Synchronization of Interconnected Systems with an Input-Output Approach. Part I: Main Results

L. Scardovi, M. Arcak, and E. D. Sontag

Abstract—This paper provides synchronization conditions for networks of nonlinear systems, where each component of the network itself consists of subsystems represented as operators in the extended $L_2$ space. The synchronization conditions are provided by combining the input-output properties of the subsystems with information about the structure of network. The work is motivated by cellular networks where signaling occurs both internally, through interactions of species, and externally, through intercellular signaling. The companion paper [1] explores results for state-space models as well as biochemical applications.

I. INTRODUCTION

The analysis of synchronization phenomena has become an important topic in systems and control theory, motivated by many applications in physics, biology, and engineering. Emerging results in this area show that, in addition to the individual dynamics of the components, the network structure plays an important role in determining conditions leading to synchronization [2], [3], [4], [5], [6].

In this paper, we study synchronization in networks of nonlinear systems, by making use of the input-output properties of the subsystems comprising the network. Motivated by cellular networks where signaling occurs both internally, through interactions of species, and externally, through intercellular signaling, we assume that each component of the network (referred to as a “compartment” in the paper) itself consists of subsystems (referred to as “species”) represented as operators in the extended $L_2$ space. The input to the operator includes the influence of other species within the compartment as well as a diffusion-like coupling term between identical species in different compartments.

A similar input-output approach was taken in [7], [8], [9] to study stability properties of individual compartments, rather than synchronization of compartments. These studies verify an appropriate passivity property [10], [11] for each species and form a “dissipativity matrix”, denoted here by $E$, that incorporates information about the passivity of the subsystems, the interconnection structure of the species, and the signs of the interconnection terms. To determine the stability of the network, [8], [9] check the diagonal stability of the dissipativity matrix, that is, the existence of a diagonal solution $D > 0$ to the Lyapunov equation $E^T D + DE < 0$, similarly to classical work on large-scale systems by Vidyasagar and others, see [12], [13], [14].

In the special case of a cyclic interconnection structure with negative feedback, this diagonal stability test encompasses the classical secant criterion [15], [16] used frequently in mathematical biology. Following [7], [8], reference [17] investigated synchronization of cyclic feedback structures using an incremental variant of the passivity property. This reference assumes that only one of the species is subject to diffusion and modifies the secant criterion to become a synchronization condition.

With respect to previous work, the main contributions of the present paper are as follows: i) The results are obtained by using a purely input/output approach. This approach requires in principle minimal knowledge of the physical laws governing the systems, and is therefore particularly well-suited to applications displaying high uncertainty on parameters and structure, such as (molecular) biological systems. Results for systems with an “internal description”, i.e. in state space form, are derived as corollaries in the companion paper [1]; ii) The individual species are only required to satisfy an output-feedback passivity condition, compared to the stronger output-strict passivity condition in [17]; iii) The interconnections among the subsystems composing each network are not limited to cyclic topologies, thus enlarging the class of systems for which synchronization can be proved; iv) The diffusive coupling can involve more than one species; and v) The new formulation allows exogeneous signals, and studies their effect on synchronization.

The paper is organized as follows. In Section II the notation used throughout the paper is summarized, and the model under study is introduced. In Section III the main result is presented, and its proof can be found in Section IV. Finally, in Section V, we show that the synchronization condition can be expressed in terms of algebraic inequalities, for particular classes of interconnection structures. In a companion paper [1], the results of the present paper are extended to systems described by state-space models, and an application to synchronization analysis in biochemical networks is also discussed (see [18] for an online version).

II. PRELIMINARIES AND PROBLEM STATEMENT

We denote by $L_{2e}$ the extended space of signals $w : [0, \infty) \rightarrow \mathbb{R}$ which have the property that each restriction
where \( w_T = w|_{[0,T]} \) is in \( L_2(0,T) \), for every \( T > 0 \). Given an element \( w \in L_2e \) and any fixed \( T > 0 \), we write \( ||w||_T \) for the \( L_2 \) norm of the restriction \( w_T \), and given two functions \( v, w \in L_2e \) and any fixed \( T > 0 \), the inner product of \( v_T \) and \( w_T \) is denoted by \( \langle v, w \rangle_T \). The same notation is used for vector functions \(^1\).

Consider \( n \) identical \textit{compartments}, each composed of \( N \) subsystems that we refer to as \textit{species}. The input-output behavior of species \( k \) in compartment \( j \) is described by

\[
y_{k,j} = H_k v_{k,j}, \quad k = 1, \ldots, N, \quad j = 1, \ldots, n, \tag{1}
\]

where \( H_k \) is an operator to be further specified. The interconnections among species and compartments are given by:

\[
v_{k,j} = w_{k,j} + \sum_{i=1}^{N} \sigma_{k,i} y_{i,j} + \sum_{z=1}^{n} a_{k,z}^j (y_{k,z} - y_{k,j}), \tag{2}
\]

\( k = 1, \ldots, N, \quad j = 1, \ldots, n \), where the coefficients \( \sigma_{k,i} \in \mathbb{R} \), \( k, i = 1, 2, \ldots, N \), represent the interconnection between different species, and are identical in each compartment. These coefficients are grouped into an \( N \times N \) matrix:

\[
\Sigma := [\sigma_{k,i}], \quad k, i = 1, 2, \ldots, N, \tag{3}
\]

and the resulting interconnection is called \textit{species coupling}.

The scalars \( a_{k,z}^j, \quad k = 1, 2, \ldots, N, \quad j, z = 1, 2, \ldots, n \) are nonnegative and represent the interconnection among systems of the same species in different compartments. We will call this interconnection \textit{compartamental coupling}. We assume that there are no self-loops, i.e. \( a_{k,j}^j = 0 \), \( k = 1, 2, \ldots, N \), \( j = 1, 2, \ldots, n \). Note that different species can possess different coupling structures (as implied by the superscript \( k \) in \( a_{k,j}^j \)). The compartamental coupling is expressed in a diffusive-like form, as a function of the differences between species in the respective compartments, and not the species themselves. This is more general than true diffusion, which would correspond to the special case in which \( a_{k,z}^j = a_{z,j}^k \) for all \( k, j, z \); under this symmetry condition, the fluxes \( a_{k,z}^j (y_{k,z} - y_{k,j}) \) and \( a_{k,j}^z (y_{k,j} - y_{k,z}) \) (between the \( k \)th species in the \( j \)th and the \( z \)th compartments) would cancel each other out. Finally, the scalars \( w_{k,j} \) are external inputs that can model e.g., \( L_{2e} \) disturbances acting on the systems.

The resulting interconnected system can be represented as a graph as illustrated in Figure 1.

We denote by \( Y_k = [y_{k,1}, \ldots, y_{k,n}]^T \), \( V_k = [v_{k,1}, \ldots, v_{k,n}]^T \) and \( W_k = [w_{k,1}, \ldots, w_{k,n}]^T \) the vectors of the outputs, inputs and external signals of systems of the same species \( k \). Given a set of vectors \( Z_k, k = 1, 2, \ldots, N \), we indicate the stacked vector by \( Z := \text{col}(Z_1, \ldots, Z_N) \), e.g. we will indicate the stacked vector of outputs by \( Y := \text{col}(Y_1, \ldots, Y_N) \).

We then rewrite the feedback law (2) as

\[
V_k(t) = W_k(t) + \sum_{i=1}^{N} \sigma_{k,i} Y_i(t) - L_k Y_k(t), \quad k = 1, 2, \ldots, N, \tag{4}
\]

\(^1\)We will denote by \( L_{2e}^m \) the extended space of \( m \) dimensional signals.

![Fig. 1. Example of interconnection structure. Each compartment is composed by \( 3 \) subsystems (represented as nodes of a graph) each characterized by an operator \( H_k \), \( k = 1, 2, 3 \). Two subsystems of the same species in different compartments are connected by an edge whenever the corresponding coefficient \( a_{k,j}^z \) is positive.](image)

where \( L_k, k = 1, \ldots, N \), are Laplacian matrices associated to the compartamental coupling:

\[
l_{i,j}^k = \begin{cases} 
\sum_{z=1}^{n} a_{i,z}^k, & i = j \\
-a_{i,j}^k, & i \neq j.
\end{cases}
\]

The connectivity properties of the corresponding graphs are related to the algebraic properties of the Laplacian matrices and, in particular, to the notion of \textit{algebraic connectivity} extended to directed graphs in [19]:

\textbf{Definition 1:} For a directed graph with Laplacian matrix \( L_k \), the \textit{algebraic connectivity} is the real number defined as:

\[
\lambda_k := \min_{z \in \mathbb{R}^n} \frac{z^T L_k z}{z^T z}
\]

where \( \mathcal{P} = \{ z \in \mathbb{R}^n : z \perp 1_n, ||z|| = 1 \} \) and where \( 1_n := [1, 1, \ldots, 1]^T \in \mathbb{R}^n \).

To characterize synchronization, we denote the average of the outputs of the \( n \) copies of the species \( k \) by:

\[
\bar{Y}_k := \frac{1}{n} Y_k^\Delta,
\]

where \( 1_n := [1, 1, \ldots, 1]^T \in \mathbb{R}^n \), \( k = 1, 2, \ldots, N \), and define:

\[
Y_k^\Delta := \text{col}(y_{k,1} - \bar{Y}_k, \ldots, y_{k,n} - \bar{Y}_k). \tag{5}
\]

Because \( Y_k^\Delta \) is equal to zero if and only if \( Y_k = \alpha_k 1_n \) for some \( \alpha_k \geq 0 \) \( ||Y_k^\Delta||_T \) measures the synchrony of the outputs of the species \( k \) in the time interval \([0, T]\). The same notation is used to define the vectors \( V_k^\Delta \) and \( W_k^\Delta \).

We recall now an operator property that will be extensively used in the paper (the definitions are slightly adapted versions of those in [20], [21] and [11]).
Definition 2: Let $H : L^m_{2e} \to L^m_{2e}$. Then $H$ is relaxed cocoercive if there exists some $\gamma_c \in \mathbb{R}$ such that for every pair of inputs $u, v \in L^m_{2e}$
\[ \gamma_c \| Hu - Hv \|^2_T \leq (Hu - Hv, u - v)_T, \quad \forall T \geq 0. \] (6)
If (6) holds with $\gamma_c \geq 0$, then $H$ is called monotone. If (6) holds with $\gamma_c > 0$, then $H$ is called cocoercive.

Cocoercivity implies monotonicity and monotonicity implies relaxed cocoercivity. We refer to the maximum possible $\gamma_c$ with which (6) holds as the cocoercivity gain, and denote it as $\gamma$. The existence of $\gamma$ follows because the set of $\gamma_c$’s that satisfy (6) is closed from above. In particular, we will call $\gamma$-relaxed cocoercive the operators with a cocoercivity gain $\gamma \in \mathbb{R}$. Sometimes in the literature the terms relaxed cocoercive, monotone, and cocoercive are often referred as incrementally output-feedback passive, incrementally passive and incrementally output-passively respectively [4].

When there is no coupling between the compartments, i.e., $L_k = 0$, $k = 1, 2, \ldots, N$, the compartments are isolated and their stability depends on the species coupling. Stability with species coupling has been studied in [7] with an input-output approach, and in [8], [9] with a Lyapunov approach.

III. MAIN RESULT

The following theorem relates the properties of the interconnections and the operators to the synchrony of the outputs in the closed-loop system. In particular we show that, if the operators describing the open-loop systems are $\gamma$-relaxed cocoercive and the interconnection matrices satisfy certain algebraic conditions, the closed loop system has the property that external inputs with a “high” level of synchrony (as implied by a small $\|W^\Delta\|_T$) produce outputs with the same property (small $\|Y^\Delta\|_T$).

Theorem 1: Consider the closed loop system defined by (1)-(2). Suppose that the following assumptions are verified:
1) Each operator $H_k$ is $\gamma_k$-relaxed cocoercive as in Definition 2, $k = 1, 2, \ldots, N$.
2) For $k = 1, \ldots, N$, $\tilde{\gamma}_k := \lambda_k + \gamma_k > 0$, where $\lambda_k$ is the algebraic connectivity in Definition 1 associated to the matrix $L_k$ that describes the compartmental coupling of species $k$.
3) The dissipativity matrix \( E_{\tilde{\gamma}} := \Sigma - \Gamma_{\tilde{\gamma}} \) (7)
where $\Gamma_{\tilde{\gamma}} := \text{diag}(\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_N)$, $\tilde{\gamma} = \text{col}(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_N)$, is diagonally stable, i.e., there exists a diagonal matrix $D > 0$ such that
\[ E_{\tilde{\gamma}}^T D + D E_{\tilde{\gamma}} < 0. \]
Then, for all $w_{k,j}, y_{k,j}, k = 1, 2, \ldots, N, j = 1, 2, \ldots, n$ that satisfy (1) and (2) we have
\[ \|Y^\Delta\|_T \leq \rho \|W^\Delta\|_T, \quad \forall T \geq 0, \]
for some $\rho > 0$, and all $W \in L^{nN}_{2e}$. Moreover, if $W^\Delta \in L^{nN}_{2e}$, then $Y^\Delta \in L^{nN}_{2e}$, and we have $\|Y^\Delta\| \leq \rho \|W^\Delta\|$. \( \Box \)

From Theorem 1 we observe that the compartmental coupling increases the co-coercivity gain of a species whenever the algebraic connectivity is strictly positive. Since high values of cocoercivity gains tend to improve diagonal stability the compartmental coupling has a beneficial effect on synchronization. The algebraic connectivity and other properties of the Laplacian matrices can be related to topological properties of the underlying interconnection graph associated to the compartmental coupling. For example the condition $1^T L_k = 0$ is equivalent to assuming that the underlying graphs are balanced (i.e. that for each vertex the sum of the weights of the edges entering in one vertex is equal to the sum of the weights of the edges exiting from the same vertex). Under this condition the resulting Laplacian matrices are doubly hyperdominant with zero excess and $L_k + L^T_k$ are guaranteed to be positive semidefinite. This implies also that the algebraic connectivity $\lambda_k$ are non negative. It is also known that if there exists a spanning tree in the reversal of the graph, then the algebraic connectivity is guaranteed to be strictly positive (see e.g., [19]). Another condition for the algebraic connectivity to be strictly positive is that the graph is strongly connected and balanced.

IV. PROOF OF THE MAIN RESULT

We define the $(n-1) \times n$ matrix $Q$ that satisfies $Q_1 \equiv 0$, $QQ^T = I_{n-1}$, and $Q^T Q = I_n - \frac{1}{n} 1_n 1_n^T$.

By observing that
\[ \tilde{Y}_k := QY_k \]
is equal to zero for every $k = 1, \ldots, n$ if and only if $Y_k = \alpha_k 1_n$ for every $k = 1, \ldots, N$, for some $\alpha_k \geq 0$, it is evident that also $\|Y_k\|$ is a measure of synchrony for the outputs of the species (in different compartments) denoted by the index $k$. Moreover, since $Q^T Q Y_k = Y^\Delta_k$, $\tilde{Y}_k$ and $Y^\Delta_k$ are related by $Y^\Delta_k = Q^T \tilde{Y}_k$ and thus,
\[ \|Y_k\|^2_T = \int_0^T \tilde{Y}_k^T Q Q^T \tilde{Y}_k dt = \|\tilde{Y}_k\|^2_T, \quad k = 1, 2, \ldots, N. \] (8)

Accordingly we define $\tilde{U}_k = QU_k$, $\tilde{W}_k = QW_k$, $\tilde{V}_k = QV_k$.

Before proving Theorem 1 we present a preliminary Lemma:

Lemma 1: Consider the open-loop systems (1). If the operators $H_k$, $k = 1, 2, \ldots, N$ are $\gamma_k$-relaxed cocoercive then
\[ \gamma_k \|\tilde{Y}_k\|^2_T \leq (\tilde{Y}_k, \tilde{V}_k)_T, \quad k = 1, \ldots, N, \] for each $T > 0$ and every $V_k \in L^m_{2e}$. \( \Box \)

Proof: Consider the scalar product
\[ (\tilde{V}_k, \tilde{Y}_k)_T \]
and define $z_{k,j} = v_{k,j} - \gamma_k y_{k,j}$ for every $k, j$, that in vector form reads
\[ Z_k = V_k - \gamma_k Y_k. \] (11)
Define $\hat{Z}_k = QZ_k$. By substituting (11) in (10) we obtain
\[ (\hat{V}_k, \hat{Y}_k)_T = (\hat{Z}_k, \hat{Y}_k)_T + \gamma_k (\hat{Y}_k, \hat{V}_k)_T. \] (12)
We first claim that the term $\langle \tilde{Z}_k, \tilde{Y}_k \rangle_T$ is nonnegative. To show this, we use the $\gamma_k$-relaxed cocoercivity property of $H_k$ and obtain:

$$\langle z_{k,i} - z_{k,j}, y_{k,i} - y_{k,j} \rangle_T = \langle v_{k,i} - v_{k,j}, y_{k,i} - y_{k,j} \rangle_T - \gamma_k \langle y_{k,i} - y_{k,j}, y_{k,i} - y_{k,j} \rangle_T \geq 0,$$

for $i, j = 1, 2, \ldots, n$. By summing (13) over $i, j = 1, 2, \ldots, n$ and by dividing by a normalization constant we get

$$\frac{1}{2n} \sum_{i,j=1}^{n} \langle z_{k,i} - z_{k,j}, y_{k,i} - y_{k,j} \rangle_T = \langle Z_k, Y_k \rangle_T n - \langle \tilde{Z}_k, \tilde{Y}_k \rangle_T \geq 0,$$

which proves the claim. Finally, from (12) and (14) we conclude that

$$\langle \tilde{Z}_k, \tilde{Y}_k \rangle_T = \langle Z_k, Y_k \rangle_T n - \langle \tilde{Z}_k, \tilde{Y}_k \rangle_T \geq 0,$$

which is the desired inequality (9).

We are now ready to prove Theorem 1.

**Proof of Theorem 1:** Consider the inputs

$$V_k(t) = U_k(t) - L_k Y_k(t),$$

where $L_k$ are the Laplacian matrices representing the coupling between the compartments and the $U_k(t)$ are for now thought as external inputs. From Lemma 1 and substituting (15) in (9) we get,

$$\gamma_k \frac{1}{2} \| \tilde{Y}_k \|^2_T \leq \langle \tilde{Y}_k, \tilde{U}_k \rangle_T - \langle \tilde{Y}_k, QL_k Y_k \rangle_T.$$

Next, we note that $I_n - Q^T Q = \frac{1}{2} 1_n 1_n^T$ is a projection matrix onto the span of $1_n$. Because $L_k 1_n = 0$, it follows that $L_k (I_n - Q^T Q) Y_k = 0$ and thus,

$$L_k Y_k = L_k Q^T Q Y_k = L_k Q^T \tilde{Y}_k.$$

Using (17) as well as the fact that

$$Y_k^T Q L Q^T Q Y_k = Y_k^T Q L Q^T Q Y_k,$$

(because this expression is a scalar), we observe that:

$$\langle \tilde{Y}_k, Q L_k Y_k \rangle_T = \frac{1}{2} \int_0^T \tilde{Y}_k(t) Q (L_k + L_k^T) Q \tilde{Y}_k(t) dt \geq \lambda_k \int_0^T \tilde{Y}_k(t) \tilde{Y}_k(t) dt = \lambda_k \| \tilde{Y}_k \|^2_T.$$

were $\lambda_k$ are the smallest eigenvalues of the symmetric part of the “reduced Laplacian matrices”, i.e., of the matrices $(1/2) (Q (L_k + L_k^T) Q)^T$. By using the properties of the matrix $Q$ it is straightforward to check that $\lambda_k$ is the algebraic connectivity as defined in Definition 1. Combining (16) and (18) we obtain

$$\gamma_k \| \tilde{Y}_k \|^2_T \leq \langle \tilde{Y}_k, \tilde{U}_k \rangle_T - \lambda_k \| \tilde{Y}_k \|^2_T.$$

From Assumption 2 we have that $\lambda_k > -\gamma_k$ for $k = 1, 2, \ldots, n$. We conclude that

$$\| \tilde{Y}_k \|^2_T \leq \frac{1}{\gamma_k} \langle \tilde{Y}_k, \tilde{U}_k \rangle_T, \quad k = 1, 2, \ldots, N$$

where $\tilde{X}_k = X_k + \lambda_k$. The rest of the proof follows by applying the feedback

$$U_k = W_k + \sum_{j=1}^{N} \sigma_k Y_j, \quad k = 1, 2, \ldots, N$$

to the resulting system, where $\sum_{j=1}^{N} \sigma_k Y_j$ represents the interconnection between the different species. By defining $U = \text{col}(U_1, \ldots, U_N)$, we rewrite (20) as

$$U = W + (\Sigma \otimes I_n) Y.$$

From Assumption 3 the matrix $E_{\tilde{X}}$ defined in (7) is diagonally stable i.e. there exist positive constants $d_i, i = 1, \ldots, N$ such that $DE_{\tilde{X}} + E_{\tilde{X}}^T D < 0$, where $D = \text{diag}(d_1, \ldots, d_N)$. Choose $\alpha > 0$ such that $DE_{\tilde{X}} + E_{\tilde{X}}^T D < -2\alpha I_N$ and observe that

$$\langle Dz, E_{\tilde{X}} z \rangle_T = \frac{1}{2} \int_0^T z^T(t) (DE_{\tilde{X}} + E_{\tilde{X}}^T D) z(t) dt \leq -\alpha \| z \|^2_T.$$n

From (19) we can write $\langle \tilde{d}_k \tilde{Y}_k, \tilde{U}_k - \gamma_k \tilde{Y}_k \rangle_T \geq 0$, for $k = 1, 2, \ldots, N$, and therefore

$$\langle (D \otimes I_{n-1}) \tilde{Y}, \tilde{U} - (\Gamma_{\tilde{X}} \otimes I_{n-1}) \tilde{Y} \rangle_T \geq 0,$$

where $\tilde{U} = \text{col}(\tilde{U}_1, \ldots, \tilde{U}_N)$ and $\tilde{Y} = \text{col}(\tilde{Y}_1, \ldots, \tilde{Y}_N)$. Substituting $U = W + (\Sigma \otimes I_{n-1}) \tilde{Y}$, where $W = \text{col}(W_1, \ldots, W_N)$, we obtain

$$\langle (D \otimes I_{n-1}) \tilde{Y}, W + (E_{\tilde{X}} \otimes I_{n-1}) \tilde{Y} \rangle_T \geq 0,$$

and using the Cauchy-Schwartz inequality we write

$$\beta \| \tilde{W} \|_T \| \tilde{Y} \|_T \geq \langle (D \otimes I_n) \tilde{Y}, W \rangle_T \geq -\langle (D \otimes I_n) \tilde{Y}, (E_{\tilde{X}} \otimes I_n) \tilde{Y} \rangle_T \geq \| \tilde{Y} \|_T^2,$$

for some $\beta > 0$. We conclude that

$$\| \tilde{Y} \|_T \leq \rho \| \tilde{W} \|_T, \quad \forall T \geq 0,$$

for any $W \in L^{2n}_{2e}$, where $\rho = \beta/\alpha$. As a direct consequence, if $W \in L^2_{2e}$ then $\| \tilde{Y} \|_T \leq \rho \| \tilde{W} \|_T$. We conclude by observing that $\| \tilde{Y}_k \|_T = \| \tilde{Y}_k \|_T$ and $\| \tilde{W}_k \|_T = \| W^\Delta_k \|_T$ for every $T \geq 0$ and $k = 1, 2, \ldots, N$. ■
Since corresponds to 1 coupling (through the interconnection matrix \( \Sigma \)) and the species coupling (through the algebraic connectivity) and the species synchronally stable, which is related to both the compartmental systems are coupled through the output of the first system in \([17]\), for the special case in which the interconnected synchronization problem for cyclic systems was studied analyzed in \([8]\). Extending the work in \([8]\), the output and provide conditions for the matrix \( E_\gamma \) to be diagonally stable. These conditions take the form of inequalities that link the algebraic connectivities of the compartmental coupling with the relaxed cocoercivity gains of the operators \( H_k \).

Cyclic systems: Stability of isolated cyclic systems is analyzed in \([8]\). Extending the work in \([8]\), the output synchronization problem for cyclic systems was studied in \([17]\), for the special case in which the interconnected systems are coupled through the output of the first system only. Our approach is suitable for more general coupling (the interconnection structure is depicted in Figure 2). The interconnection matrix is

\[
\Sigma_{\text{cyclic}} = \begin{bmatrix}
0 & 0 & \cdots & 0 & -1 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
\end{bmatrix}
\]

and the dissipativity matrix is therefore

\[
E_\gamma = \begin{bmatrix}
-\gamma_1 & 0 & \cdots & 0 & -1 \\
1 & -\gamma_2 & 0 & \cdots & 0 \\
0 & 1 & -\gamma_3 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 -\gamma_N \\
\end{bmatrix}
\]

For this matrix to be diagonally stable the following secant condition must be satisfied \([8]\):

\[
\frac{1}{\gamma_1} \frac{1}{\gamma_2} \cdots \frac{1}{\gamma_N} < \sec(\pi/N)^N.
\]

Since \( \gamma_k = \gamma_k + \lambda_k > 0 \), the secant condition leads to:

\[
\prod_{k=1}^{N} \frac{1}{\gamma_k + \lambda_k} < \sec(\pi/N)^N. \quad (21)
\]

Our approach generalizes the result of \([17]\) (note that \( \gamma_k \) correspond to \( 1/\gamma_k \) in \([17]\)) where the coupling among the systems is limited to the first system (i.e., when \( \lambda_j = 0 \), \( j = 2, \ldots, N \) in \((21))\). In fact, in this case \((21)\) reduces to

\[
\lambda_1 > \frac{\cos(\pi/N)^N}{\gamma_2 \cdots \gamma_N} - \gamma_1, \quad (22)
\]

which is the expression provided in \([17]\).

As an example, consider the case where each species in a compartment is directly connected to the respective species in each other compartment with the same weight \( q \), i.e. \( a_{ij}^k = q \) for every \( i, j = 1, \ldots, n \) and \( k = 1, \ldots, N \). This implies that the Laplacian matrices are

\[
L_k = q n \left( I_n - \frac{1}{n} I_n^T \right), \quad k = 1, 2, \ldots, N,
\]

and that \( \lambda_k = q n \), \( k = 1, \ldots, N \). In this case, \((21)\) specializes to:

\[
\prod_{k=1}^{N} \frac{1}{\gamma_k + q n} < \sec(\pi/N)^N,
\]

where, since \( \lambda_k + \gamma_k \) must be strictly positive, the condition \( q > -\gamma_k/n \), \( k = 1, 2, \ldots, N \) must be satisfied. If we restrict the compartmental coupling to only the first species, \((22)\) takes the simple form

\[
qn > \frac{\cos(\pi/N)^N}{\gamma_2 \cdots \gamma_N} - \gamma_1.
\]

Branched structures: In \([9]\) several interconnection structures have been analyzed and diagonal stability is proven for the associated dissipative matrices.

i) For the interconnection structure depicted in Figure 3, the interconnection and dissipativity matrices are, respectively,

\[
\Sigma_{bl} = \begin{bmatrix}
0 & 0 & 0 & -1 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix},
\]

\[
E_{\gamma}^{bl} = \begin{bmatrix}
-\gamma_1 & 0 & 0 & -1 & 0 & 0 & -1 \\
1 & -\gamma_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -\gamma_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -\gamma_4 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -\gamma_5 & 0 & 0 \\
0 & 0 & 0 & 1 & -\gamma_6 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -\gamma_7 \\
\end{bmatrix}.
\]


Lemma 2 in [22] shows that $E^{b1}_\gamma$ is diagonally stable iff the condition:

$$ \frac{1}{\gamma_1 \gamma_2 \gamma_3 \gamma_4} + \frac{1}{\gamma_1 \gamma_5 \gamma_6 \gamma_7} < \sec(\pi/4)^4 $$

holds. Since $\gamma_k = \gamma_k + \lambda_k$, the synchronization condition becomes:

$$ \frac{1}{\gamma_1 + \lambda_1} \left( \prod_{k=2}^{4} \frac{1}{\gamma_k + \lambda_k} + \prod_{k=5}^{7} \frac{1}{\gamma_k + \lambda_k} \right) < \sec(\pi/4)^4. $$

(23)

If we limit the coupling to the first species only, (23) reduces to:

$$ \lambda_1 > \frac{7 \gamma_3 \gamma_4 + \gamma_5 \gamma_6 \gamma_7}{\gamma_2 \gamma_3 \gamma_4 \gamma_5 \gamma_6 \gamma_7} \cos(\pi/4)^4 - \gamma_1. $$

(24)

ii) For the interconnection structure depicted in Figure 4, the interconnection matrix is:

$$ \Sigma_{b2} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, $$

and therefore the dissipativity matrix is:

$$ E^{b2}_\gamma = \begin{bmatrix} -\gamma_1 & 0 & 0 & 1 & 0 \\ 1 & -\gamma_2 & -1 & 0 & 0 \\ 0 & 1 & 0 & -\gamma_3 & 0 \\ 0 & 0 & 1 & 0 & -\gamma_4 \end{bmatrix}. $$

The analysis in [9] gives the sufficient condition

$$ \frac{1}{\gamma_1 \gamma_2 \gamma_4} + \frac{1}{\gamma_4 \gamma_5} < 1, $$

which leads to

$$ \frac{1}{\gamma_4 + \lambda_1} \left( \frac{1}{(\gamma_1 + \lambda_1)(\gamma_2 + \lambda_2)} + \frac{1}{\gamma_5 + \lambda_5} \right) < 1. $$

(25)

If we limit the coupling to the first species only, (25) reduces to:

$$ \lambda_1 > \frac{\gamma_5}{\gamma_2(\gamma_4 \gamma_5 - 1)} - \gamma_1. $$

(26)

VI. CONCLUSION AND FUTURE WORK

Synchronization conditions for networks of nonlinear systems have been investigated combining the input-output properties of the subsystems with the information about the structure of network. The proposed model is motivated by cellular networks where signaling occurs both internally, through interactions of species, and externally, through intercellular signaling. The companion paper [1] explores results for state-space models as well as biochemical applications.

REFERENCES


