An Analysis of Cross Points in the Low-Degree Polynomial Gains of $p$-lag Unbiased Smoothing FIR Filters

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Abstract—We address an analysis of the low-degree polynomials gains of the $p$-lag unbiased smoothing FIR filters. It is shown that such a gain is uniquely specified for unbiased FIR filters depending on the number of system states. An important feature of the $p$-lag gain is that, at the cross points, it converges to the reduced degree gain. The cross points between the uniform, linear, quadratic, and cubic smoothing unbiased FIR filter gains have therefore been investigated in detail. At these points, a number of simulations have been provided and advantages demonstrated.

Keywords—FIR filter, smoothing, power gain, cross point

I. INTRODUCTION

Design of the discrete-time finite impulse response (FIR) filters for state space applications has gained currency in recent years owing to some important inherent properties, such as the bounded input/bounded output (BIBO) stability and better robustness against uncertainties. That extended applications of such filter in different areas of signal, image, and speech processing. Accordingly, a number of algorithms for smoothing, filtering, and prediction of random processes and fields have been designed for decades.

One of the most universal solutions for the problem has been proposed by Shmaliy in [1] to obtain the $p$-step predictive unbiased FIR filtering of discrete-time state-space models with applications to the global position system (GPS)-based timekeeping. It turns out that such a predictive filter becomes a pure filter with $p = 0$, as described in [2] with applications to noise reduction in GPS-based measurement of time errors of local clocks. Allowed $p = 1$, the predictive filter becomes applicable for receding horizon control [3]. Of applied importance is that the $p$-step filter proposed can also be used to have negative $p$, in which case we have a $p$-lag unbiased smoothing FIR filter applied in [4] and [5] to ultrasound image processing. A complete analysis of the $p$-lag unbiased smoothing FIR filter is given in [6].

In this paper, we give an analysis of the $p$-lag unbiased smoothing FIR filter paying prime attention to the cross points, at which the degree-polynomial gains can be represented with the reduced degree gains. Accordingly, the computational complexity is reduced in practical implementations of such filters.

II. SIGNAL MODEL AND PROBLEM FORMULATION

A polynomial signal $x_{n}$ can be represented in state space with the state and observation equations as follows, respectively

$$x_{n} = A^{n-1+p}x_{n-N+1-p}, \quad (1)$$
$$y_{n} = Cx_{n} + v_{n}, \quad (2)$$

where $x_{n} = [x_{n}, x_{2n}, \ldots, x_{Kn}]$ is the $K \times 1$ vector of the states, $y_{n}$ is the measurements, $v_{n}$ is the measurement noise, the $1 \times K$ measurement matrix is $C = [1 \ 0 \ \ldots \ 0]$, and the $K \times K$ triangular matrix $A^{i}$ is specified as

$$A^{i} = \begin{bmatrix}
1 & \tau & \frac{1}{2!} (\tau)^{2} & \cdots & \frac{1}{(K-1)!} (\tau)^{K-1} \\
0 & 1 & \tau & \cdots & \frac{1}{(K-2)!} (\tau)^{K-2} \\
0 & 0 & 1 & \cdots & \frac{1}{(K-3)!} (\tau)^{K-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}, \quad (3)$$

where, $\tau$ is the sampling time. It follows that the noiseless model (1) projects ahead from $n - N + 1 - p$ to $n$ with the finite degree Taylor polynomial as
\[
x_n = \sum_{q=0}^{K-1} x_{n+q} \tau^q (N-1+p)^q / q!
\]  

(4)

If we introduce the \( l \)-degree polynomial \( p \)-dependent filter gain \( h_{il}(N, p) \), then the estimate of \( x_{1n} \) can be obtained on an averaging horizon of \( N \) points by the convolution

\[
\hat{x}_n = \sum_{i=p}^{N-1+p} h_{il}(N, p) y_{n-i}
\]

(5)

where \( p > 0 \) is allowed for predictive FIR filtering, \( p = 0 \) for FIR filtering, and \( p < 0 \) for smoothing FIR filtering. It has been shown in [1] and [2] that, in order for the estimate to be unbiased, the gain \( h_{il}(N, p) \) must satisfy the following conditions:

\[
\sum_{i=p}^{N-1+p} h_{il}(N, p) = 1,
\]

(6)

\[
\sum_{i=p}^{N-1+p} h_{il}(N, p) i^u = 0, \quad u \in [1, l].
\]

(7)

One knows from Kalman filter theory that the order of the optimal (and so unbiased) filter is the same as that of the system. That means that, for the \( K \) state model, the gain can be represented with the \( l \)-degree polynomial such that

\[
h_{il}(N, p) = \sum_{j=0}^{l} a_{jl}(N, p) i^j
\]

(8)

where, \( a_{jl}(N, p) \) is the polynomial coefficient and the degree \( l \) must be chosen such that \( l = K - 1 \).

The coefficients for the polynomial (8) have been found in [1] in the form of

\[
a_{jl}(N, p) = (-1)^j M_{(j+1)l}(N, p) / |D(N, p)|
\]

(9)

where a short \((l+1) \times (l+1)\) symmetric matrix \( D(N, p) \) is

\[
D = \begin{bmatrix}
d_0 & d_1 & \cdots & d_l \\
d_1 & d_2 & \cdots & d_{l+1} \\
\vdots & \vdots & \ddots & \vdots \\
d_l & d_{l+1} & \cdots & d_{2l}
\end{bmatrix}
\]

(10)

\( |D| \) is the determinant of (10), and \( M_{(j+1)l}(N, p) \) is the minor of (10). The component in (10) can be determined using the Bernoulli polynomials \( B_{(j+1)l}(x) \) as follows

\[
d_r(N, p) = \sum_{i=p}^{N-1+p} i^r = \frac{1}{r+1} [B_{r+1}(N+p) - B_{r+1}(p)].
\]

(11)

III. LOW DEGREE POLYNOMIAL GAINS

In FIR filtering, an estimate is obtained via the discrete convolution applied to measurement. That can be done if to represent the state space model on an averaging interval of some \( N \) points. Referring to (8), the low-degree polynomial gains can thus be defined with

\[
h_{il}(p) = \sum_{j=0}^{l} a_{jl}(p) i^j,
\]

(12)

Below, we derive and investigate the relevant unique gains for the uniform, linear, quadratic and cubic models covering an overwhelming majority of practical needs.

A) Uniform model

A model that is uniform over an averaging horizon of \( N \) points is the simplest one. The relevant signal is characterized with one state and the filter gain is represented, by (5), with the 0-degree polynomial as

\[
h_{0l}(p) = \begin{cases} 
\frac{1}{N}, & p \leq N - 1 + p \\
0, & \text{otherwise}
\end{cases}
\]

(13)

The noise power gain (NPG) of this filter is \( p \)-invariant, \( g_0(p) = 1/N \). Because this gain is associated with simple averaging, it is also optimal for a common task: reducing random noise while retaining a sharp step response.

B) Linear Model

For the linear prediction, \( l = 1 \), the gain is ramp

\[
h_{1l}(p) = a_{01}(p) + a_{11}(p)i
\]

(14)

having the coefficients

\[
a_{01}(p) = \frac{2(N-1)(N-1)+12p(N-1+p)}{N^{N^2-1}},
\]

(15)

\[
a_{11}(p) = - \frac{6(N+2)p}{N^{N^2-1}}.
\]

(16)

C) Quadratic and Cubic Models

For the quadratic and cubic models, the gains of the unbiased FIR filters become, respectively

\[
h_{2l}(p) = a_{02}(p) + a_{12}(p)i + a_{22}(p)i^2,
\]

(17)

\[
h_{3l}(p) = a_{03}(p) + a_{13}(p)i + a_{23}(p)i^2 + a_{33}(p)i^3
\]

(18)
where the coefficients are defined in [6]. In the Fig. 1, we sketch (13), (14), (17) and (18) for $N = 31$ and $p = 0$.

IV. **NPGS OF THE LOW-DEGREE POLYNOMIAL GAINS**

Noise in FIR estimates if often evaluated in terms of the NPG

$$g_l = \sum_{i=p}^{N-1-p} h_i^2(N, p).$$

(19)

Referring to (13), (14), (17) and (18), the NPG of the relevant FIR filters can be found to be, respectively,

$$g_0 = \frac{1}{N},$$

(20)

$$g_1 = \frac{2(2N-1)(N-1)+12p(N-1+p)}{N(N^2-1)},$$

(21)

$$g_2 = \frac{(3N^2-3N+2)(N-1)(N-2)}{N(N^2-1)} + 12(N-1)(2N^2-5N+2)p + 60p^4$$

$$+ 12(7N^2-15N+7)p^2 + 120(N-1)p^3,$$

(22)

$$g_3 = \frac{2(N^2-N+3)(N-1)(N-2)(N-3)}{N(N^2-1)(N^2-4)}$$

$$+ 5(6N^2-6N+5)(N-1)(N-2)(N-3)p$$

$$+ 5(42N^2-213N^3+378N^2-288N+91)p^2$$

$$+ 10(71N^2-175N+96)(N-1)p^3 + 350p^6$$

$$+ 5(246N^2-525N+271)p^4 + 1050(N-1)p^5,$$

(23)

In Fig. 2, we show the NPGs of the low-degree polynomial approximation.

V. **CROSS POINTS OF THE NPGS**

To find the cross point between the uniform gain (20) and ramp gain (21), we solve

$$g_0 = g_1$$

(24)

and obtain

$$p = -\frac{N-1}{2},$$

(25)

meaning that (25) makes the ramp gain (14) uniform (13). At this point, the unbiased ramp gain is also optimal with its zero bias and minimum possible noise produced by simple averaging (Fig. 3).

Similarly, the cross points of the ramp gain (21) and quadratic gain (22) can be found. Namely, by the lags

$$p_{21} = -\frac{N-1}{2} + \sqrt{\frac{N^2-1}{12}},$$

(26)

$$p_{22} = -\frac{N-1}{2} - \sqrt{\frac{N^2-1}{12}}.$$  

(27)

The quadratic gain simplifies to the ramp one and, at the middle of the averaging horizon, it becomes symmetric.
Unlike the $p$-lag linear gain (14) having a lower bound for the NPG at $1/N$, the relevant bound for the $p$-lag quadratic gain (17) ranges upper and is given by

$$G_{2\text{min}} = \frac{3(3N^2 - 2)}{5N(N^2 - 1)}. \quad (28)$$

This value appears if to put to zero the derivative of $g_2(N, p)$ with respect to $p$ and find the roots of the polynomial. Two lags correspond to (28), namely

$$P_{23} = -\frac{N - 1}{2} + \frac{1}{10} \sqrt{N^2 + 1}, \quad (29)$$

$$P_{24} = -\frac{N - 1}{2} - \frac{1}{10} \sqrt{N^2 + 1}. \quad (30)$$

Like the ramp gain case, here noise in the smoothing estimate is lower than in the filtering estimate, if $p$ does not exceed and averaging horizon. Otherwise, we watch for the increase in the error that can be substantial.

As well as the ramp and quadratic gains, the cubic one demonstrates several important features, including an ability of converting to the quadratic gain. Special values of $p$ associated with this gain are listed below:

$$P_{31} = -\frac{N - 1}{2} + \frac{1}{10} \sqrt{3N^2 - 7}, \quad (31)$$

$$P_{32} = -\frac{N - 1}{2} + \frac{105}{210} \sqrt{33N^2 - 17 + 2B}, \quad (32)$$

$$P_{33} = -\frac{N - 1}{2} + \frac{105}{210} \sqrt{33N^2 - 17 - 2B}, \quad (33)$$

where $B$ is defined as

$$B = \sqrt{36N^4 + 507N^2 - 2579} \quad (37)$$

The lags $P_{31}$ and $P_{36}$ convert the cubic gain to the quadratic one (Fig. 5a). These lags are therefore preferable from the standpoint of filtering accuracy owing to lower noise.

The lags $P_{32}$ and $P_{35}$ correspond to minima on the NPG characteristic (Fig. 5b). The remaining lags, $P_{33}$ and $P_{34}$, cause two maxima in the range of $-N + 1 < p < 0$ (Fig. 5c). This NPG ranges above the lower bound

$$G_{3\text{min}} = \frac{3(3N^2 - 7)}{4N(N^2 - 4)} \quad (38)$$

and, with $p = \text{const}$, it asymptotically approaches $g_4(N, 0)$, by increasing $N$. As well as in the quadratic gain case, noise in the cubic gain can be much lower than in the relevant filter ($p = 0$). On the other hand, the range of uncertainties is broadened here to $N = 3$ and the smoothing filter
becomes low inefficient at short horizons. In fact, when the gain (23) exceeds unity, the 3-degree unbiased smoothing FIR filter loses an ability of denoising and its use becomes hence meaningless.

_A) Generalizations_

Several important common properties of the unbiased smoothing FIR filters can now be outlined as in the following. Effect of the lag \( p \) on the NPG of low-degree unbiased smoothing FIR filters is reflected in Fig. 2. As can be seen, with (25), the NPG of the ramp gain is exactly that of the uniform gain. By \( p = p_{21} \) and \( p = p_{22} \), where \( p_{21} \) and \( p_{22} \) are specified by (29) and (30), respectively, the NPG of the quadratic gain simplifies to that of the ramp gain. Also, by \( p = - (N - 1)/2 \), and \( p_{36} \), the NPG of the cubic gain becomes that of the quadratic gain.

The following generalization can now be provided for a two-parameter family of the \( l \)-degree and \( p \)-lag, \( p < 0 \), unbiased smoothing FIR filters specified with the gain \( h_l(p) \) and NPG \( g_l(p) \):

i. Any FIR filter with the lag \( p \) lying on the averaging horizon \(- (N - 1) < p < 0 \), produces smaller errors than the relevant FIR filter with \( p = 0 \).

ii. Without loss in accuracy, the \( l \)-degree unbiased FIR filter can be substituted, for some special values of \( p \), with a reduced \((l - 1)\)-degree one. Namely, the 1-degree gain can be substituted with the 0-degree gain for \( p = - (N - 1)/2 \), the 2-degree gain with the 1-degree gain for \( p_{21} \) and \( p_{22} \), and the 3-degree gain with the 2-degree gain, if \( p_{31}, p = - (N - 1)/2, \) or \( p_{36} \).

iii. Beyond the averaging horizon, the error in the smoothing FIR filter with \( p < - N + 1 \) is equal to that in the predictive FIR filter with \( p > 0 \).

iv. The error lower bounds for the ramp gain, \( g_{1\text{min}} \), quadratic gain \( g_{2\text{min}} \), and cubic gain, \( g_{3\text{min}} \), are given by, respectively, (see Fig. 2)

\[
g_{1\text{min}} = \frac{1}{N}, \quad (39)
\]

\[
g_{2\text{min}} = \frac{3(3N^2 - 2)}{5N(N^2 - 1)} \bigg|_{N > 1} \approx \frac{9}{5N}, \quad (40)
\]

\[
g_{3\text{min}} = \frac{3(3N^2 - 7)}{4N(N^2 - 4)} \bigg|_{N > 4} \approx \frac{9}{4N}, \quad (41)
\]

v. With large \( N \), errors in the \( l \)-degree \( p \)-lag unbiased FIR filter for \( p = - N + 1 \) are defined by

\[
g_l(N, -N + 1) \bigg|_{N > 1} = \frac{(l + 1)^2}{N}. \quad (42)
\]

\[\text{Fig. 6: Cross point between the low-degree polynomial gains for } N = 31.\]

\[\text{Fig. 7: Cross flags to the particular gains with } N = 31.\]

The initial conditions can hence be ascertained using the ramp and quadratic gains with the NPGs \( = 4/N \) and \( = 9/N \), respectively.

vi. By increasing \( N \) for a constant \( p \) such that \( p < \frac{N}{2} \), the error in the polynomial gains asymptotically approaches that in the relevant gains with \( p = 0 \).

VI. EXPERIMENTAL RESULTS

In this section, we present the results of experimental investigations of the cross points. To develop the procedure, we first plot the gain functions defined from (20) to (23) with \( N = 31 \) such as in the Fig. 2. Next, we find the cross points between two functions (e.g. between \( g_1 \) and \( g_3 \) or \( g_2 \)). These points are also analytically calculated using (25)-(27) and (29)-(37). Finally, we compare the values in order to establish a correspondence.

True cross points are shown in the Fig. 6. Here, one can see that only six points exist: \( p, p_{21}, p_{22}, p_{31}, \) and \( p_{36} \). The cross flags for the particular gains are shown in Fig. 7. In the first case when the ramp gain transforms to the uniform
one (Fig. 7a and 7b), one find only one flag corresponding to a cross point \( p \) defined by (25). In the second case of conversion of the quadratic gain to the ramp one (Fig. 7b and 7c), we have two flags corresponding to points \( p_{21} \) and \( p_{22} \) defined by (26) and (27), respectively. Finally, in the third case, when the cubic gain transforms to the quadratic gain (Fig. 7c and Fig. 7d), there exist three flags associated with the cross point \( p \), \( p_{31} \) and \( p_{32} \) defined by (25), (29) and (36), respectively.

VII. CONCLUSIONS

In this paper, we investigated the cross points of the \( p \)-step unbiased smoothing FIR filter with low-degree polynomials gains. A practical importance of these points resides in the fact that the \( l \)-degree FIR filter gain reduces to the lower-degree one, thereby reducing the computational complexity.

REFERENCES