial_k)u, (K^R)^*u \rangle$$

which using Theorem 3.3.1 in [21] and the  $L^2$ -continuity of  $K^R$  can be bounded by

$$TR^{N_0} ||u(t)||_2^2, \quad \forall t \in (0,T)$$

To handle  $\Theta_1$  we use again Theorem 3.3.1 in [21] to write that

(2.43) 
$$\beta_{jk}(x,t)\partial_{jk}^2 K^R - K^R \beta_{jk}(x,t)\partial_{jk}^2 = \Psi_{d^R} + \text{ zero order terms},$$

#### SCHRÖDINGER EQUATION

with

$$d^{R}(x,t,\xi) = \beta_{jk}(x,t)\partial_{\xi_{l}}(\xi_{j}\xi_{k})\partial_{x_{l}}K^{R} - \partial_{x_{l}}\beta_{jk}(x,t)\xi_{j}\xi_{k}\partial_{\xi_{l}}K^{R}$$

We recall that a symbol in our class  $a(x,\xi)$  when multiplied by  $\phi(x)$ , a fast decaying function in the x variable, becomes a classical symbol of the same order.

To each term in the symbol of  $d^R(x, t, \xi)$  we can apply the following argument.

<u>Claim</u>: Let  $\phi(x,t)$  be an smooth function with strong decay at infinity in the xvariable uniformly in  $t \in [0,T]$ , with a sufficient number of semi-norms in the Schwartz class bounded by cT. Then

(2.44) 
$$|\langle \phi(x,t)\partial_j K^R u, (K^R)^* u \rangle| \le cTR^{2N_0} ||(J^{1/2}u(t))\langle x \rangle^{-\bar{N}}||_2^2 + ||u(t)||_2^2$$

<u>Notation</u> We shall use the notation  $\simeq$  to denote terms whose difference can be bounded by a multiple of  $||u||_2^2$  or by an operator of order zero.

*Proof.* We have

$$\begin{split} \langle \phi(x,t)\partial_{j}K^{R}u, (K^{R})^{*}u \rangle \\ &= \langle \langle x \rangle^{-\tilde{N}} \langle x \rangle^{2\tilde{N}} \phi(x,t)\partial_{j}K^{R}J^{1/2}J^{-1}J^{1/2}u, \langle x \rangle^{-\tilde{N}}(K^{R})^{*}u \rangle \\ &\simeq \langle \langle x \rangle^{-\tilde{N}}J^{1/2} \langle x \rangle^{2\tilde{N}} \phi(x,t)\partial_{j}K^{R}J^{-1}J^{1/2}u, \langle x \rangle^{-\tilde{N}}(K^{R})^{*}u \rangle \\ &\simeq \langle \langle x \rangle^{-\tilde{N}}J^{1/2} \langle x \rangle^{3\tilde{N}} \phi(x,t)\partial_{j}K^{R}J^{-1} \langle x \rangle^{-\tilde{N}}J^{1/2}u, \langle x \rangle^{-\tilde{N}}(K^{R})^{*}u \rangle \\ &\simeq \langle J^{1/2} \langle x \rangle^{-\tilde{N}} \langle x \rangle^{3\tilde{N}} \phi(x,t)\partial_{j}K^{R}J^{-1} \langle x \rangle^{-\tilde{N}}J^{1/2}u, \langle x \rangle^{-\tilde{N}}(K^{R})^{*}u \rangle \\ &\simeq \langle \langle x \rangle^{3\tilde{N}} \phi(x,t)\partial_{j}K^{R}J^{-1} \langle x \rangle^{-\tilde{N}}J^{1/2}u, \langle x \rangle^{-\tilde{N}}(K^{R})^{*}u \rangle \\ &\simeq \langle \langle x \rangle^{3\tilde{N}} \phi(x,t)\partial_{j}K^{R}J^{-1} \langle x \rangle^{-\tilde{N}}J^{1/2}u, \langle x \rangle^{-\tilde{N}}(K^{R})^{*}u \rangle \\ &= \Lambda_{3}. \end{split}$$

Since  $\langle x \rangle^{3\tilde{N}} \phi(x,t) \partial_j K^R J^{-1} \in S^0_{1,0}$ , i.e. a classical  $\psi$ .d.o. of order zero, it follows that

$$|\Lambda_3| \le cTR^{N_0} \|\langle x \rangle^{-\tilde{N}} J^{1/2} u \|_2 \|\langle x \rangle^{-\tilde{N}} J^{1/2} \langle x \rangle^{-\tilde{N}} (K^R)^* u \|_2$$

Finally, using that

$$(\langle x \rangle^{-\tilde{N}} J^{1/2} \langle x \rangle^{-\tilde{N}} (K^R)^*)^* = K^R \langle x \rangle^{-\tilde{N}} J^{1/2} \langle x \rangle^{-\tilde{N}},$$

we obtain the desired inequality (2.44).

Returning to the operator  $\Psi_{d^R}$  whose symbol is  $d^R(x,\xi)$  and applying our claim one gets that

$$|\langle \Psi_{d^R} u, (K^R)^* u \rangle| \le cTR^{N_0} ||\langle x \rangle^{-\tilde{N}} J^{1/2} u(t)||_2^2 + ||u(t)||_2^2,$$

which proves (2.39).

Integration by parts yields (2.40).

To prove (2.41) we write

$$((K^R)^*(b_1(x,t) - b_1(x,0)) \cdot \nabla))^* \simeq ((b_1(x,t) - b_1(x,0)) \cdot \nabla)K^R = G_R^* \in S_{1,0}^1,$$

and  $\Psi_{G_R^*} \simeq \Psi_{\bar{G}_R}$ , so using an argument similar to that given in the proof of the claim above we get (2.41). Similarly for (2.42).

To bound the terms in  $I_{E;2}$  we have the following estimates.

**Proposition 2.3.** There exists  $N_0 \in \mathbb{Z}^+$  depending only on the dimension n such that for any  $t \in (0, T_0]$ 

(2.45) 
$$|\langle [(\mathcal{L}(x,t) - \mathcal{L}(x,0)); E^R] u, E^R u \rangle| \le cT R^{N_0} ||u(t)||_2^2,$$

and

(2.46) 
$$\Re e \, i \langle (\mathcal{L}(x,t) - \mathcal{L}(x,0)) E^R u, E^R u \rangle = 0$$

*Proof.* With the same notation that in the previous proof we have

 $(\mathcal{L}(x,t) - \mathcal{L}(x,0))u = \partial_j(\beta_{jk}(x,t)\partial_k u).$ 

So using that  $E^R = I - \tilde{K}^R (K^R)^*$ , one sees that

$$[(\mathcal{L}(x,t) - \mathcal{L}(x,0)); E^R]u$$
  
=  $\tilde{K}^R[(K^R)^*; \partial_j\beta_{jk}(x,t)\partial_k] + [\tilde{K}^R; \partial_j\beta_{jk}(x,t)\partial_k](K^R)^*$ 

First we consider the term involving  $[(K^R)^*; \partial_j \beta_{jk}(x, t)\partial_k]$ . Taking the adjoint it follows that  $([(K^R)^*; \partial_j \beta_j (x, t)\partial_j])^* = -K^R \partial_j \beta_j (x, t)\partial_j + \partial_j \beta_j (x, t)\partial_j K^R$ 

$$([(K^{R})^{*};\partial_{j}\beta_{jk}(x,t)\partial_{k}])^{*} = -K^{R}\partial_{j}\beta_{jk}(x,t)\partial_{k} + \partial_{j}\beta_{jk}(x,t)\partial_{k}K^{R}$$
$$= -K^{R}\partial_{j}(\beta_{jk}(x,t))\partial_{k} + \partial_{j}(\beta_{jk}(x,t))\partial_{k}K^{R}$$
$$-K^{R}\beta_{jk}(x,t)\partial_{jk}^{2} + \beta_{jk}(x,t)\partial_{jk}^{2}K^{R}$$
$$\simeq -K^{R}\beta_{jk}(x,t)\partial_{jk}^{2} + \beta_{jk}(x,t)\partial_{jk}^{2}K^{R},$$

since  $-K^R \partial_j(\beta_{jk}) \partial_k + \partial_j(\beta_{jk}) \partial_k K^R$  is an operator of order zero, see Theorem 3.3.1 in [21].

Up to symbols of order zero (bounded operator in  $L^2$ ) the symbol of  $\beta_{jk}(x,t)\partial_{jk}^2 K^R - K^R \beta_{jk}(x,t)\partial_{jk}^2$  is equal to

$$\eta(x,t,\xi) = \beta_{jk} \partial_{\xi_l}(\xi_k \xi_j) \partial_{x_l} k^R(x,\xi) - \partial_{x_l} \beta_{jk} \xi_k \xi_j \partial_{\xi_l} k^R(x,\xi).$$

So  $\eta(x,t,\xi) \in S_{1,0}^1$  uniformly in  $t \in [0,T]$  and  $\eta(x,t,\xi) \simeq \Psi_d \tilde{\beta}_l(x,t) \partial_l$  with  $d = d(x,\xi)$  in our class. Similarly for  $\eta^*(x,t,\xi)$ , then  $\eta^*(x,t,\xi) \simeq \tilde{\beta}_l(x,t) \partial_l \Psi_{d_1}$ . Inserting this in (2.45) we get

$$\begin{split} \langle \tilde{K}^{R}[(K^{R})^{*};\partial_{j}\beta_{jk}(x,t)\partial_{k}]u, E^{R}u\rangle &\simeq \langle \tilde{K}^{R}\tilde{\beta}_{l}(x,t)\partial_{l}\Psi_{d_{1}(x,\xi)}u, E^{R}u\rangle \\ &\simeq \langle \Psi_{b}\Psi_{d_{1}(x,\xi)}u, E^{R}u\rangle \simeq \langle \Psi_{d_{1}(x,\xi)}u, (\Psi_{b})^{*}E^{R}u\rangle, \end{split}$$

which can be bounded by  $||u(t)||_2^2$ , by using that  $\Psi_b$  is a classical  $\psi$ .d.o. of order 1 and the continuities properties of  $E^R$  deduce in Lemma 5.2.6 in [21].

A similar argument provides the bound for the term  $[\tilde{K}^R, \partial_j \beta_{jk}(x, t)\partial_k](K^R)^*$ . Collecting this information we get (2.45).

The proof of (2.46) follows by integration by parts.

The terms coming from the artificial viscosity term  $\epsilon \Delta^2$ , i.e. the terms in  $I_{K;3}$  and  $I_{E;3}$  will be handled by using the following inequalities.

### Proposition 2.4.

(2.47) 
$$\langle \epsilon(K^R)^* \Delta^2 u, (K^R)^* u \rangle = \epsilon \langle \Delta(K^R)^* u, \Delta(K^R)^* u \rangle + \Lambda_1,$$

and

(2.48) 
$$\langle \epsilon E^R \Delta^2 u, E^R u \rangle = \epsilon \langle \Delta E^R u, \Delta E^R u \rangle + \Lambda_2,$$

where

$$|\Lambda_j| \le \epsilon R^{2N_0} \|\Delta u(t)\|_2 (\|u(t)\|_2 + \|\nabla u(t)\|_2), \quad j = 1, 2.$$

*Proof.* To obtain (2.47) we write

(2.49)  

$$\langle (K^{R})^{*} \Delta^{2} u, (K^{R})^{*} u \rangle$$

$$= \langle [(K^{R})^{*}; \Delta] \Delta u, (K^{R})^{*} u \rangle + \langle \Delta (K^{R})^{*} \Delta u, (K^{R})^{*} u \rangle$$

$$= \langle [(K^{R})^{*}; \Delta] \Delta u, (K^{R})^{*} u \rangle + \langle (K^{R})^{*} \Delta u, \Delta (K^{R})^{*} u \rangle$$

$$= \langle [(K^{R})^{*}; \Delta] \Delta u, (K^{R})^{*} u \rangle + \langle (K^{R})^{*} \Delta u, [\Delta; (K^{R})^{*}] u \rangle$$

$$+ \langle (K^{R})^{*} \Delta u, (K^{R})^{*} \Delta u \rangle$$

$$\equiv \Omega_{1} + \Omega_{2} + \Omega_{3}.$$

Thus, we consider

$$[\Delta; (K^R)^*] = \Delta (K^R)^* - (K^R)^* \Delta = \mathfrak{T},$$

and its adjoint

$$\mathfrak{T}^* = -(\Delta K^R - K^R \Delta).$$

Let  $k^R(x,\xi)$  denote the symbol of  $K^R$  such that

$$K^{R}f(x) = \int e^{ix\cdot\xi}k^{R}(x,\xi)\hat{f}(\xi)d\xi,$$

 $\mathbf{SO}$ 

$$\Delta K^R f(x) = \int e^{ix \cdot \xi} \{-|\xi|^2 k^R(x,\xi) + 2i\xi_j \partial_{x_j} k^R(x,\xi) + \Delta_x k^R(x,\xi)\} \hat{f}(\xi) d\xi.$$

Therefore,

$$\mathfrak{I}^*f(x) = \int e^{ix\cdot\xi} \{2i\xi_j\partial_{x_j}k^R(x,\xi) + \Delta_x k^R(x,\xi)\}\hat{f}(\xi)d\xi.$$

From Lemma 2.2 above and Theorem 3.2.1 in [21] one has that the operators with symbols  $\partial_{x_j} k^R(x,\xi)$  and  $\Delta_x k^R(x,\xi)$  are bounded in  $L^2$ . Hence, we can write that (2.50)  $\Upsilon^* = C_j \partial_{x_j} + C_0$ , and  $\Upsilon = C_0^* + \partial_{x_j} (-C_j^*)$ ,

with  $C_j$ , j = 0, 1, .., n denoting  $L^2$ -bounded operators.

Also we have that

$$\int e^{ix\cdot\xi} i\xi_j \partial_{x_j} k^R(x,\xi) \hat{f}(\xi) d\xi$$
  
=  $\partial_{x_j} (\int e^{ix\cdot\xi} \partial_{x_j} k^R(x,\xi) \hat{f}(\xi) d\xi) - \int e^{ix\cdot\xi} \partial_{x_j x_j}^2 k^R(x,\xi) \hat{f}(\xi) d\xi$ 

so that

(2.51) 
$$\mathfrak{T}^* = \partial_{x_j} \tilde{C}_j + \tilde{C}_0, \quad \text{and} \quad \mathfrak{T} = \tilde{C}_0^* - \tilde{C}_j^* \partial_{x_j},$$

with  $\tilde{C}_j, \ j = 0, 1, .., n$  denoting  $L^2$  bounded operators. To bound  $\Omega_1$  we see that

$$\langle [(K^R)^*; \Delta] \Delta u, (K^R)^* u \rangle = -\langle \mathfrak{T} \Delta u, (K^R)^* u \rangle$$
  
=  $-\langle (C_0^* + \partial_{x_j} (-C_j^*)) \Delta u, (K^R)^* u \rangle$   
=  $-\langle C_0^* \Delta u, (K^R)^* u \rangle - \langle (-C_j^*) \Delta u, \partial_{x_j} (K^R)^* u \rangle.$ 

Since an explicit computation shows that

$$\partial_{x_j}(K^R)^* \simeq (K^R)^* \partial_{x_j},$$

i.e. their difference is an  $L^2$ -bounded operator, it follows that

$$\begin{aligned} |\Omega_1| &\leq c_0 \|\Delta u\|_2 (\|u\|_2 + \|(K^R)^* \nabla u\|_2) \\ &\leq c_0 \|\Delta u\|_2 (\|u\|_2 + R^{N_0} \|\nabla u\|_2). \end{aligned}$$

Using again (2.50)-(2.51) we have that

$$\langle (K^R)^* \Delta u, [\Delta; (K^R)^*] u \rangle = \langle (K^R)^* \Delta u, \Im u \rangle = \langle (K^R)^* \Delta u, (\tilde{C}_0^* - \tilde{C}_j^* \partial_{x_j}) u \rangle,$$

so  $\Omega_2$  can be bounded as

$$\begin{aligned} |\Omega_2| &\leq c_0 \| (K^R)^* \Delta u \|_2 (\|u\|_2 + \|\nabla u\|_2) \\ &\leq c_0 R^{N_0} \|\Delta u\|_2 (\|u\|_2 + \|\nabla u\|_2). \end{aligned}$$

Inserting this information in (2.49) we obtain (2.47). To obtain (2.48) we recall that  $E^R = I - \tilde{K}^R (K^R)^*$  so that

$$E^{R}\Delta^{2} = \Delta E^{R}\Delta + [E^{R}; \Delta]\Delta,$$

with

$$[E^{R}; \Delta] = -[\tilde{K}^{R}(K^{R})^{*}; \Delta] = -(\tilde{K}^{R}(K^{R})^{*}\Delta - \Delta \tilde{K}^{R}(K^{R})^{*})$$
  
=  $-\tilde{K}^{R}[(K^{R})^{*}; \Delta] + [\tilde{K}^{R}; \Delta](K^{R})^{*} \equiv \Gamma_{1} + \Gamma_{2}.$ 

Using (2.50)-(2.51) and an explicit computation it follows that

$$\Gamma_1 = \tilde{K}^R \mathfrak{T} = \tilde{K}^R (C_0^* - \partial_{x_j} C_j^*) = -\partial_{x_j} \tilde{K}^R C_j^* + Q_1,$$

where  $Q_j$ , j = 1, .., 4 will denote  $L^2$  bounded operators. So

$$\begin{aligned} |\langle \Gamma_1 \Delta u, E^R u \rangle| &= |\langle (\partial_{x_j} \tilde{K}^R C_j^* + Q_1) \Delta u, E^R u \rangle| \\ &\leq c_0 ||\Delta u||_2 (||\partial_{x_j} E^R u||_2 + ||E^R u||_2). \end{aligned}$$

Combining  $\partial_{x_j} E^R = \partial_{x_j} - \partial_{x_j} \tilde{K}^R (K^R)^*$  and an explicit computation one gets that

$$\begin{split} \|\partial_{x_j} E^R u\|_2 &= \|(\partial_{x_j} - \partial_{x_j} \tilde{K}^R (K^R)^*) u\|_2 \\ &\leq \|u\|_2 + \|\partial_{x_j} \tilde{K}^R (K^R)^* u\|_2 \simeq \|u\|_2 + \|\tilde{K}^R \partial_{x_j} (K^R)^* u\|_2 \\ &\leq \|u\|_2 + R^{N_0} \|\partial_{x_j} (K^R)^* u\|_2 = \|u\|_2 + R^{N_0} \|K^R \partial_{x_j} u\|_2 \\ &\leq \|u\|_2 + R^{2N_0} \|\partial_{x_j} u\|_2, \end{split}$$

and consequently

$$|\langle \Gamma_1 \Delta u, E^R u \rangle| \le c_0 R^{2N_0} ||\Delta u||_2 (||u||_2 + ||\nabla u||_2)$$

For  $\Gamma_2$  we reproduce the argument in (2.50)-(2.51) for  $\tilde{K}^R$  instead of  $K^R$ . So using the same notation we have

$$\Gamma_2 = [\tilde{K}^R; \Delta] (K^R)^* = (C_0 + \partial_{x_j} C_j) (K^R)^*,$$

so as before

$$\begin{split} |\langle \Gamma_2 \Delta u, E^R u \rangle| &= |\langle (C_0 + \partial_{x_j} C_j) (K^R)^* \Delta u, E^R u \rangle| \\ &\leq |\langle C_0 (K^R)^* \Delta u, E^R u \rangle| + |\langle C_j (K^R)^* \Delta u, \partial_{x_j} E^R u \rangle| \\ &\leq \|\Delta u\|_2 (\|E^R u\|_2 + \|\partial_{x_j} E^R u\|_2) \leq c_0 R^{2N_0} \|\Delta u\|_2 (\|u\|_2 + \|\nabla u\|_2). \end{split}$$

Collecting the results in Propositions 2.1-2.4 we get that

(2.52) 
$$\frac{d}{dt} \| (K^R)^* u(t) \|_2^2 + \epsilon \| (K^R)^* \Delta u(t) \|_2^2 \\
\leq (c_0 R^{-M} + cT R^{N_0}) \| (J^{1/2} u)(t) \langle x \rangle^{-\tilde{N}} \|_2^2 + c(R^{N_0 + N_1(M)} + T R^{N_0}) \| u(t) \|_2^2 \\
+ c_0 \| f \|_2^2 + c_0 \epsilon R^{2N_0} \| \Delta u(t) \|_2 (\| u(t) \|_2 + \| \nabla u(t) \|_2),$$

and

(2.53)  
$$\frac{d}{dt} \|E^{R}u\|_{2}^{2} + \epsilon \|E^{R}\Delta u\|_{2}^{2} \\
\leq c(R^{N_{0}} + TR^{N_{0}})\|u(t)\|_{2}^{2} + c\|f\|_{2}^{2} \\
+ c_{0}\epsilon R^{2N_{0}}\|\Delta u(t)\|_{2}(\|u(t)\|_{2} + \|\nabla u(t)\|_{2})$$

We will use that  $E^R = I - \tilde{K}^R (K^R)^*$  so that

(2.54) 
$$\begin{aligned} \|v\|_{2} &= \|(E^{R} + \tilde{K}^{R}(K^{R})^{*})v\|_{2} \\ &\leq \|E^{R}v\|_{2} + R^{N_{0}}\|(K^{R})^{*}v\|_{2}. \end{aligned}$$

Also, we need the following interpolation estimates : for any  $\, l \in \mathbb{Z}^+$ 

(2.55) 
$$R^{l} \|v\|_{2} \|\Delta v(t)\|_{2} \leq R^{2l} \|v\|_{2}^{2} + \|\Delta v\|_{2}^{2},$$

and

(2.56) 
$$R^{l} \|\nabla v\|_{2} \|\Delta v\|_{2} \leq c R^{l} \|v\|_{2}^{1/2} \|\Delta v\|_{2}^{3/2} \leq c R^{4l} \|v\|_{2}^{2} + \|\Delta v\|_{2}^{2}.$$
  
So combining (2.52)-(2.53) with (2.54)-(2.56) we find that

$$\frac{d}{dt} (\|(K^{R})^{*}u(t)\|_{2}^{2} + \|E^{R}u(t)\|_{2}^{2}) + \epsilon R^{-2N_{0}} \|\Delta u(t)\|_{2}^{2} \\
\leq (cTR^{N_{0}} + c_{0}R^{-M}) \|(J^{1/2}u)\langle x \rangle^{-\tilde{N}}\|_{2}^{2} + c(R^{N_{0}+N_{1}(M)} + TR^{N_{0}})\|u(t)\|_{2}^{2} \\
+ c_{0} \|f(t)\|_{2}^{2} + c_{0}\epsilon R^{2N_{0}} \|\Delta u(t)\|_{2} (\|u(t)\|_{2} + \|\nabla u(t)\|_{2}) \\
\leq (cTR^{N_{0}} + c_{0}R^{-M}) \|(J^{1/2}u)\langle x \rangle^{-\tilde{N}}\|_{2}^{2} + c(R^{N_{0}+N_{1}(M)} + TR^{N_{0}})\|u(t)\|_{2}^{2} \\
+ c_{0} \|f(t)\|_{2}^{2} + \frac{\epsilon}{2}R^{-2N_{0}} \|\Delta u(t)\|_{2}^{2} + c_{0}\epsilon R^{14N_{0}} \|u(t)\|_{2}^{2}.$$

Thus,

(2.58) 
$$\frac{d}{dt} (\|(K^R)^* u(t)\|_2^2 + \|E^R u(t)\|_2^2) + \frac{\epsilon}{2} R^{-2N_0} \|\Delta u(t)\|_2^2 \\ \leq (cTR^{N_0} + c_0 R^{-M}) \|(J^{1/2} u)\langle x \rangle^{-\tilde{N}}\|_2^2 + c(R^{N_0 + N_1(M)} + TR^{N_0}) \|u(t)\|_2^2 \\ + c_0 \|f(t)\|_2^2 + c_0 \epsilon R^{14N_0} \|u(t)\|_2^2.$$

Integrating the above inequality in the time interval (0, T) and inserting in the result the estimate (2.19) we get

$$(2.59) \begin{aligned} \sup_{0 \le t \le T} \left( \| (K^R)^* u(t) \|_2^2 + \| E^R u(t) \|_2^2 \right) + \frac{\epsilon}{2} R^{-2N_0} \int_0^T \| \Delta u(t) \|_2^2 dt \\ \le \| (K^R)^* u(0) \|_2^2 + \| E^R u(0) \|_2^2 \\ + (cTR^{N_0} + c_0 R^{-M}) \int_0^T \| (J^{1/2} u) \langle x \rangle^{-\tilde{N}} \|_2^2 dt \\ + (cR^{N_0 + N_1(M)} + TR^{N_0} + c_0 \epsilon R^{14N_0}) \int_0^T \| u(t) \|_2^2 dt + c_0 \int_0^T \| f(t) \|_2^2 dt \\ \le \| (K^R)^* u(0) \|_2^2 + \| E^R u(0) \|_2^2 \\ + (cTR^{N_0} + c_0 R^{-M}) ((c_0 + cT) \sup_{0 \le t \le T} \| u(t) \|_2^2 \\ + c_0 \epsilon \int_0^T \| \Delta u(t) \|_2^2 dt + c_0 \int_0^T \| f(t) \|_2^2 dt ) \\ + c(R^{N_0 + N_1(M)} + TR^{N_0} + c_0 \epsilon R^{14N_0}) T \sup_{0 \le t \le T} \| u(t) \|_2^2 + c_0 \int_0^T \| f(t) \|_2^2 dt. \end{aligned}$$

Since  $M(N) \uparrow \infty$  as  $N \uparrow \infty$ , we take N in our hypotheses large enough such that  $M = 100N_0$  (we recall that  $N_0$  depends only on the dimension n). Next, we fix  $R \ge R_0$  sufficiently large and then  $T = T(N_0, M, R) > 0$  small enough such that the following inequalities holds

$$(cTR^{N_0} + c_0R^{-M})(c_0 + cT) \le \frac{R^{-2N_0}}{4},$$
$$(cR^{N_0 + N_1(M)} + cTR^{N_0} + c_0R^{14N_0})T \le \frac{R^{-2N_0}}{4}$$

and

$$c_0(cTR^{N_0} + c_0R^{-M}) \le R^{-N_0}/4.$$

Combining these inequalities, (2.59), and (2.54) we get the estimate

$$\sup_{0 \le t \le T} \|u(t)\|_2^2 + \frac{\epsilon}{4} R^{-2N_0} \int_0^T \|\Delta u(t)\|_2^2 dt \le c_0 R^{2N_0} \|u(0)\|_2^2 + c_0 \int_0^T \|f(t)\|_2^2 dt.$$

which proves (2.12).

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