Discontinuous Galerkin time-domain method for GPR simulation of conducting objects

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ABSTRACT
In this paper we describe the discontinuous Galerkin time-domain method and apply it to the simulation of ground-penetrating radar (GPR) problems in 3D. The method is first validated with analytical solutions and we show its superior behaviour when compared to the classical finite-difference time-domain method, widely employed in GPR simulation.

INTRODUCTION
Numerical techniques are an indispensable tool in the analysis and design of all kinds of electromagnetic systems. In particular, they have been successfully applied to the simulation and optimization of ground-penetrating radar (GPR) systems (Fernández Pantoja et al. 2000; López et al. 2001; Diamanti and Giannopoulos 2009). Among them, time-domain methods are especially suitable for GPR simulation, since they are able to provide the full transient response of the system on a single run, allowing the user to analyze the system response in a causal way. The finite-difference time-domain (FDTD) method has been the most employed one, mainly because of its simplicity, ease of implementation and simulation speed (Giannopoulos 2005).

However, FDTD has severe drawbacks related to the staircased approximation it employs for curved boundaries. A recent alternative of FDTD is given by the discontinuous Galerkin time-domain method, which is experimenting an increasing development in computational electromagnetics (Hesthaven and Warburton 2002; Gedney et al. 2007; Pebernet et al. 2008; García et al. 2008; Alvarez et al. 2010).

Discontinuous Galerkin time-domain employs a discontinuous Galerkin weighting procedure to handle the spatial part of time-domain Maxwell’s curl equations. Like in the finite elements method, the space is divided into \(M\) non-overlapping elements (e.g., curvilinear tetrahedra), in each of which the solution is expanded in a set of nodal (Bernacki et al. 2006) or vector (Gedney et al. 2007) basis functions of arbitrary order. The temporal part of Maxwell curl equations can be handled by finite differences or by any other finite differentation technique.

In the discontinuous Galerkin time-domain the solution is allowed to be discontinuous at the boundaries between adjacent elements (unlike in finite elements) and continuous numerical fluxes are employed at the interface to connect the solution between them. The resulting algorithm is quasi-explicit in space, only requiring the inversion of \(M\) square matrices of \(Q \times Q\) elements (with \(Q\) the number of basis functions).

A two-dimensional discontinuous Galerkin time-domain approach has been successfully applied to GPR simulations involving buried objects in a lossy half-space (Lu et al. 2005). In this paper, we present a general description of a three-dimensional discontinuous Galerkin time-domain method including both nodal and vector formulations and show an application to the simulation of a full GPR scenario. Validations of the method with benchmark problems serve to prove the superior accuracy of this technique compared to the classical FDTD, outperforming the later in computer requirements.

DISCONTINUOUS GALERKIN TIME-DOMAIN THEORY
Vector elements formulation
Let us assume Maxwell’s curl equations for linear isotropic homogeneous media in Cartesian coordinates. Now, let us divide the space in \(M\) non-overlapping elements \(V_\alpha\), each bounded by \(S_\alpha\) and enforce a weak form of them by performing the inner product of each equation with a basis of local continuous vector test functions. The term ‘weak’ here means that we no longer require the equation to hold absolutely and we search for ‘weak’ solutions with respect to certain test functions to be defined later (Hesthaven and Warburton 2007).

\[
\int_{V_\alpha} \frac{\partial \mathbf{E}^\alpha}{\partial t} \cdot (\mathbf{\alpha} \mathbf{H}^\alpha + \mathbf{\sigma} \mathbf{E}^\alpha + \mathbf{J} - \nabla \times \mathbf{\mathbf{\mathbf{H}}}^\alpha) \, dV = 0 \\
\int_{V_\alpha} \frac{\partial \mathbf{H}^\alpha}{\partial t} \cdot (\mathbf{\mu} \mathbf{E}^\alpha + \nabla \times \mathbf{\mathbf{\mathbf{E}}}^\alpha) \, dV = 0.
\]
\( \Phi^{\tau+h} \in B^{\tau+h} = \{ \Phi_0^{\tau+h}, \Phi_1^{\tau+h}, \ldots, \Phi_Q^{\tau+h} \} \),
(2)

with \( E, H, J, \sigma, \varepsilon, \mu \) being, respectively: electric field, magnetic field, electric current density, electric conductivity, permittivity and permeability.

Integrating the curl terms by parts we can write equation (1) as

\[
\int_T \left( \vec{\phi}^\nu \cdot (\partial_t \vec{E} + \sigma \vec{E} + J) - \nabla \times \vec{\phi}^\nu \times \vec{H} \right) dV = \frac{\partial}{\partial \tau} \int_{\partial_T} \vec{\phi}^\nu \cdot (\vec{n} \times \vec{H}) dS.
\]
(3)

\[
\int_{S_m} \left( \vec{\phi}^{\nu+} \cdot (\mu \vec{H}^{\nu+}) + \nabla \times \vec{\phi}^{\nu-} \cdot \vec{E} \right) dV = -\partial_T \int_{\partial_{S_m}} \vec{\phi}^{\nu-} \cdot (\vec{n} \times \vec{E}) dS.
\]
(4)

The core idea of the discontinuous Galerkin time-domain is to only require the weak form of the tangential fields on the faces of adjacent elements \( S_m \) (right-hand side of equations (3) and (4)) to be continuous, instead of requiring full continuity of the solution as in the classical finite elements method. Since the fields are allowed to be different at each side of the interface, a trade-off value must be taken to evaluate the right-hand side of equations (3) and (4). This trade-off value (denoted with an added superscript *) cannot be written, in general, as a function of the fields at each sides interface

\[
\vec{n} \times \vec{E}^{\nu+} = \vec{n} \times \left( \vec{f}^{\nu+} \left( \vec{E}^{\nu+}, \vec{H}^{\nu+} \right) + \vec{f}^{\nu-} \left( \vec{E}^{\nu-}, \vec{H}^{\nu-} \right) \right),
\]
\[
\vec{n} \times \vec{H}^{\nu+} = \vec{n} \times \left( \vec{f}^{\nu+} \left( \vec{E}^{\nu+}, \vec{H}^{\nu+} \right) + \vec{f}^{\nu-} \left( \vec{E}^{\nu-}, \vec{H}^{\nu-} \right) \right),
\]
where we have added the superscript * to the fields on \( S_m \) in the element adjacent to \( m \) and the superscript t to the fields calculated in \( m \). These terms \( \vec{n} \times \vec{E}^{\nu+} \) and \( \vec{n} \times \vec{H}^{\nu+} \), so-called numerical fluxes, are used in the right-hand side of equations (21) and (22)) instead of \( \vec{n} \times \vec{E}^{\nu} \) and \( \vec{n} \times \vec{H}^{\nu} \).

Two common choices of the numerical flux are reported in the literature:

1. A centred flux (Bernacki et al. 2006) found by averaging the solutions at both sides of the interface.

\[
\vec{n} \times \vec{E}^{\nu+} = \frac{\vec{n} \times \vec{E}^{\nu+} + \vec{n} \times \vec{E}^{\nu-}}{2},
\]
\[
\vec{n} \times \vec{H}^{\nu+} = \frac{\vec{n} \times \vec{H}^{\nu+} + \vec{n} \times \vec{H}^{\nu-}}{2}.
\]

2. The upwind flux usually employed in the finite volume time-domain (Mohammadian et al. 1991) arising from the solution of the advection equations with discontinuous initial values (Riemann problem) (Hesthaven and Warburton 2002)

\[
\vec{n} \times \vec{E}^{\nu+} = \frac{\vec{n} \times \vec{E}^{\nu+} + \vec{n} \times \vec{E}^{\nu-}}{2} + \frac{\vec{n} \times \vec{E}^{\nu+} + \vec{n} \times \vec{H}^{\nu-}}{2},
\]
\[
\vec{n} \times \vec{H}^{\nu+} = \frac{\vec{n} \times \vec{H}^{\nu+} + \vec{n} \times \vec{E}^{\nu-}}{2} + \frac{\vec{n} \times \vec{H}^{\nu+} + \vec{n} \times \vec{E}^{\nu-}}{2},
\]

with \( Z^{\nu+} = \frac{\mu^{\nu+}}{\varepsilon^{\nu+}} = \frac{1}{Y^{\nu-}} \) being the intrinsic impedance of the element \( m \) and \( Z^{\nu+} = \frac{1}{Y^{\nu+}} \) being that of the adjacent one.

Notice, that boundary conditions between different dielectric/magnetic media are naturally handled in a weaker manner in the discontinuous Galerkin time-domain formulation, thanks to taking the same tangential components of the fields \( \vec{n} \times \vec{E}^{\nu+} \) and \( \vec{n} \times \vec{H}^{\nu+} \) in the flux integrals for two adjacent elements. Perfect electric conductor boundary conditions are also enforced in a weak manner by requiring the tangential electric field employed in the flux integrals to be null and the tangential magnetic field to be continuous (Alvarez et al. 2010)

\[
\vec{n} \times \vec{E}^{\nu+} = 0, \vec{n} \times \vec{H}^{\nu+} = 0.
\]

Regarding the truncation conditions, perfectly matched layers are successfully implemented in the discontinuous Galerkin time-domain following the formulation given in Lu and Cai (2004).

The semi-discrete algorithm\(^1\) is found by assuming that the space and time dependencies of the fields can be separated and that the spatial part is expanded within each element in a set of basis functions equal to the set of test functions (Galerkin method)

\[
\vec{E}^{\nu+} = \sum_{q=0}^{Q} \vec{E}_q^{\nu+}(\bar{r}) \vec{\phi}_q^{\nu+}(\bar{r}), \vec{H}^{\nu+} = \sum_{q=0}^{Q} \vec{H}_q^{\nu+}(\bar{r}) \vec{\phi}_q^{\nu+}(\bar{r}).
\]

The final form of the semi-discrete algorithm at element \( m \) is

\[
\varepsilon M^{\nu+}_m \frac{\partial}{\partial \tau} \vec{E} + \left( \sigma M^{\nu+}_m - F^{\nu+}_m \right) \vec{E} = -\varepsilon J^{\nu+}_m \vec{H} + F^{\nu+}_m \vec{H}^{\nu+} + H^{\nu+} - F^{\nu+}_m \varepsilon \vec{E},
\]
\[
\mu \varepsilon M^{\nu+}_m \frac{\partial}{\partial \tau} \vec{H} + \left( -F^{\nu+}_m \varepsilon \vec{E} \right) \vec{H} = +F^{\nu+}_m \vec{E} + F^{\nu+}_m \vec{E}^{\nu+} + F^{\nu+}_m \vec{E} - F^{\nu+}_m \varepsilon \vec{H},
\]

where \( \vec{E} \) and \( \vec{H} \) are the field coefficients

\[
\bar{E} = (E^{\nu+}_0(t), \ldots, E^{\nu+}_Q(t))^T,
\]
\[
\bar{H} = (H^{\nu+}_0(t), \ldots, H^{\nu+}_Q(t))^T,
\]
\( \vec{J} \) is the weak form of the source terms

\[
\vec{J} = \left( \int_{S_0} \vec{J}(\bar{r}, t) \cdot \vec{\phi}_q^{\nu+} \, dV \right)^T,
\]
\( \vec{M} \) is the mass matrix

\[
\left[ \vec{M}^{\nu+}_m \right]_{ij} = \int_T \vec{\phi}_i^{\nu+} \cdot \vec{\phi}_j^{\nu+} \, dV,
\]
\( \vec{S} \) is the stiffness matrix

\[
\left[ \vec{S}^{\nu+}_m \right]_{ij} = \int_T (\nabla \times \vec{\phi}_i^{\nu+}) \cdot \vec{\phi}_j^{\nu+} \, dV,
\]

\(^1\) Semi-discrete means that up to this point, the spatial part is discretized and the temporal part is not.

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A common choice for the basis functions (Hesthaven and Warburton 2002), is the set of 3D Lagrange interpolating \( n \)th order polynomials with an equal set of electric and magnetic basis functions. They are first defined in a standard reference element (Silvester and Ferrari 1990) as a function of the simplex coordinates \( \xi \) by
\[
\Phi^e_{\xi}(\xi) = \frac{(\xi+1)(\xi+2)(\xi+3)}{6} \text{ nodal points in the element,}
\]
requiring \( Q = (n+1)(n+2)(n+3)/6 \) nodal points in the element to form a complete basis. The local basis for each element is found by computing the mapping of the transformation from the reference element to the actual one. The case \( n = 0 \) leads to the classical finite volume time-domain algorithm (Mohammadian et al. 1991).

A common choice for the basis functions is the hierarchical high-order vector-basis functions, widely used in finite elements methods (Gedney et al. 2007; Webb 1999). The resulting system of ordinary differential equations in time can be solved in a number of ways: second-order leap-frog (Garcia et al. 2008), 4th order Runge-Kutta (Gedney et al. 2007), implicit Crank-Nicolson (Catella et al. 2008), etc.

**Nodal elements formulation**

The fundamentals of the scalar formulation are similar to those of the vector one. Now the basis and test functions are chosen to be scalar \( B^\text{sc} = \{\Phi^{\text{sc}}_{\nu^e}, \Phi^{\text{sc}}_{\nu^m} \}, m = 1, \ldots, M \).

Where \( B \) denotes the space basis and the superscripts \( e \) and \( m \) are used to distinguish between the basis employed for the electric and magnetic fields, respectively. The weak form of Maxwell curl equations become
\[
\int_{\xi^e} \left( \nabla \times \Phi^{\text{sc}}_{\nu^e} \right) \cdot \nabla \times \mathbf{E}^e d\xi^e = \int_{\xi^m} \left( \nabla \times \Phi^{\text{sc}}_{\nu^m} \right) \cdot \nabla \times \mathbf{H}^m d\xi^m,
\]
\[
\int_{\xi^m} \left( \nabla \times \Phi^{\text{sc}}_{\nu^e} \right) \cdot \nabla \times \mathbf{E}^e d\xi^e = -\int_{\xi^e} \left( \nabla \times \Phi^{\text{sc}}_{\nu^m} \right) \cdot \nabla \times \mathbf{H}^m d\xi^m,
\]
where we already assumed the fluxes in the right-hand side to be the numerical ones.

Comparing equations (21) and (22) and equations (3) and (4) we find similar flux-density integrals in their right-hand sides. Thus the same upward and centred fluxes of the scalar case can be used here.

For scalar-basis functions, expansion (23) now becomes
\[
\mathbf{E}^e = \sum_{i=1}^{Q} \Phi_{\nu^e}^i(\xi) \mathbf{E}^{\text{sc}}_i(\xi), \mathbf{H}^m = \sum_{i=1}^{Q} \Phi_{\nu^m}^i(\xi) \mathbf{H}^{\text{sc}}_i(\xi).
\]

The semi-discrete algorithm is formulated by plugging equation (9) into equations (3) and (4). The resulting equations are formally equal to equations (10) and (11), now with
\[
\mathbf{E} = \left( \mathbf{E}^e(t), \ldots, \mathbf{E}^m(t) \right)^T,
\]
\[
\mathbf{H} = \left( \mathbf{H}^e(t), \ldots, \mathbf{H}^m(t) \right)^T,
\]
\[
\mathbf{J} = \left( \mathbf{J}^e(t), \ldots, \mathbf{J}^m(t) \right)^T \Phi_{\nu^e}^i d\xi^e,
\]
\[
\mathbf{M} = \int_{\xi^e} \Phi_{\nu^e}^i \Phi_{\nu^e}^j d\xi^e.
\]
APPLICATION TO GPR PROBLEMS

As demonstrated, the discontinuous Galerkin time-domain is a numerical technique that achieves a superior accuracy with less computational requirements than FDTD. In this section, as a proof of concept, we show results for the simulation of simple GPR systems.

Object presence discrimination

A transient electromagnetic (TEM) horn antenna has been excited with a z-directed Gaussian current source near its shortcut wall with a half-width half-amplitude (Fig. 3) with a –3 dB bandwidth of 97.6 MHz. The antenna is placed in free-space 0.3 m away from a dielectric soil with electric relative permittivity 2, inside which, a 0.15 m radius perfect electric conductor sphere is buried at a depth of 0.5 m (Fig. 4).

Figure 5 shows a snapshot of the $E_z$ field after 16.7 nsec, computed with a vector discontinuous Galerkin time-domain with basis $(G_1, R_2)$ and a central numerical flux. Figure 5 shows the time evolution of $E_z$ observed at a point close to the source point inside the antenna (with coordinates $x = 0, y = 0, z = 0$), compared with that in the absence of a buried sphere. The subtractions of both values (magnified by a factor 10) is also shown to identify the effect of the presence of a sphere. The normalized $E_z$ field values at the observation point show the result of multiple reflections between the soil and the antenna.

Radargram simulation

As a more GPR oriented application we performed a series of simulations to elaborate a radargram. As the simulation of a 4 No dispersion/losses have been considered in this simplified problem.
Conclusions

In this paper, we have described and validated a discontinuous Galerkin time-domain method, suitable to become an efficient radargram requires several runs the computational time can become a critical parameter where the discontinuous Galerkin time-domain can overcome FDTD. These simulations can be critical to the design of antennas able to find objects in environments where the near-field coupling with the surface is not negligible (Uduwawala 2005). Moreover, when a certain number of iterations is needed to achieve an optimal result the computational time becomes a critical factor.

The case presented here models a TM-62M landmine as described in Kositsky (2002). The landmine is assumed to be 5 cm below the surface, located in a dry (non-dispersive, non-dissipative) medium with a constant $\varepsilon = 5$. The landmine is then moved along 5 cm steps simulating the radar passing above it. The air part of the geometry remains the same as in the previous case.

The radargram obtained is shown in Fig. 7. The simulated signal was subtracted with respect to a signal in a medium without a mine buried. The simulated radargram results are similar to the ones that can be obtained from direct experimental data.

CONCLUSIONS

In this paper, we have described and validated a discontinuous Galerkin time-domain method, suitable to become an efficient
and accurate alternative to FDTD. As a proof of concept, we have simulated a simple GPR scenario with a TEM-horn antenna illuminating soil with an object buried in it. Although formulated for non-dispersive media, the discontinuous Galerkin time-domain can easily be extended to handle these media using the auxiliary differential equation technique (Joseph et al. 1991), for instance. Actually, the discontinuous Galerkin time-domain can be extended to handle any type of material for which FDTD is already formulated, in a similar manner since the discontinuous Galerkin time-domain and FDTD only differ in the treatment of the space variations.

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REFERENCES


FIGURE 6
$E_z$ observed at a point with coordinates $(0, 0.5, 0)$ m, both with/without a buried sphere. The subtraction of both signals is shown magnified by 10. As can be seen an noticeable reflection peak from the surface is observed at time 21 ns. The subtraction of both signals produces a Gaussian modulated differential response (black line) produced by the presence of the buried sphere.

FIGURE 7
$E_z$ observed at a point with coordinates $(0, 0.5, 0)$ (same as in Fig. 6). Due to the model symmetries the value of $|E_z|$ is the same as the electric field magnitude, so the value represented can be representative of the induced signal in a hypothetically receiving antenna.


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