Robust mask-constrained linear array synthesis through an interval-based particle SWARM optimisation

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Abstract: An innovative strategy for the robust design of linear antenna arrays is presented. Being the array elements characterised by tolerance errors, the synthesis is aimed at determining the intervals of values fitting the user-defined mask constraints on the radiated power pattern. With reference to the upper and lower bounds of the power pattern analytically determined for given tolerances through interval analysis, the nominal excitations of the array elements are then optimised by means of a global stochastic optimiser suitably customised to deal with interval numbers. A set of numerical examples is reported to show the behaviour of the proposed method as well as to assess its potentials in dealing with the robust synthesis of pencil and shaped beams.

1 Introduction

Antenna array synthesis problems are aimed at defining the geometrical and electrical descriptors of the radiating structure to fit the required radiation properties and, if needed, additional user-defined design constraints. The synthesis parameters usually are the excitation weights (i.e. amplitude, phase coefficients and time delays) of the beam forming network (BFN) and the element positions or spacing in case of elements located on a regular grid. In several decades of research activity, a wide set of analytical and numerical techniques has been proposed based on closed-form mathematical relationships [1–4] and iterative optimisation strategies of deterministic [5–10], stochastic [11–19] or hybrid [20–23] nature. Usually, these methods assume that the values of the synthesis parameters vary with an arbitrary level of accuracy and with continuity. In some cases, a quantisation of the excitation values has been taken into account to encompass the use of digital amplifiers or phase shifters [24]. In both cases, ad hoc design strategies have been implemented and tested where the unknowns have been properly coded using either real values of a discrete alphabet (e.g. binary) or symbols. Moreover, suitable operators able to iteratively generate new trial solutions have been adopted to directly address continuous [16, 17] or combinatorial problems [11, 13, 14].

Despite the potentialities of such approaches and the possibility to ideally yield whatever beam shape is, the realisation of the antenna implies unavoidable inaccuracies, tolerances and errors in the manufacturing processes [25]. Therefore, it is almost impossible from a practical viewpoint to set the array control points to the desired excitations defined according to standard synthesis techniques. As a consequence, the field radiated by the real array can have a pattern different from the expected one. For example, arrays designed to afford low/very-low secondary lobes or nulls are extremely sensitive to errors in the levels of amplitude or phase shifts occurring in the BFN [25, 26]. To cope with these drawbacks, time-consuming and complex antenna calibrations are generally needed. Moreover, each antenna sample has to be calibrated although in a mass-production process when the requirements on the pattern tolerance are strict as in some challenging applications.

Since the tolerances in realising real antenna arrays are not exactly known, statistical strategies have been exploited for evaluating the impact on the principal pattern features, namely the sidelobe level (SLL), the directivity and the mainlobe direction. In [27], the influence of random errors in equally spaced linear arrays has been evaluated supposing all elements having the same excitation amplitude errors, the phase errors being equally probable. The method has been then extended to consider errors proportional to the nominal excitation amplitudes generating a reference/nominal pattern [28]. Afterwards, the effects of mechanical positioning errors have been analysed [29, 30] and the analysis of two-dimensional scanning arrays has been carried out [31], as well. Many other studies have been performed and presented in the state-of-the-art literature dealing with the dependence of the far-field pattern on random errors in the currents of an array of thin-wire dipoles [32], the effects on the SLL and gain of a space-fed array [33] and the degradation of the effective isotropic radiated power [34]. Moreover, techniques for robust source localisation [35] have also been proposed.

The impossibility to control the values of the array control points with an arbitrary accuracy and the unavoidable presence of unknown realisation errors in the manufacturing...
processes have induced researchers to define ad hoc robust beam-forming techniques to keep reliable communication channels also in the presence of random errors on the excitation weights [36–39] and the element positions [40]. Probabilistic synthesis methods have also been introduced [41, 42]. In this latter framework, a Monte Carlo method has been recently proposed to estimate the maximum tolerance of the amplitudes and phases for fitting tolerance requirements on the arising radiation pattern [43]. The effects of each error have been statistically evaluated [43], as well. Despite the enhanced efficiency with respect to enumerative and deterministic techniques aimed at computing the pattern tolerance by subdividing the interval errors in multiple subsets and then evaluating the resulting beam by testing all possible combinations of error values in the array elements, the number of beam pattern evaluations still turns out being non-negligible [43] despite for small/medium arrays. Moreover, because of the statistical selection of the combinations of excitation errors used to evaluate the effects on the radiation pattern, the method is not able to analyse all possible solutions with an intrinsic limitation in predicting the robustness of the final design.

To overcome such negative features, an innovative synthesis method based on interval analysis (IA) is presented in this work where the bounds of all possible patterns generated by an array, within given tolerances on the element amplitudes, are computed by means of analytical rules. As a matter of fact, the arithmetic of intervals, available in IA, allows one to analytically address arithmetic problems where the variables at hand are not characterised by single points but intervals. Initially introduced to compute the error bounds on the rounding operations in numerical computation [44, 45], IA has been successively used to deal with the analysis of complex intervals [46, 47], the solution of linear and non-linear systems [48] and the optimisation of functions [49, 50]. In electromagnetics, despite the wide number of problems where tolerances and errors can cause severe degradations of the performances, IA has been applied, to the best of authors’ knowledge, in only few cases (e.g. [51–55]).

In this work, the mask-constrained power synthesis of linear arrays is reformulated within the IA framework. Accordingly, the amplitude excitations are defined in terms of intervals of width corresponding to the maximum manufacturing tolerance around the nominal values. These latter are successively optimised by means of a stochastic global optimiser [56] to yield, at the convergence, the bounds of the arising power patterns laying within user-defined masks.

The paper is organised as follows: The array synthesis is formulated in Section 2, where the IA-based analysis tool (Section 2.1) and the interval-based optimisation algorithm (Section 2.2) are presented, as well. A set of numerical examples is reported and discussed in Section 3 to show the effectiveness of the proposed method. Eventually, some conclusions are drawn in Section 4.

## 2 Mathematical formulation

The mathematical expression of the array factor of a linear array with $N$ elements and uniform spacing $d$ along the array axis is

$$ AF(u) = \sum_{n=0}^{N-1} a_n e^{j2\pi d n u} $$

(1)

where $\Theta_n(u) = (n\delta d u + \phi_n)$, $\beta = (2\pi/\lambda)$ being the free-space wavenumber, $\lambda$ is the wavelength, and $u = \sin \phi$, $\phi$ being the angle measured from the direction orthogonal to the array axis. Moreover, $\Delta_n$, $n = 0, \ldots, N - 1$ are the amplitude and phase weights of the array elements.

Dealing with a mask-constrained power synthesis problem and a fixed antenna geometry (i.e. given $d$), the objective is to determine the values of the array excitations such that the power pattern, $PP(u) = |AF(u)|^2$, satisfies the following relationship

$$ \text{LM}(u) \leq PP(u) \leq \text{UM}(u) $$

(2)

LM$(u)$ and UM$(u)$ being positive functions mathematically describing the user-defined lower and upper constraints. The solution of such a problem can be yielded by minimising the cost function $\Phi(I) = \Phi^\text{int}(I) + \Phi^\text{sup}(I)$, where

$$ \Phi^\text{int}(I) = \int_{-1}^{1} \{ \text{LM}(u) - PP(u) \} H \{ \text{LM}(u) - PP(u) \} du $$

(3)

and

$$ \Phi^\text{sup}(I) = \int_{-1}^{1} \{ PP(u) - \text{UM}(u) \} H \{ PP(u) - \text{UM}(u) \} du $$

(4)

where $I = \{ I_n = a_n e^{j\phi_n}; n = 1, \ldots, N \}$ and $H\{ \circ \}$ is the Heaviside step function defined as $H\{ \circ \} = 1$ when $\circ \geq 0$ and $H\{ \circ \} = 0$, otherwise.

Owing to the presence of unavoidable manufacturing errors, the optimal nominal values, $I^\text{nom}$, determined from the minimisation of $\Phi$ cannot be exactly realised. This latter inconvenience unavoidably impacts on the shape of the radiated power pattern as well as on the fitting of the pattern constraints. In order to guarantee that the array affords a pattern laying within the desired masks despite the presence of excitation tolerances, a different synthesis approach is needed. More specifically, the proposed synthesis method is aimed at determining the values of the nominal excitations, $a^\text{best}_n$, for all $n = 0, \ldots, N - 1$, such that the arising power pattern satisfies the pattern constraints in the presence of manufacturing tolerances within the intervals $a^\text{best}_n - \varepsilon^\text{int}_n \leq a^\text{best}_n \leq a^\text{best}_n + \varepsilon^\text{sup}_n$, $n = 0, \ldots, N - 1$. Towards this aim, a suitable analysis tool for the fast (to enable the use of iterative optimisation techniques) and exhaustive (to encompass all possible patterns that can be generated by the array) computation of the tolerances on the power pattern in terms of the maximum errors on the array excitation is first required. Accordingly, the arithmetic of intervals is adopted [46, 47]. Moreover, an ad hoc optimisation technique able to generate a set of trial solutions represented by intervals of real values instead of real numbers, where the IA-based analysis tool could be easily integrated, and to deal with the optimisation of multiple-minima functionals is necessary. These steps will be detailed in the following.

### 2.1 Analysis method: the interval analysis approach

The peculiarity of IA is the capability to analytically deal with intervals of values [50]. As for the problem at hand, the levels of amplification/attenuation implemented through the BFN are characterised by known tolerances, $\varepsilon^\text{int}$ and $\varepsilon^\text{sup}$.
n = 0, . . . , N − 1, around the nominal values \( a_n, n = 0, . . . , N − 1 \) where the amplifiers/attenuators are supposed to work. Hence, the amplitude weights that are actually realised by the array can assume whatever value within the intervals (Fig. 1)

\[
[a_n] = [a_n^{-\text{inf}}, a_n^{+\text{sup}}], \quad n = 0, . . . , N − 1
\]  

(5)

\( [a_n] \) being real-valued intervals. Each nth interval in (5) is completely characterised by its end-points, namely \( a_n^{-\text{inf}} \triangleq a_n - e_n^{-\text{inf}} \) and \( a_n^{+\text{sup}} \triangleq a_n + e_n^{+\text{sup}} \), respectively. The expression of the array factor (1) when considering (5) is given by

\[
[\text{AF}(u)] = \sum_{n=0}^{N-1} [a_n] e^{j\phi_n(u)}
\]

(6)

and the corresponding power pattern turns out to be

\[
[\text{PP}(u)] = \text{Re}^2\{[\text{AF}(u)]\} + \text{Im}^2\{[\text{AF}(u)]\}
\]

(7)

By exploiting the rules of complex interval arithmetic [46, 47], it is possible to explicit [\text{PP}(u)], that is, its end-points \([\text{PP}^{-\text{inf}}(u)]\) and \([\text{PP}^{+\text{sup}}(u)]\), in terms of the nominal amplitudes, \( a = \{a_n; n = 0, . . . , N - 1\} \) and the tolerance sets, \( e^{+\text{sup}} = \{e_n^{+\text{sup}}; n = 0, . . . , N - 1\} \) and \( e^{-\text{inf}} = \{e_n^{-\text{inf}}; n = 0, . . . , N - 1\} \), as summarised in Appendix [57].

2.2 Optimisation strategy: the interval-based particle swarm optimiser

As for the optimisation of the nominal values of the amplitude weights \( a = \{a_n; n = 0, . . . , N - 1\} \) and the phase delays \( \phi = \{\phi_n; n = 0, . . . , N - 1\} \) to fit the power pattern constraints \( \text{LM}(u) \) and \( \text{UM}(u) \), an interval-based particle swarm optimisation (PSO) algorithm is used. Accordingly, let us consider a swarm of \( P \) particles \( p_i, i = 1, . . . , I \). Each particle \( p_i = \{([a_{i,n}], \phi_{i,n}); n = 0, . . . , N - 1\} \) is descriptively of the set of \( N \) amplitudes, with tolerances included, \( [a_n] = [a_n^{-\text{inf}}, a_n^{-\text{sup}}] \), and \( N \) phase weights, \( \phi_{i,n}, n = 0, . . . , N - 1 \). The particles are iteratively updated by changing their positions within the solution space from \( p_i^{(k)} \) and \( p_i^{(k+1)} \), \( i = 1, . . . , I \), \( k \) being the iteration index, according to the following procedure:

Step 0: Initialisation \((k = 0)\) – Generate a swarm of \( I \) trial solutions \( p_i^{(0)}, i = 1, . . . , I \) with associated positions \( ([a_{i,n}], \phi_{i,n}), n = 0, . . . , N - 1, i = 1, . . . , I \). These latter are randomly generated within user-defined boundaries, \( a_n^{-\text{inf}} \in [a_n^{-\text{min}}, a_n^{-\text{max}}], \phi_{i,n} \in [\phi_{n,i}^{-\text{min}}, \phi_{n,i}^{-\text{max}}], n = 0, . . . , N - 1, i = 1, . . . , I \) or defined around a set of reference weights, \( (a_n^{\text{ref}}, \phi_{n,i}^{\text{ref}}), n = 0, . . . , N - 1 \), but still lying within the admissibility bounds. Moreover, randomly define the particle velocities \( v_i^{(k)}(0), i = 1, . . . , I \) with the same process used for the randomly generated particle vectors. Set the values of the inertial weight \( w \), the cognitive \( C_1 \) and the social \( C_2 \) acceleration coefficients.

Step 1: Interval-based optimisation process

Step 1.1: Cost function evaluation – For each particle \( p_i^{(k)}, i = 1, . . . , I \), evaluate the upper \( (\text{PP}^{+\text{sup}}(u))^{(k)} \) and lower \( (\text{PP}^{-\text{inf}}(u))^{(k)} \) bounds of the corresponding power pattern according to (12)–(16). Analogously to (3) and (4), compute the cost function value

\[
\Phi_i^{(k)} = \Phi(p_i^{(k)}) = \Phi^{-\text{inf}}(p_i^{(k)}) + \Phi^{+\text{sup}}(p_i^{(k)})
\]

being

\[
\Phi^{-\text{inf}}(p_i^{(k)}) = \int_{-\infty}^{\infty} \left[ \text{LM}(u) - (\text{PP}^{-\text{inf}}(u))^{(k)}_{i} \right] \times H(\text{LM}(u) - (\text{PP}^{-\text{inf}}(u))^{(k)}_{i}) du
\]

\[
\Phi^{+\text{sup}}(p_i^{(k)}) = \int_{-\infty}^{\infty} \left[ (\text{PP}^{+\text{sup}}(u))^{(k)} - \text{UM}(u) \right] \times H((\text{PP}^{+\text{sup}}(u))^{(k)} - \text{UM}(u)) du
\]

(8)

the two terms measuring the deviation of the upper and lower bounds of the power pattern from the upper and lower mask-constraints, respectively, as shown in Fig. 2.

Step 1.2: Personal and global solution update – Compare the cost function value of each particle \( \Phi_i^{(k)}, i = 1, . . . , I \), to the best value previously achieved by the same particle, \( \Phi(b_i^{(k-1)}) = \min_{i=1}^{I} \Phi_i^{(k-1)} \) and set the personal best as \( b_i^{(k)} = p_i^{(k)} \) if \( \Phi_i^{(k)} \leq \Phi(b_i^{(k-1)}) \) and \( b_i^{(k)} = b_i^{(k-1)} \) otherwise.
Moreover, update the global best solution found by the swarm, $g^{(k)} = \arg\{\min_{i=1,...,N} \Phi(h^{(k)})\}$.

Step 1.3: Convergence check – Stop the optimisation process when $\Phi(h^{(k)}) < \Phi_{th}$, $\Phi_{th}$ being a user-defined threshold, or a maximum number of iterations $K_{\text{max}}$ is reached. Then, go to Step 2. Otherwise, update the iteration index $k = k + 1$ and go to Step 1.4.

Step 1.4: Particle position update – Update the velocity of each particle [57]

$$v_{n,i}^{(k)} = wv_{n,i}^{(k-1)} + C_{1}r_{1}(b_{n,i}^{(k-1)} - p_{n,i}^{(k-1)}) + C_{2}r_{2}(g_{n}^{(k-1)} - p_{n,i}^{(k-1)})$$  \(\text{(9)}\)

where $r_{1}, r_{2} \in [0; 1]$ are uniform random numbers. Then, compute the new trial solutions by modifying the particle positions

$$p_{n,i}^{(k)} = p_{n,i}^{(k-1)} + v_{n,i}^{(k)}, \quad n = 0, \ldots, N - 1; \quad i = 1, \ldots, I.$$  \(\text{(10)}\)

Step 1.5: Admissibility check – To consider feasible solutions that can be implemented in practice, the values of the amplitude and phase weights are set to the boundaries of the solution space, namely $a_n^{\text{min}}$ and $a_n^{\text{max}}$ and/or $\psi_n^{\text{min}}$ and $\psi_n^{\text{max}}$, when $a_n^{(k)}$, $\psi_n^{(k)}$ are outside the admissible solution space. In such case, the sign of $v_{n,i}^{(k)}$ is inverted by applying the ‘reflecting wall’ boundary condition [58]. Then, go to Step 1.1.

Step 2: Setup array configuration – Set the array control points to the values $(a_n^{\text{best}}, \psi_n^{\text{best}})$, $n = 0, \ldots, N - 1$, defined in $g^{(k)}$.
3 Numerical results

The proposed synthesis approach is validated by reporting and discussing representative results from a wide set of numerical simulations. In the former part, the behaviour of the interval-based PSO algorithm is analysed throughout the iterative optimisation process. Successively, the robust design of sum beams is taken into account by considering different tolerance levels of the array control points in correspondence with the same power pattern constraints. Finally, flat-top beams are also synthesised to point out the flexibility of the method in dealing with different pattern masks. Without loss of generality, the condition

\[ \inf_n \alpha_n = 1, \sup_n \alpha_n = d, n = 0, \ldots, N - 1 \]

has been assumed throughout the whole numerical assessment. Moreover, the boundaries of the solution space have been always set to

\[ \alpha_\min_n = 0, \alpha_\max_n = 1, \\omega_\max_n = -\omega_\min_n \]

3.1 Interval-based PSO validation

Let us consider the mask constraints on the power pattern shown in Fig. 3a where the width of the main lobe of the upper mask UM(u) has been set to \( BW_{UM} = 0.46[a] \) and the level of the secondary lobes to \( SLL_{UM} = -20 \text{ dB} \). Concerning the lower mask, LM(u), the lower bound in the mainlobe region has been chosen \( \Gamma_{LM} = 5 \text{ dB} \) below that of the upper mask, while the inner mask width has been assumed equal to \( BW_{LM} = 0.25[a] \). As for the linear array, \( N = 10 \) elements spaced by \( d = (\lambda/2) \) have been considered with amplifiers at the BFN control points that guarantee a tolerance of \( \delta \alpha_n = (1/100) \alpha_n, n = 0, \ldots, N - 1 \), around the nominal amplitude \( \alpha_n \). Because of the symmetric masks, the phases have been set to \( \varphi_n = 0, n = 0, \ldots, N - 1 \). As far as the interval-based PSO is concerned, it has been run with a swarm of \( I = 20 \) particles, inertial weight set to \( w = 0.4 \) and cognitive and social acceleration coefficients equal to \( C_1 = C_2 = 2 \). Moreover, the maximum number of iterations and the threshold on the cost function have been chosen equal to \( K_{\text{max}} = 200 \) and \( \Phi_{\text{th}} = 10^{-5} \), respectively.

The simulation stopped after \( K = 19 \) iterations, when \( \Phi_{(k)}^{\text{best}} < \Phi_{\text{th}} \), in a total computational time of 33.7 [sec] by using a standard CPU (2.4 GHz PC with 2 GB of RAM) and a non-optimised source code. The values of the cost function in correspondence with the global best solution, \( \Phi_{(k)}^{\text{best}} = \Phi_{(k)}^{\text{g}} \), are shown in Fig. 4 where the two terms

\[ (\Phi_{(k)}^{\text{inf}})^{\text{m}} \approx \Phi_{(k)}^{\text{inf}}(g^{(k)}) \text{ and } (\Phi_{(k)}^{\text{sup}})^{\text{m}} \approx \Phi_{(k)}^{\text{sup}}(g^{(k)}) \]

and, \( k = 1, \ldots, K \) are explicitly indicated. As it can be observed, \( \Phi_{(k)}^{\text{inf}} \approx (\Phi_{(k)}^{\text{inf}})^{\text{m}}, k = 1, \ldots, K \), while \( (\Phi_{(k)}^{\text{sup}})^{\text{m}} \) has values smaller than \( 10^{-20} \) for most of the iterations. Lower
and upper bounds of the power pattern generated by the best PSO-solution are reported in Fig. 3a, while the distribution of the corresponding nominal amplitudes, \( a_{\text{best}} = a_n, n = 0, \ldots, N - 1 \) and interval errors \( [a_n] = [a_n - \delta a_n, a_n + \delta a_n], n = 0, \ldots, N - 1 \), are given in Fig. 3b and Table 1. Fig. 3a shows that the impact of tolerances equal to \( \delta a_n = 1\% \), \( n = 0, \ldots, N - 1 \), is negligible on the mainlobe, while the effects are much more important in the sidelobe region, especially close to the end-fire directions (i.e. \( u \approx \pm 1 \)), where the distance between the upper and the lower bounds of the power pattern interval increases up to 3 dB.

For illustrative purposes, trial power pattern intervals corresponding to the best solutions found by the PSO at the initialisation \( (k = 0) \) and at the iteration \( k = 4 \) are shown in Fig. 5. The cost function value in correspondence with these intermediate solutions amounts to \( \Phi_{k=0} = 2.39 \times 10^{-2} \) and \( \Phi_{k=4} = 2.50 \times 10^{-4} \), while \( \Phi_{k=0} < 10^{-22} \). As a matter of fact, the power pattern intervals are mainly outside the constraints at the initialisation, while they fully fit the masks at convergence.

### 3.2 Robust design of pencil beams

To assess the effectiveness and the limitations of the synthesis method in determining suitable solutions for different array configurations with different tolerances on the excitations under desired mask constraints, let us consider the masks characterised by \( BW_{\text{UM}} = 0.22 [u] \), \( SL_{\text{LM}} = 5 \text{ dB} \), \( BW_{\text{LM}} = 0.10 [u] \), and various tolerance intervals, \([a_n], n = 0, \ldots, N - 1\). As for the PSO, \( I = 2 \times N = 32 \) particles have been used with the same control parameters of Section 3.1, but now setting \( K_{\text{max}} = 400 \). At \( K = K_{\text{max}} \), the best value of the cost function is equal to \( \Phi_{\text{best}} = 2.8 \times 10^{-3} \) much larger than \( \Phi_{k=0} \). Indeed, the power pattern interval generated by the distribution in Fig. 7a does not satisfactorily match the pattern constraints as shown in Fig. 6a where the highest peaks of the

![Figure 8](image1.png)

**Fig. 8** Example 3 (\( d = \lambda/2 \)), \( \delta a_n = 5\% \), \( SL_{\text{UM}} = -20 \text{ dB} \), \( I_{\text{LM}} = 5 \text{ dB} \), \( BW_{\text{UM}} = 0.22 [u] \), \( BW_{\text{LM}} = 0.10 [u] \). (a) Upper (PP\text{sup}(u)) and lower (PP\text{inf}(u)) bounds of the optimised power pattern interval. (b) When \( N = 18 \), \( b, N = 20 \)

![Figure 9](image2.png)

**Fig. 9** Example 3 (\( d = \lambda/2 \)), \( \delta a_n = 5\% \), \( SL_{\text{UM}} = -20 \text{ dB} \), \( I_{\text{LM}} = 5 \text{ dB} \), \( BW_{\text{UM}} = 0.22 [u] \), \( BW_{\text{LM}} = 0.10 [u] \). Nominal amplitudes \( a_n \), \( n = 0, \ldots, N - 1 \), and tolerance intervals, \([a_n], n = 0, \ldots, N - 1\) (Fig. 6). At first, let the linear array be a \( N = 16 \) half-wavelength \( (d = \lambda/2) \) spaced arrangement and the maximum amplitude tolerance equal to \( \delta a_n = 1\% \), \( n = 0, \ldots, N - 1 \). As for the PSO, \( I = 2 \times N = 32 \) particles have been used with the same control parameters of Section 3.1, but now setting \( K_{\text{max}} = 400 \). At \( K = K_{\text{max}} \), the best value of the cost function is equal to \( \Phi_{\text{best}} = 2.8 \times 10^{-3} \) much larger than \( \Phi_{k=0} \). Indeed, the power pattern interval generated by the distribution in Fig. 7a does not satisfactorily match the pattern constraints as shown in Fig. 6a where the highest peaks of the

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**Table 2** Example 2-3-4 (\( d = \lambda/2 \)); \( SL_{\text{UM}} = -20 \text{ dB} \), \( I_{\text{LM}} = 5 \text{ dB} \), \( BW_{\text{UM}} = 0.22 [u] \), \( BW_{\text{LM}} = 0.10 [u] \). Values of \( \Phi_{\text{best}} \) at the convergence for different array sizes, \( N \), and various tolerance errors, \( \delta a_n \).
secondary lobes of $\text{PP}^{\text{sup}}(u)$ are equal to $-17.4$ dB, that is 2.6 dB above the desired level.

On the contrary, the solution optimised after $K=275$ iterations for an $N=18$ array allows one to yield a suitable power pattern interval as indicated in Fig. 6b and confirmed by the corresponding cost function value ($\Phi^{(K)}_{\text{best}} = 2.1 \times 10^{-6}$). The distributions of the nominal amplitudes and error tolerances are shown in Fig. 7b. In a successive test, still considering $N=18$ elements, amplifiers with higher tolerances (i.e. lower costs) have been considered by setting $\delta_{a_n} = 5\%$, $n = 0, \ldots, N-1$. Because of the complexity of the problem at hand, the PSO synthesis is not able to synthesise amplitudes (Fig. 9a) affording a pattern fully fitting the mask constraints (Fig. 8a) since $\Phi^{(K)}_{\text{best}} = 9.4 \times 10^{-6}$, and the pattern interval (Fig. 8b) radiated by the optimal excitation weights (Fig. 9b) lay within the desired masks although the peak values of $\text{PP}^{\text{sup}}(u)$ and $\text{PP}^{\text{inf}}(u)$ differ by 0.86 dB. To quantitatively evaluate the whole gap between the upper and the lower limits of the power pattern interval, let us define the pattern tolerance index $\Delta$

$$\Delta = \int_{1}^{-1} (\text{PP}^{\text{sup}}(u) - \text{PP}^{\text{inf}}(u)) \, du \quad (11)$$

As expected, the tolerance on the power pattern increases with the amplitude error

$$\Delta_{N=20}^{\delta_{a_n}=5\%} = 3.36 \times 10^{-2} \text{ against } \Delta_{N=18}^{\delta_{a_n}=1\%} = 0.83 \times 10^{-2}$$

Further increasing the amplifiers tolerance (i.e. $\delta_{a_n} = 8\%$, $n = 0, \ldots, N-1$), an array with $N=26$ elements is enough to satisfy the mask constraints on the power pattern with a value of the pattern tolerance further increased to $\Delta_{N=26}^{\delta_{a_n}=8\%} = 4.30 \times 10^{-2}$.

In summary, it results that more array elements are needed to balance the use of low complex amplifiers for fitting the required radiation performance as pointed out in Table 2 and Fig. 10 where the convergence values of $\Phi^{(K)}_{\text{best}}$ are reported.

3.3 Robust design of shaped beams

To analyse the method performance in dealing with flat-top beams, let the mask constraints be $\text{BW}_{\text{UM}} = 0.829[u]$, $\text{SLL}_{\text{UM}} = -20$ dB, $\text{BW}_{\text{LM}} = 0.716[u]$ and $\Gamma_{\text{LM}} = 4$ dB.

![Image](image-url)

**Fig. 10** Example 2-3-4 ($d = (\lambda/2)$; $\text{SLL}_{\text{UM}} = -20$ dB, $\Gamma_{\text{LM}} = 5$ dB, $\text{BW}_{\text{UM}} = 0.22[u]$, $\text{BW}_{\text{LM}} = 0.10[u]$) – behaviour of the value of $\Phi^{(K)}_{\text{best}}$ at the convergence against the array size, $N$, for different tolerance errors, $\delta_{a_n} = [1, 5, 8\%]$

![Image](image-url)

**Fig. 11** Example 5 ($N = 30$, $d = (\lambda/2)$, $\delta_{a_n} = 3\%$, $\text{SLL}_{\text{UM}} = -20$ dB, $\Gamma_{\text{LM}} = 4$ dB, $\text{BW}_{\text{UM}} = 0.829[u]$, $\text{BW}_{\text{LM}} = 0.716[u]$)

a) Distribution of the optimised phase weights $\phi_n$, $n = 0, \ldots, N-1$

b) Nominal amplitudes, $a_n$, $n = 0, \ldots, N-1$, with the corresponding tolerance intervals, $[a_n]$, $n = 0, \ldots, N-1$ together the solution derived from [7]

**Fig. 12** Example 5 ($N = 30$, $d = (\lambda/2)$, $\delta_{a_n} = 3\%$, $\text{SLL}_{\text{UM}} = -20$ dB, $\Gamma_{\text{LM}} = 4$ dB, $\text{BW}_{\text{UM}} = 0.829[u]$, $\text{BW}_{\text{LM}} = 0.716[u]$) – plot of the upper (PP$^{\text{sup}}(u)$) and lower (PP$^{\text{inf}}(u)$) bounds of the optimised power pattern
Concerning the interval-based PSO initialisation \( (k = 0) \), the architectural solution with \( N = 30 \) elements and spacing \( d = (2/2) \) yielded according to [7] has been taken as reference. Accordingly, one particle of the swarm has been set to the excitation weights derived as in [7] and shown in Figs. 11a and b, while the others have been randomly generated. As a representative experiment, the maximum tolerance on the amplitudes has been set to \( \Delta a_n = 3\% \), \( n = 0, \ldots, N - 1 \), while the phase weights have been assumed being correctly implemented. After \( K_{\text{max}} = 2000 \) iterations, the synthesised solution (Figs. 11a and b) corresponds to a cost function value equal to \( \Phi_{\text{best}} = 4.4 \times 10^{-3} \) and the arising power pattern interval is given in Fig. 12. As it can be noticed, the mask fitting turns out to be very accurate (especially in the mainlobe region) even though some undesired spikes of the upper power pattern appear in the sidelobe region.

4 Conclusions

The robust design of linear antenna arrays has been addressed through an innovative optimisation algorithm based on IA. The nominal values of the amplification levels have been determined such that the corresponding interval power patterns lay within the user-defined mask constraints. The obtained results have been shown below

- the potentialities and the intrinsic robustness of the IA-based synthesis method in defining the intervals of values for the array excitations still guaranteeing the generation of power patterns laying within the desired mask constraints;
- the computational efficiency of the IA in determining the pattern bounds by using close-form relationships defined by means of the arithmetic of intervals, thus enabling the use of iterative optimisation algorithms like the PSO;
- the possibility to select suitable trade-off solutions (size of the array against complexity/cost of the feeding network) to match the power pattern constraints;
- the versatility of the IA-based pattern synthesis in dealing with different beam shapes.

5 References

The upper bound is equal to

\[
PP_{\text{sup}}(u) = \left( \sum_{n=0}^{N-1} \left( a_n + \frac{(e_n^{\text{sup}} - e_n^{\text{inf}})}{2} \right) \cos \Theta_n(u) \right) + \frac{1}{2} \sum_{n=0}^{N-1} (e_n^{\text{sup}} + e_n^{\text{inf}}) \left| \sin \Theta_n(u) \right| \tag{12}
\]

As for the lower bound, by denoting as (see equation at the bottom of the page)

\[
PP_{\text{inf}}(u) = \left( \sum_{n=0}^{N-1} \left( a_n + \frac{(e_n^{\text{sup}} - e_n^{\text{inf}})}{2} \right) \cos \Theta_n(u) \right) - \frac{1}{2} \sum_{n=0}^{N-1} (e_n^{\text{sup}} + e_n^{\text{inf}}) \left| \cos \Theta_n(u) \right| - \frac{1}{2} \sum_{n=0}^{N-1} (e_n^{\text{sup}} + e_n^{\text{inf}}) \left| \sin \Theta_n(u) \right| \tag{13}
\]

If \((\Omega_1 > 0 \text{ or } \Omega_2 < 0)\) and \((\Omega_3 > 0 \text{ or } \Omega_4 < 0)\), then

\[
PP_{\text{inf}}(u) = \left( \sum_{n=0}^{N-1} \left( a_n + \frac{(e_n^{\text{sup}} - e_n^{\text{inf}})}{2} \right) \cos \Theta_n(u) \right) - \frac{1}{2} \sum_{n=0}^{N-1} (e_n^{\text{sup}} + e_n^{\text{inf}}) \left| \cos \Theta_n(u) \right| \tag{14}
\]

If \((\Omega_1 \leq 0 \text{ or } \Omega_2 < 0)\) and \((\Omega_3 > 0 \text{ or } \Omega_4 < 0)\), then

\[
PP_{\text{inf}}(u) = \left( \sum_{n=0}^{N-1} \left( a_n + \frac{(e_n^{\text{sup}} - e_n^{\text{inf}})}{2} \right) \sin \Theta_n(u) \right) - \frac{1}{2} \sum_{n=0}^{N-1} (e_n^{\text{sup}} + e_n^{\text{inf}}) \left| \sin \Theta_n(u) \right| \tag{15}
\]

If \((\Omega_1 \leq 0 \text{ or } \Omega_2 < 0)\) and \((\Omega_3 \leq 0 \text{ or } \Omega_4 < 0)\), then

\[
PP_{\text{inf}}(u) = 0.0 \tag{16}
\]

### Appendix

#### 6.1 Power pattern bounds

The upper bound is equal to

\[
PP_{\text{sup}}(u) = \left( \sum_{n=0}^{N-1} \left( a_n + \frac{(e_n^{\text{sup}} - e_n^{\text{inf}})}{2} \right) \cos \Theta_n(u) \right) + \frac{1}{2} \sum_{n=0}^{N-1} (e_n^{\text{sup}} + e_n^{\text{inf}}) \left| \cos \Theta_n(u) \right| \tag{12}
\]

As for the lower bound, by denoting as (see equation at the bottom of the page)

\[
PP_{\text{inf}}(u) = \left( \sum_{n=0}^{N-1} \left( a_n + \frac{(e_n^{\text{sup}} - e_n^{\text{inf}})}{2} \right) \cos \Theta_n(u) \right) - \frac{1}{2} \sum_{n=0}^{N-1} (e_n^{\text{sup}} + e_n^{\text{inf}}) \left| \cos \Theta_n(u) \right| - \frac{1}{2} \sum_{n=0}^{N-1} (e_n^{\text{sup}} + e_n^{\text{inf}}) \left| \sin \Theta_n(u) \right| \tag{13}
\]

If \((\Omega_1 > 0 \text{ or } \Omega_2 < 0)\) and \((\Omega_3 > 0 \text{ or } \Omega_4 < 0)\), then

\[
PP_{\text{inf}}(u) = \left( \sum_{n=0}^{N-1} \left( a_n + \frac{(e_n^{\text{sup}} - e_n^{\text{inf}})}{2} \right) \cos \Theta_n(u) \right) - \frac{1}{2} \sum_{n=0}^{N-1} (e_n^{\text{sup}} + e_n^{\text{inf}}) \left| \cos \Theta_n(u) \right| \tag{14}
\]

If \((\Omega_1 \leq 0 \text{ or } \Omega_2 < 0)\) and \((\Omega_3 > 0 \text{ or } \Omega_4 < 0)\), then

\[
PP_{\text{inf}}(u) = \left( \sum_{n=0}^{N-1} \left( a_n + \frac{(e_n^{\text{sup}} - e_n^{\text{inf}})}{2} \right) \sin \Theta_n(u) \right) - \frac{1}{2} \sum_{n=0}^{N-1} (e_n^{\text{sup}} + e_n^{\text{inf}}) \left| \sin \Theta_n(u) \right| \tag{15}
\]

If \((\Omega_1 \leq 0 \text{ or } \Omega_2 < 0)\) and \((\Omega_3 \leq 0 \text{ or } \Omega_4 < 0)\), then

\[
PP_{\text{inf}}(u) = 0.0 \tag{16}
\]