ON PSEUDO-INVERSES AND DUALITY OF FRAMES IN HILBERT SPACES

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Abstract

In this paper, we show how to find dual frames using the notion of singular value decomposition and pseudo-inverses of an operator in a Hilbert space. We will also show how properties of dual frames are linked to the spectral properties of the dual frame operator and the Grammian.

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1 Introduction

The concept of frames was introduced by Duffin and Schaefer [9] to address some deep questions in non-harmonic Fourier series. They were interested in Fourier type series for functions in $L^2[-\pi,\pi]$ involving functions $f_k : t \longmapsto e^{i\lambda_k t}$, for frequencies $\lambda_k \in \mathbb{R}$, which might not be integers. Today frame theory has applications in a variety of areas of mathematics, physics and engineering such as signal and image processing , wireless communications [14] and many other fields. Signal processing has become very important in todays life, for example, in mobile telephony, xDSL and digital television. By a signal, we mean a complex-valued function $f : X \longrightarrow \mathbb{C}$, where X is a Banach or Hilbert space and $\mathbb{C}$ denotes the field of complex numbers. The choice of the time domain X determines different types of signals. For instance, $X = \mathbb{R}$ describes a time-continuous signal, $X = \mathbb{Z}$ or $X = \mathbb{N}$ describes a discrete signal in time, $X = [a, b]$ describes a signal that is time-limited. Signals may be distinguished into the following classes: *τ-periodic* if $f(t) = f(t + \tau)$ where $\tau > 0$, *finite-energy* if $f \in L^2(\mathbb{R})$ or $f \in \ell^2(\mathbb{Z})$, *bounded* if $f \in L^\infty(\mathbb{R})$ or $f \in \ell^\infty(\mathbb{Z})$ and *integrable* or *summable* if $f \in L^1(\mathbb{R})$ or $f \in \ell^1(\mathbb{Z})$.

From a practical point of view, the use of orthonormal bases for signal expansion is non-redundant in the sense that the expansion coefficients equals the dimension of the Hilbert space and corruption or loss of expansion coefficients can result in significant reconstruction errors. Second, the reconstruction process is very rigid. Frames are usually preferred because of their redundancy, yet providing stable decompositions, resilience or robustness to additive noise and erasure(see [7],[5]), resilience to quantization (see [10]), their numerical stability of reconstruction (see [7]), and greater freedom to capture signal characteristics (see [2],[3],[5]). A special type of frames, the equal-norm Parseval tight frames has found applications in the design of multiple-antenna codes(see [11]).

2 Preliminaries and Notations

Definition 2.1 ([5], Definition 1.7) *A* `singular value decomposition` *(SVD) of an $M \times N$ matrix A is a factorization $A = U\Sigma V^*$, where $\Sigma = diag(\sigma_1, \sigma_2, ..., \sigma_p, 0..., 0)$ is an $M \times N$ real matrix, $p = \min\{M, N\}$ and $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_p \geq 0$ are the singular values of A, $U = [u_1, u_2, ..., u_M]$ is an $M \times M$ real unitary matrix, $V = [v_1, v_2, ..., v_n]$ is an $N \times N$ real unitary matrix.*

Theorem 2.2 (Singular Value Decomposition, SVD) *Let A be an $M \times N$ real matrix with $M \geq N$. Then there exists a real unitary $M \times M$ matrix U, a real unitary $N \times N$ matrix V and a diagonal $M \times N$ real matrix $\Sigma = diag(\sigma_1, \sigma_2, ..., \sigma_N)$ with $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_N \geq 0$ such that $A = U\Sigma V^*$ holds. Moreover, the column vectors of V are the eigenvectors of A^*A associated with the eigenvalues σ_i^2, $i = 1, 2, ..., N$. The columns of u are the eigenvectors of the matrix AA^*.*

Proof. The existence claim is trivial. We prove the second claim. First note that $A^*A = (U\Sigma V^*)(U\Sigma V^*)^* = VDV^*$, where $D = \Sigma^*\Sigma = diag(\sigma_1^2, \sigma_2^2, ..., \sigma_N^2)$ is $N \times N$. Thus $A^*AV = VD$. This shows that σ_i^2 is an eigenvalue of A^*A. Similarly, $AA^* = U\Sigma V^*(U\Sigma V^*)^* = U\Sigma\Sigma^*U^*$, where $\Sigma\Sigma^* = diag(\sigma_1^2, \sigma_2^2, ..., \sigma_N^2, 0, ..., 0)$ is $M \times M$ Clearly if $U\Sigma V^*$ is a singular value decomposition, then $V\Sigma^*U^*$ is a singular value decomposition of A^*. The non-zero singular values of A are the square roots of the non-zero eigenvalues of A^*A or AA^*.

Definition 2.3 *A* `Moore-Penrose pseudo-inverse` *of an $M \times N$ matrix A is an $N \times M$ matrix $A^\dagger$ that satisfies the four Penrose conditions:*

$$AA^\dagger A = A; \quad A^\dagger AA^\dagger = A^\dagger; \quad (AA^\dagger)^* = AA^\dagger; \quad (A^\dagger A)^* = A^\dagger A.$$

Theorem 2.4 ([5], Theorem 1.2) *If A an $M \times N$ matrix has SVD $A = U\Sigma V^*$, then its pseudo-inverse is $A^\dagger = V\Sigma^\dagger U^*$, where $\Sigma^\dagger = diag(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, ..., \frac{1}{\sigma_p}, 0, ..., 0)$ is $N \times M$.*

The notion of pseudo-inverse can be extended to any bounded linear operators. Let $A \in B(\mathcal{H}, \mathcal{K})$. If AA^* is invertible, then $B = A^*(AA^*)^{-1}$ is the pseudo-inverse of A. Equivalently, if A^*A is invertible, then $B = (A^*A)^{-1}A^*$ is the pseudo-inverse of A. From this definition, it is succinctly clear that the pseudo-inverse of a bounded linear operator need not be unique. That is, bounded linear operator may admit infinitely many pseudo-inverses. In fact, if an operator has more than one pseudo-inverse, then it has infinitely many(see [12]).

3 Hilbert space frames and their associated operators

Theorem 3.1 *(Parseval Identity) Let $\{f_k\}_{k=1}^n$ be an orthonormal basis for an n-dimensional Hilbert space $\mathcal{H}$. Then for any $f \in \mathcal{H}$,*

$$\sum_{k=1}^{n} |\langle f, f_k\rangle|^2 = \|f\|^2.$$

We note that the Parseval Identity also holds in infinite dimensional Hilbert spaces.

A subset $\{f_k\}_{k\in J}$ of a Hilbert space $\mathcal{H}$ is said to be *complete* if every element $f \in \mathcal{H}$ can be represented arbitrarily well in norm by linear combinations of the elements in $\{f_k\}_{k\in J}$. A complete set $\{f_k\}_{k\in J}$ is said to be *over-complete* or *redundant* if removal of an element f_j from the set results in a complete set or system. That is, if $\{f_k\}_{k\in J\setminus\{j\}}$ is still complete.

Definition 3.2 *A sequence of vectors $\{f_k\}$ in a Hilbert space $\mathcal{H}$ is a* frame *for $\mathcal{H}$ if there exists real numbers $0 < \alpha \leq \beta < \infty$ called* `frame bounds` *such that for all $f \in \mathcal{H}$*

$$\alpha\|f\|^2 \leq \sum_k |\langle f, f_k\rangle|^2 \leq \beta\|f\|^2.$$

The numbers α and β are called the *lower bound and upper bound* of the frame, respectively. They are, respectively, the smallest and largest eigenvalues of the frame operator. The numbers $(\langle f, f_k\rangle)$ are called the `frame coefficients`. A frame is a redundant or over-complete (i.e. not linearly independent) coordinate system for a vector space that satisfies a Parseval-type norm inequality. A set of vectors in a finite dimensional Hilbert space is a frame if and only if it is (just) a spanning set.

Let J be an indexing set. If $\alpha = \beta$, then the frame $\{f_k\}_{k\in J}$ is called *tight* and if $\alpha = \beta = 1$, the frame is called a *normalized tight frame* or *Parseval*. If $\|f_i\| = \|f_j\|$, for all $i, j \in J$, then $\{f_k\}_{k\in J}$ is called an *equal-norm* or *uniform norm frame*, and if in addition $\alpha = \beta = 1$, we have a *uniform normalized tight frame* (UNTF). If a frame is equal-norm and if there exists a $c \geq 0$ such that $|\langle f_j, f_k\rangle| = c$, for all j, k with $j \neq k$, then the frame is said to be *equiangular*. A frame $\{f_k\}$ that ceases to be a frame when an arbitrary element $\{f_j\}$ is removed is called an `exact frame`. For more exposition about these classes of frames(see [9],[7],[8]).

Definition 3.3 *Let $\{f_k\}$ be a frame for a Hilbert space $\mathcal{H}$. The operator $A : \mathcal{H} \to \ell^2(\mathbb{Z})$ defined by $Af = \{\langle f, f_k\rangle\}$, for all $f \in \mathcal{H}$ and $k \in \mathbb{Z}$ is called the* `analysis operator` *of the frame $\{f_k\}$.*

Definition 3.4 *Let $\{f_k\}$ be a frame for a Hilbert space $\mathcal{H}$ with analysis operator A. The Hilbert space adjoint of the analysis operator $A^* : \ell^2(\mathbb{Z}) \to \mathcal{H}$ defined by $A^*(\{\langle f, f_k\rangle\}) = \sum_k \langle f, f_k\rangle f_k$ is called the* `synthesis operator` *of the frame $\{f_k\}$.*

Remark. The analysis and synthesis operators of a frame play a central role in the analysis, reconstruction and recovery of any function or signal $f \in \mathcal{H}$. The analysis operator analyzes a signal in terms of the frame by computing its frame coefficients.

Definition 3.5 *Given a frame $\{f_k\}$ in a Hilbert space $\mathcal{H}$ with analysis operator A, another frame $\{g_k\}$ with analysis operator B is said to be a* `dual frame` *of $\{f_k\}$ if the following reproducing formula or frame decomposition formula holds*

$$f = \sum_k \langle f, f_k\rangle g_k, \qquad \forall f \in \mathcal{H}. \tag{3.1}$$

We call $\{f_k\}$ and $\{g_k\}$ a `pair of dual frames` or a `dual frame pair`.

Remark. Equation (3.1) says that $B^*A = I$, where I denotes the identity operator in $\mathcal{H}$. This means that a frame $\{g_k\}$ with analysis operator B is dual to a frame $\{f_k\}$ with analysis operator A if and only if $B^*A = I$ or equivalently $(B^*A)^* = A^*B = I$. Therefore all the duals of $\{f_k\}$ are left inverses B^* to A (or equivalently, right inverses to A^*). Dual frames are not unique. However, it has been shown that if the frame is exact, then the dual is unique(see [5]).

Definition 3.6 *Let $\{f_k\}$ be a frame in a Hilbert space $\mathcal{H}$ with analysis operator A. The operators $S = A^*A$ and $G = AA^*$ are called the* `frame operator` *and* `Grammian`*, respectively.*

The frame operator $S : \mathcal{H} \to \mathcal{H}$ is positive and invertible, while the Grammian $G : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ need not be invertible, since its range need not be all of $\ell^2(\mathbb{Z})$. The Grammian operator and its pseudo-inverse $G^\dagger$ play a crucial role in the process of recovery of $f \in \mathcal{H}$ from frame representation.

Proposition 3.7 *Suppose that $\{f_k\}$ is a frame for the Hilbert space $\mathcal{H}$ with analysis operator A and Grammian $G = AA^*$ and frame bounds α and β.*

(i). If the set $\{f_k\}$ is an orthonormal basis for $\mathcal{H}$, then the Grammian operator G is the identity.

(ii). The frame $\{f_k\}$ is a Parseval frame if and only if the Grammian operator G is an orthogonal projection.

Proof.
(i). Since $\{f_k\}$ is an orthonormal basis for $\mathcal{H}$, we have that $A = A^* = I$. Therefore $G = AA^* = I$.

(ii). Clearly $\{f_k\}$ is Parseval if and only if the frame operator $S = A^*A = I$. It is easily verified that Grammian $G = AA^*$ is self-adjoint and that

$$G^2 = (AA^*)(AA^*) = A(A^*A)A^* = A(I)A^* = AA^* = G.$$

Proposition 3.8 (Frame Reconstruction/Reproducing Formula) *Let $\{f_k\}$ be a frame in a Hilbert space $\mathcal{H}$ with analysis operator A and frame operator $S = A^*A$. Then*

$$f = \sum_k \langle S^{-1}f, f_k\rangle = \sum_k \langle f, S^{-1}f_k\rangle f_k = \sum_k \langle f, f_k\rangle S^{-1}f_k = \sum_k \langle f, S^{-1/2}f_k\rangle S^{-1/2}f_k, \qquad f \in \mathcal{H}.$$

Moreover, the series converges to f unconditionally in the induced norm on $\mathcal{H}$.

Proof. Let $f \in \mathcal{H}$. By definition and self-adjointness of the frame operator S, we have

$$f = SS^{-1}f = \sum_k \langle S^{-1}f, f_k\rangle f_k = \sum_k \langle f, S^{-1}f_k\rangle f_k.$$

Similarly,

$$f = S^{-1}Sf = S^{-1}\sum_k \langle f, f_k\rangle f_k = \sum_k \langle f, f_k\rangle S^{-1}f_k = \sum_k \langle f, S^{-1/2}f_k\rangle S^{-1/2}f_k.$$

Finally, using the fact that $I = S^{-1/2}SS^{-1/2}$, we have

$$f = S^{-1/2}SS^{-1/2}f = S^{-1/2}\sum_k \langle S^{-1/2}f, f_k\rangle f_k = \sum_k \langle f, S^{-1/2}f_k\rangle S^{-1/2}f_k.$$

Remark. The reconstruction formula shows that all information about a given vector or signal $f \in \mathcal{H}$ is contained in the sequence $\{\langle f, S^{-1}f_k\rangle\}$. We note that the choice of coefficients in Proposition 3.8 is not unique, in general. If the frame $\{f_k\}$ is linearly dependent(redundant or over-complete), a typical phenomenon in applications, then there exist infinitely many choices of coefficients $c_k = \langle f, S^{-1}f_k\rangle$ in the expansion of $f \in \mathcal{H}$ as $f = \sum_k c_k f_k$. This possibility ensures resilience to erasures or noise in a signal $f \in \mathcal{H}$. A new approach (see [4]) has emerged recently, and has received increasing attention, namely choose the coefficient sequence to be sparse in the sense of having only few non-zero entries, thereby allowing data compression while preserving perfect reconstruction or recoverability.

The sequence $\{S^{-1}f_k\}$ is called the `canonical dual` of $\{f_k\}$. Bijectivity of S clearly implies that the canonical dual $\{S^{-1}f_k\}$ is also a frame in $\mathcal{H}$ with frame bounds $\frac{1}{\beta}$ and $\frac{1}{\alpha}$ and frame operator S^{-1}.
The sequence $\{S^{-1/2}f_k\}$ is also frame(by the bijectivity of $S^{-1/2}$), called the `canonical tight frame` associated with the frame $\{f_k\}$(see [1], [5]). By Definition 3.5, we note that the canonical dual frame is the pseudo-inverse of A, which we write $(A^*)^\dagger = (A^*A)^{-1}A^* = S^{-1}A^*$.(see also [1]).

Proposition 3.9 *Let $\{f_k\}$ be a frame in a Hilbert space $\mathcal{H}$ with analysis operator A and frame operator $S = A^*A$. Then the frame operator provides a stable reconstruction process*

$$Sf = \sum_k \langle f, f_k\rangle f_k, \qquad f \in \mathcal{H}.$$

Proof. Follows immediately from the definition of S and Proposition 3.8.

Remark. Notice from Proposition 3.9 that

$$\langle Sf, f\rangle = \langle A^*Af, f\rangle = \langle Af, Af\rangle = \|Af\|^2 = \sum_k \langle f, f_k\rangle\langle f_k, f\rangle = \sum_k \langle f, f_k\rangle\overline{\langle f, f_k\rangle} = \sum_k |\langle f, f_k\rangle|^2, \ \forall f \in \mathcal{H}.$$

Therefore if α and β are the frame bounds, we have

$$\langle \alpha f, f\rangle = \alpha\|f\|^2 \leq \sum_k |\langle f, f_k\rangle|^2 = \langle Sf, f\rangle \leq \beta\|f\|^2 = \beta\langle f, f\rangle, \quad \forall f \in \mathcal{H}.$$

This says that

$$\alpha I \leq S \leq \beta I.$$

It has been shown in ([1], Theorem 2.2) that if $\{f_k\}$ is a frame for a Hilbert space $\mathcal{H}$ with frame operator S and $T \in B(\mathcal{H})$, then the frame operator for $\{Tf_k\}$ equals TST^*. Using this result, we conclude that the canonical tight frame has frame operator $S^{-1/2}SS^{-1/2} = I$. This means that $\{S^{-1/2}f_k\}$ is a Parseval frame.

Theorem 3.10 *Let $\{f_k\}$ be a tight frame in a Hilbert space $\mathcal{H}$. Then the canonical dual frame $\{S^{-1}f_k\} = \{\frac{1}{\alpha}f_k\}$. Moreover, $f = \frac{1}{\alpha}\sum_k \langle f, f_k\rangle f_k$ and α is the tight frame bound.*

Proof. Suppose that $\{f_k\}$ is a tight frame with frame bound α and frame operator S. Then by definition of S and the reconstrution formula in Proposition 3.9, we have

$$\langle Sf, f\rangle = \sum_k |\langle f, f_k\rangle| = \alpha\|f\| = \langle \alpha f, f\rangle.$$

Since S is self-adjoint, this implies that $S = \alpha I$. Thus S^{-1} is the multiplication by $\frac{1}{\alpha}$ operator. The rest of the proof follows from application of Proposition 3.8 and definition of a frame.

Frames having Gabor structure or wavelet structure involve translations and modulations of a fixed function $g \in L^2(\mathbb{R})$,called the window function. A Gabor frame is a sequence for $L^2(\mathbb{R})$ of the form $\{M_{mb}T_{na}g\}_{n,m\in\mathbb{Z}}$, where $M_{mb}T_{na}g(x) = e^{2\pi imbx}g(x - na)$, $\;a, b > 0$, $T_a, M_b : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ are the *translation by a* and *modulation by b operators* defined by $(T_a f)(x) = f(x - a)$ and $(M_b f)(x) = e^{2\pi bx}f(x)$, respectively, where $x \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$. Gabor frames are overcomplete frames for $L^2(\mathbb{R})$. A wavelet system takes the form $\{2^{j/2}\psi(2^j x - k)\}_{j,k\in\mathbb{Z}}$, where D is the *dilation operator* $D : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ defined by $(Df)(x) = 2^{j/2}f(2x)$, which are orthonormal bases for $L^2(\mathbb{R})$. Wavelet frames are used to obtain Fourier expansion for $f \in L^2(\mathbb{R})$. It is known (see [13], [2] and [5]) that most Gabor frames are overcomplete and that if $ab > 1$, then any Gabor system is incomplete, if $ab = 1$, then a Gabor frame is a Riesz basis, and if $ab < 1$, then a Gabor frame is overcomplete.
Over-complete Gabor frames and wavelet frames have been used in signal detection, image representation, object recognition, noise reduction, sampling theory, wireless communications, filter banks and quantum computing(see [6]).

4 The Singular Value Decomposition, Pseudo-inverses and Dual Frames

When designing frames with prescribed properties, it is important to check the behavior of the canonical dual frame $\{S^{-1/2}f_k\}$. In some cases, especially in high dimensional settings, however, the complicated structure of the frame operator and its inverse make this a difficult task. For instance, if $\{f_k\}$ is a frame in the Hilbert space $L^2(\mathbb{R})$ consisting of functions with exponential decay, there is no guarantee that the functions in the canonical dual frame $\{S^{-1}f_k\}$ have exponential decay.

Some frames have advantages over others. For tight frames, by Proposition 3.8, Proposition 3.9 and Proposition 3.10, the canonical dual frame automatically has the same structure as the frame itself. If the frame has a wavelet structure or a Gabor structure, the same is the case for the canonical dual frame. In contrast, there are non-tight wavelet frames which lack this special property. We use the singular value decomposition to avoid inverting the frame operator S.

Proposition 4.1 *Let $\{f_k\}$ be a frame in a Hilbert space $\mathcal{H}$ and suppose that $\{g_k\}$ is its dual frame. Then*

$$f = \sum_k \langle f, g_k\rangle f_k = \sum_k \langle f, f_k\rangle g_k, \qquad \forall f \in \mathcal{H}.$$

It is clear that if $\{g_k\}$ is a dual frame for $\{f_k\}$, then $\{f_k\}$ is also a dual of $\{g_k\}$.(see [5]). If the frame $\{f_k\}_{k=1}^{M}$ for a Hilbert space of dimension N and $M > N$(that is the frame contains more vectors than is needed for the spanning property-that is, it is over-complete or redundant), there exists infinitely many dual frames(no rigidity as is the case of bases or when $M = N$)(see [1]).

We find the $SVD(A)$ as $A = U\Sigma V^*$, where A is any $M \times N$ matrix of real numbers with rank k, U is a matrix whose columns are the M orthonormal eigenvectors associated with the non-zero eigenvalues of the self-adjoint matrix $G = AA^*$. On the other hand, matrix V is formed with the orthonormal eigenvectors associated with the non-zero eigenvalues of the self-adjoint operators $S = A^*A$. In a frame, S is invertible, and hence has no zero eigenvalues(see [12]).

Remark. For computational purposes, it is important to notice that the pseudo-inverse of an operator T can be found by the singular value decomposition of T. We will explore the use of MAPLE software to find the duals of frames and avoid finding the inverse of the frame operator S. We will explore the use of MAPLE software to find the duals of frames and avoid finding the inverse of the frame operator S.

Example 4.2 *For the frame* $\{f_k\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ *for* $\mathcal{H} = \mathbb{R}^2$,

$$G = (\langle f_m, f_n \rangle)_{1\le m,n\le 3} = \begin{bmatrix} \langle f_1, f_1\rangle & \langle f_1, f_2\rangle & \langle f_1, f_3\rangle \\ \langle f_2, f_1\rangle & \langle f_2, f_2\rangle & \langle f_2, f_3\rangle \\ \langle f_3, f_1\rangle & \langle f_3, f_2\rangle & \langle f_3, f_3\rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

A simple calculation shows that G is not invertible. A simple computation shows that $S = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, and S is invertible. It is easy to show that $\sigma(G) = \{0, 1, 3\}$ and $\sigma(S) = \{1, 3\}$.

Theorem 4.3 *Let $\{f_n\}$ be a frame for a Hilbert space $\mathcal{H}$ with analysis operator A and frame operator S and Grammian G. Let the associated canonical dual frame be $\{\widetilde{f_n}\}$, where $\widetilde{f_n} = S^{-1}f_n$ with an associated analysis operator $\widetilde{A}$. Then $\widetilde{A} = (G|_{Ran(A)})^{-1}A$.*

Proof. We first note that $\widetilde{A}f = (\langle f, \widetilde{f_n}\rangle) = (\langle f, S^{-1}f_n\rangle)$. Thus $Ran(A) = Ran(\widetilde{A})$, since S is invertible. Thus

$$A^*\widetilde{A} = \widetilde{A}^*A = I_{\mathcal{H}},$$

where $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. On $Ran(A), A, \widetilde{A}$, and hence the Gramian G for$\{f_n\}$ are invertible and we have that $A^{-1} = \widetilde{A}^*$ and $\widetilde{A}^{-1} = A^*$. Thus the relation between the analysis operator A and its dual $\widetilde{A}$ is

$$\widetilde{A} = (G|_{Ran(A)})^{-1}G\widetilde{A} = (G|_{Ran(A)})^{-1}AA^*\widetilde{A} = (G|_{Ran(A)})^{-1}A.$$

Proposition 4.4 ([5], Corollary 1.10) *A frame $\{f_n\}_{n=1}^{M}$ for an N-dimensional Hilbert space $\mathcal{H}$ has a unique dual frame if and only if $M = N$.*

Proposition 4.5 *Let $\{f_n\}_{n=1}^{M}$ be a frame for an N-dimensional Hilbert space $\mathcal{H}$ with frame bounds α and β. Let P be an orthogonal projection of $\mathcal{H}$ onto a subspace $\mathcal{M}$. Then $\{g_n\} = \{Pf_n\}_{n=1}^{M}$ is a frame for $\mathcal{M}$ with frame bounds α and β. In particular, if $\{f_n\}_{n=1}^{M}$ is a Parseval frame, then $\{Pf_n\}_{n=1}^{M}$ is a Parseval frame.*

Proof. For any $f \in \mathcal{M}$, we have that $f = P_{\mathcal{M}}f$ and so

$$\alpha\|f\|^2 = \alpha\|Pf\|^2 \le \sum_{n=1}^{M} |\langle Pf, f_n\rangle|^2 = \sum_{n=1}^{M} |\langle f, Pf_n\rangle|^2 \le \beta\|Pf\|^2 = \beta\|f\|^2.$$

If $\{f_n\}_{n=1}^M$ is Parseval, then we have

$$\|f\|^2 = \|Pf\|^2 = \sum_{n=1}^{M} |\langle Pf, f_n\rangle|^2 = \sum_{n=1}^{M} |\langle f, Pf_n\rangle|^2 = \sum_{n=1}^{M} |\langle f, g_n\rangle|^2.$$

The canonical coefficients from the frame expansion arise naturally by considering the pseudo-inverse of the analysis operator. The pseudo-inverse can be given by the singular value decomposition of A.

Theorem 4.6 ([1], Theorem 2.2) *If $\{f_k\}_{k=1}^n$ is a frame for an N-dimensional Hilbert space $\mathcal{H}$ with frame operator S and T is an operator on $\mathcal{H}$, then the frame operator for $\{Tf_k\}_{k=1}^n$ equals TST^*.*

Proof. The proof follows from the fact that the frame operator for $\{Tf_k\}_{k=1}^n$ is given by

$$\sum_{k=1}^{n} \langle f, Tf_k\rangle Tf_k = T(\sum_{k=1}^{n} \langle T^* f, f_k\rangle f_k) = TST^*.$$

Alternatively, from Lemma 8.20, the frame operator of $\{Tf_k\}_{k=1}^n$ is given by

$$B^*B = (AT^*)^*(AT^*) = TA^*AT^* = T(A^*A)T^* = TST^*.$$

Clearly

$$TST^*f = T(\sum_{k=1}^{n} \langle T^* f, f_k\rangle f_k) = \sum_{k=1}^{n} \langle f, Tf_k\rangle Tf_k.$$

This leads to the following consequences.

Corollary 4.7 *If $\{f_k\}_{k=1}^n$ is a tight frame for an N-dimensional Hilbert space $\mathcal{H}$ with frame operator S and T is an operator on $\mathcal{H}$, then the frame operator for $\{Tf_k\}_{k=1}^n$ is a scalar multiple of TT^*. Moreover, if $\{f_k\}_{k=1}^n$ is Parseval/normalized and tight, then the frame operator for $\{Tf_k\}_{k=1}^n$ is TT^*.*

The canonical tight frame $\{S^{-\frac{1}{2}} f_n\}_{n=1}^M$ inherits properties of the original frame $\{f_n\}_{n=1}^M$.

Proposition 4.8 *If $\{f_n\}_{n=1}^M$ is a frame for a Hilbert space $\mathcal{H}$ with frame operator S and frame bounds α and β, then $\{S^{-\frac{1}{2}} f_n\}_{n=1}^M$ is a tight frame with frame bound 1 (i.e. it is Parseval) and $f = \sum_k \langle f, S^{-1/2} f_k\rangle S^{-1/2} f_k$ for all $f \in \mathcal{H}$.*

Proof. We need to show that $\{S^{-\frac{1}{2}} f_n\}_{n=1}^M$ satisfies the reconstruction formula $f = \sum_{k=1}^n \langle f, f_k\rangle f_k$ for all $f \in \mathcal{H}$. Clearly, the operator $S^{-1/2}$ is well defined and commutes with S^{-1}. Therefore by Proposition 3.8, every $f \in \mathcal{H}$ can be reconstructed as

$$f = S^{-1/2} S S^{-1/2} f = S^{-1/2} \sum_k \langle S^{-1/2} f, f_k\rangle f_k = S^{-1/2} \sum_k \langle f, S^{-1/2} f_k\rangle f_k = \sum_k \langle f, S^{-1/2} f_k\rangle S^{-1/2} f_k.$$

This proves the Parseval reconstruction formula for f.
Taking the inner product with f we have

$$\|f\|^2 = \langle f, f\rangle = \sum_k \langle f, S^{-1/2} f_k\rangle \langle S^{-1/2} f_k, f\rangle = \sum_k \langle f, S^{-1/2} f_k\rangle \overline{\langle f, S^{-1/2} f_k\rangle} = \sum_k |\langle f, S^{-1/2} f_k\rangle|^2.$$

This shows that $\{S^{-\frac{1}{2}} f_n\}_{n=1}^M$ is a tight frame with frame bound 1.
We note that the first claim can be proved easily using Theorem 4.6 by showing that the frame operator of the canonical tight frame is $S^{can} = S^{-1/2} S S^{-1/2} = I$. This is equivalent the statement that for every $f \in \mathcal{H}$, we have

$$S^{can} f = \sum_k \langle f, \widetilde{f}_k\rangle \widetilde{f}_k = \sum_k \langle f, S^{-1/2} f_k\rangle S^{-1/2} f_k = S^{-1/2} \sum_k \langle S^{-1/2} f, \widetilde{f}_k\rangle = S^{-1/2} S S^{-1/2} f = f.$$

Therefore $S^{can} = I$.

Theorem 4.9 *Let $\Phi = \{f_k\}$ be a frame for a Hilbert space $\mathcal{H}$ with frame operator S and frame bounds α and β. The canonical dual frame $\widetilde{\Phi} = \{S^{-1}f_k\}$ has frame operator S^{-1}.*

Proof. The synthesis operator of $\widetilde{\Phi}$ is given by $B^* = S^{-1}A^*$(see [1]), where A is the analysis operator of Φ. Thus the frame operator for $\widetilde{\Phi}$ is given by

$$\widetilde{S} = B^*B = S^{-1}A^*(AS^{-1}) = S^{-1}(A^*A)S^{-1} = S^{-1}SS^{-1} = S^{-1}.$$

Alternatively, by ([1], Theorem 2.2), the frame operator of $\{S^{-1}f_k\}$ is

$$\widetilde{S} = S^{-1}S(S^{-1})^* = S^{-1}SS^{-1} = S^{-1}.$$

This result can also be proved as follows:
For every $f \in \mathcal{H}$, we have

$$\widetilde{S}f = \sum_k \langle f, \widetilde{f}_k\rangle \widetilde{f}_k = \sum_k \langle f, S^{-1}f_k\rangle S^{-1}f_k = S^{-1}\sum_k \langle S^{-1}f, \widetilde{f}_k\rangle = S^{-1}SS^{-1}f = S^{-1}f.$$

Therefore $\widetilde{S} = S^{-1}$.
We note the claim can be proved easily using Theorem 4.6 by showing that showing that the frame operator of the canonical dual frame is $\widehat{S} = S^{-1}SS^{-1} = S^{-1}$. This is equivalent the statement that for every $f \in \mathcal{H}$, we have

$$\widetilde{S}f = \sum_k \langle f, \widetilde{f}_k\rangle \widetilde{f}_k = \sum_k \langle f, S^{-1}f_k\rangle S^{-}f_k = S^{-1}\sum_k \langle S^{-1}f, \widetilde{f}_k\rangle = S^{-1}SS^{-1}f = S^{-1}f.$$

Therefore $\widetilde{S} = S^{-1}$.
Since S is the frame operator, we have that $\rangle \alpha f, f\rangle \leq \langle Sf, f\rangle \leq \langle \beta f, f\rangle$ for all $f \in \mathcal{H}$. This is equivalent to the statement that $\alpha I \leq S \leq \beta I$. We conclude that $\frac{1}{\beta}I \leq S^{-1} \leq \frac{1}{\alpha}I$.

5 Main Results

We have seen that the problem of finding duals to a frame $\Phi = \{f_k\}_{k=1}^m$ with analysis operator A boils down the problem of finding the set of matrices or operators B such that $B^*A = I$ or $A^*B = I$. Equivalently, this is the set of all left-inverses or pseudo-inverses B to A or the adjoints of all right inverses to A. Since $m > N$, the frame is redundant(consists of more vectors than needed to span $\mathcal{H}$), Gauss-Jordan elimination shows that there are infinitely many dual frames.

We give examples for the cases $m = 3, 4$ and $\mathcal{H} = \mathbb{R}^2$.

Example 5.1

Consider the sequence $\{f_k\}_{k=1}^3 := \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$. Clearly the analysis operator is $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and the synthesis operator $A^* = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. A simple calculation shows that the frame operator $S := A^*A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and the Gram matrix $G := AA^* = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. Clearly $S^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$ and hence the canonical dual frame operator $\{S^{-1}f_k\} = \{\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}, \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}\}$.

The pseudo inverse of A^* computed by singular value decomposition is $B = (A^*)^\dagger = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$ and its columns give the dual frame vectors. Notice that $BA^* = I$, and so the columns of B represent

the alternate dual frame. Notice that in this case the an alternate dual coincides with the canonical frame $\{S^{-1}f_k\}$. Notice also that the above result can be obtained from $B = S^{-1}A^*$.
However, the frame has infinitely many duals. For instance the matrix $\begin{pmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$ is another pseudo-inverse for A. This frame has a redundancy $\frac{3}{2}$.

Example 5.2

The frame $\{f_k\}_{k=1}^4 := \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$ is a tight frame for $\mathbb{R}^2$ since $S = 2I$ and hence $S^{-1} = \frac{1}{2}I$. The normalized frame is $\Psi = \{\frac{1}{\sqrt{2}}f_k\}$. A simple computation shows that Ψ is a normalized tight frame for $\mathbb{R}^2$, with Grammian $G_\Psi = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$, which is an orthogonal projection. More calculations show that the alternate dual frame consists of the columns of $\widetilde{A}^* = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$. Since $\widetilde{S} = \widetilde{A}^*\widetilde{A} = \frac{1}{2}I$, we conclude that the dual frame is also tight. This frame has redundancy 2. Another computation shows that $\widetilde{G} = G^\dagger = \frac{1}{4}G$ and $G = \widetilde{G}^\dagger$. This says that $G\widetilde{G} = G\widetilde{G} = I$.

From this example, we deduce two results.

Lemma 5.3 *Let $\Phi = \{f_k\}_{k=1}^m$ be a frame for an N-dimensional Hilbert space $\mathcal{H}$. If Φ has a redundancy greater or equal to 2, then it has a tight dual frame.*

Theorem 5.4 *The Grammian of a frame Φ and its dual $\widetilde{\Phi}$ are pseudo-inverses. That is, Gram$(\widetilde{\Phi})$ = Gram$(\Phi)^\dagger$.*

Proposition 4.1 and Theorem 5.3 leads us to a new relation, which we call `duality of finite frames`. We denote this new relation by $\Phi \overset{dual}{\sim} \Psi$ if and only if $f = \sum_{k=1}^n \langle f, g_k\rangle f_k$ for all $f \in \mathcal{H}$

Theorem 5.5 *Duality of frames $\Phi = \{f_k\}_{k=1}^n$ and $\Psi = \{g_k\}_{k=1}^n$ for a Hilbert space $\mathcal{H}$ is an equivalence relation.*

Proof. Recall that $\Phi = \{f_k\}_{k=1}^n$ and $\Psi = \{g_k\}_{k=1}^n$ are a dual pair if $f = \sum_{k=1}^n \langle f, g_k\rangle f_k$ for all $f \in \mathcal{H}$. Clearly $\Phi \overset{dual}{\sim} \Phi$, since $f = \sum_{k=1}^n \langle f, f_k\rangle f_k$. This shows that $\overset{dual}{\sim}$ is reflexive.
Suppose $\Phi \overset{dual}{\sim} \Psi$. Then $f = \sum_{k=1}^n \langle f, g_k\rangle f_k = \sum_{k=1}^n \langle f, f_k\rangle g_k$ for all $f \in \mathcal{H}$. This shows that $\Psi \overset{dual}{\sim} \Phi$ and therefore $\overset{dual}{\sim}$ is symmetric. Now, suppose $\Omega = \{h_k\}_{k=1}^n$ be a frame for $\mathcal{H}$. Suppose that $\Phi \overset{dual}{\sim} \Psi$ and $\Psi \overset{dual}{\sim} \Omega$. Then $f = \sum_{k=1}^n \langle f, g_k\rangle f_k = \sum_{k=1}^n \langle f, f_k\rangle g_k$ and $f = \sum_{k=1}^n \langle f, g_k\rangle h_k = \sum_{k=1}^n \langle f, h_k\rangle g_k$. This implies that $f = \sum_{k=1}^n \langle f, f_k\rangle g_k = \sum_{k=1}^n \langle f, h_k\rangle g_k$. Equating the coefficients we have that $\langle f, f_k\rangle = \langle f, h_k\rangle$ and therefore $f = \sum_{k=1}^n \langle f, h_k\rangle f_k$, which proves that $\Phi \overset{dual}{\sim} \Omega$. Therefore $\overset{dual}{\sim}$ is transitive. Thus $\overset{dual}{\sim}$ is an equivalence relation.

Example 5.6

Consider the frame in Example 5.1. It can be shown that the frame bounds are $\sigma_1 = 1, \sigma_2 = \sqrt{3}$ and so

$$\|f\|^2 \leq \|Af\|^2 \leq 3\|f\|^2.$$

and that the dual analysis operator is $B^* = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$. Notice that in this case the alternate dual coincides with the canonical dual frame.

Suppose we want to reconstruct $f = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$ in terms of the frame $\{f_k\}$ and int terms of the dual. Then

$$B^*f = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} -5 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ -1 \end{pmatrix}.$$

Therefore

$$f = \begin{pmatrix} -5 \\ 2 \end{pmatrix} = -4f_1 + 3f_2 - f_3.$$

To find the expansion of f in terms of the dual frame we compute the coefficients as

$$Af = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -5 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ -3 \end{pmatrix},$$

and so

$$f = \begin{pmatrix} -5 \\ 2 \end{pmatrix} = -5\widetilde{f_1} + 2\widetilde{f_2} - 3\widetilde{f_3}.$$

To find the canonical tight frame, we compute $S^{-1/2}$. To achieve this, we orthogonally diagonalize $S^{-1/2}$. Let $T = S^{-1}$. We find an orthogonal matrix U such that $UTU^{-1} = D = R^2$, where D is a diagonal matrix with diagonal entries the eigenvalues of T and R is any of the four square roots of D. We ortho-normalize the eigenvectors of $S^{-1/2}$ and let U be the matrix whose columns are the normalized vectors. A simple computation gives $\lambda_1 = 1, \lambda_2 = \frac{1}{3}$ as the eigenvalues of T with corresponding eigenvectors $[-1, 1]^t$ and $[1, 1]^t$. The vectors are already orthogonal and we only need to divide each by its length. Thus $U = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$.
Without loss of generality we let $R = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$. Then

$$S^{-1/2} = T^{1/2} = U^*R^*U = \begin{pmatrix} \frac{1}{2} + \frac{1}{\sqrt{12}} & -\frac{1}{2} + \frac{1}{\sqrt{12}} \\ -\frac{1}{2} + \frac{1}{\sqrt{12}} & \frac{1}{2} + \frac{1}{\sqrt{12}} \end{pmatrix}.$$

This means that

$$B^* = S^{-1/2}A^* = \begin{pmatrix} \frac{1}{2} + \frac{1}{\sqrt{12}} & -\frac{1}{2} + \frac{1}{\sqrt{12}} & \frac{2}{\sqrt{12}} \\ -\frac{1}{2} + \frac{1}{\sqrt{12}} & \frac{1}{2} + \frac{1}{\sqrt{12}} & \frac{2}{\sqrt{12}} \end{pmatrix}$$

is the synthesis operator for the canonical tight frame. Hence

$$\Phi^{can} = \left\{ \begin{pmatrix} \frac{1}{2} + \frac{1}{\sqrt{12}} \\ -\frac{1}{2} + \frac{1}{\sqrt{12}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} + \frac{1}{\sqrt{12}} \\ \frac{1}{2} + \frac{1}{\sqrt{12}} \end{pmatrix}, \begin{pmatrix} \frac{2}{\sqrt{12}} \\ \frac{2}{\sqrt{12}} \end{pmatrix} \right\}$$

is the canonical tight frame for Φ.
A simple calculation shows that

$$\langle f_k, \widehat{f_k} \rangle = \langle f_k, \widetilde{f_k} \rangle = \|f_k^{can}\|^2, \forall k,$$

where $\widehat{f_k}$ denotes a canonical dual vector and $\widetilde{f_k}$ denotes an alternate dual vector. This implies that

$$\sum_{k=1}^{3} \langle f_k, \widetilde{f_k} \rangle = 2 = dim(\mathcal{H}).$$

MAPLE 18 software reveals that $B^*BB^* = B^*$, which proves that B^* is a partial isometry. This agrees with an earlier remark. Further computation using MAPE 18 approximates

$$S^{can} = \begin{pmatrix} 1 & -1.899 \times 10^{-16} \\ -1.899 \times 10^{-16} & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

and

$$Gram(\Phi^{can}) = \begin{pmatrix} 0.3333 & -7.269\times 10^{-17} & 0.4714 \\ -7.269\times 10^{-17} & 1 & -1.813\times 10^{-16} \\ 0.4714 & -1.813\times 10^{-16} & 0.6666 \end{pmatrix} \approx \begin{pmatrix} 0.3333 & 0 & 0.4714 \\ 0 & 1 & 0 \\ 0.4714 & 0 & 0.6666 \end{pmatrix}.$$

Since $S^{can} = I$, we conclude that the canonical tight frame $\{S^{-1/2}f_k\}$ is a Parseval frame, which agrees with Proposition 4.5.

Theorem 5.7 *If $\Phi = \{f_k\}_{k=1}^n$ is a normalized tight frame for a Hilbert space $\mathcal{H}$ and $T : \mathcal{H} \longrightarrow \mathcal{H}$ is an invertible operator, then the frames $\{T^*f_k\}$ and $\{T^{-1}f_k\}$ are dual to each other.*

Proof. Since $\Phi = \{f_k\}_{k=1}^n$ is a normalized tight frame, its frame operator $S = I$. Using this fact together with Proposition 3.9, we have that $\{f_k\}_{k=1}^n$ is normalized tight frame if and only if $f = \sum_k \langle f, f_k\rangle f_k$, for all $f \in \mathcal{H}$. Let $\{g_k\} = \{T^*f_k\}$ and $\{h_k\} = \{T^{-1}f_k\}$. We need to show that $f = \sum_k\langle f, g_k\rangle h_k = \sum_k\langle f, h_k\rangle g_k$. Using the definition, we have

$$\begin{aligned} f = \sum_k\langle f, f_k\rangle f_k &= \sum_k\langle f, TT^{-1}f_k\rangle f_k \\ &= \sum_k\langle T^*f, T^{-1}f_k\rangle f_k \\ &= T^*\sum_k\langle f, T^{-1}f_k\rangle f_k \\ &= \sum_k\langle f, T^{-1}f_k\rangle T^*f_k \\ &= \sum_k\langle f, h_k\rangle g_k. \end{aligned}$$

Similarly,

$$\begin{aligned} f = \sum_k\langle f, f_k\rangle f_k &= \sum_k\langle f, (T^*)^{-1}T^*f_k\rangle f_k \\ &= \sum_k\langle f, (T^{-1})^*T^*f_k\rangle f_k \\ &= \sum_k\langle T^{-1}f, T^*f_k\rangle f_k \\ &= T^{-1}\sum_k\langle f, T^*f_k\rangle f_k \\ &= \sum_k\langle f, T^*f_k\rangle T^{-1}f_k \\ &= \sum_k\langle f, g_k\rangle h_k. \end{aligned}$$

This proves the claim.

Theorem 5.8 *If $\Phi = \{f_k\}_{k=1}^n$ is a frame for a Hilbert space $\mathcal{H}$ and $Q : \mathcal{H} \longrightarrow \mathcal{H}$ is an invertible operator, then the frames $\Psi = \{Qf_k\}$ is a frame for $\mathcal{H}$, and $\Psi^{can} = U\Phi^{can}$, where U is a unitary operator.*

Proof. The claim that $\Psi = \{Qf_k\}$ is a frame for $\mathcal{H}$ follows easily from the fact that Q is invertible. To prove the second claim, we let

$$g_k = Qf_k = QS^{1/2}S^{-1/2}f_k = (QS^{1/2})S^{-1/2}f_k = TS^{-1/2}f_k = Tf_k^{can} = T\Phi^{can},$$

where $T = QS^{1/2}$ is invertible. Thus the synthesis operator for Ψ is $T[f_k^{can}]$. But any canonical tight frame is Parseval by Proposition 4.8. Thus $\Psi = \{Tf_k^{can}\}$ is Parseval if and only if and only if its frame operator $S_\Psi^{can} = I$. That is if and only if $S_\Psi^{can} = T[f_k^{can}](T[f_k^{can}])^* = T[f_k^{can}][f_k^{can}]^*T^* = TT^* = I$. This means that T is an co-isometry. Since T is invertible, it must be a unitary operator. So we let $T = U$, where U is unitary. Therefore $\Psi^{can} = U\Phi^{can}$.

Remark. Let $\Phi = \{f_k\}_{k=1}^n$ be a finite frame for a Hilbert space $\mathcal{H}$ with analysis operator A and frame operator S. The Grammian of the canonical tight frame is an orthogonal projection, by ([1], Theorem 2.2), we have $P = Gram(\Phi^{can}) = AS^{-1}A^* : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$ which gives the coefficients c_k with $f = \sum_k c_k f_k$ of minimal ℓ^2-norm. This means that the canonical tight frame gives a more precise and better reconstruction than the alternate dual frame. This means that the canonical tight frame $\{S^{-1/2}f_k\}$ inherits many of the properties of the original frame $\{f_k\}$. The only problem is that it is not easy to find $\{S^{-1/2}f_k\}$ and that some nice properties of $\{f_k\}$ may not be necessarily inherited.

Lemma 5.9 *If $\{f_k\}_{k=1}^n$ is a frame for an finite dimensional Hilbert space $\mathcal{H}$ with analysis operator A and frame operator S and T is an operator on $\mathcal{H}$, then the analysis operator for $\{Tf_k\}_{k=1}^n$ equals AT^*.*

Proof. Let B be the analysis operator of $\{Tf_k\}_{k=1}^n$. Then

$$Bf = \sum_{k=1}^{n} \langle f, Tf_k \rangle f_k = \sum_{k=1}^{n} \langle T^* f, f_k \rangle f_k = AT^* f, \quad \forall f \in \mathcal{H}.$$

That is, $B = AT^*$.

APPENDIX

Maple 18 Code for Example 5.1

```
>with(MTM):
>A:=matrix([[1,0],[0,1],[1,1]]);     Enters matrix A
>svd:=svd(A);     Gives the singular values of A
>U,S,V:=svd(A);    Gives the full svd(A) and returns matrices U,S,V in that order
>PseudoInv:=MatrixInverse(A,method=pseudo);    Returns the Pseudo-inverse of A
```

The output is

$$\mathtt{A} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\mathtt{svd} := \begin{bmatrix} 0 \\ \sqrt{3} \\ 1 \end{bmatrix}$$

$$\mathtt{U,S,V} := \begin{bmatrix} 0.4082 & -0.7071 & -0.5774 \\ 0.4082 & 0.7071 & 0.5774 \\ 0.8165 & -5.5511\ 10^{-17} & 0.5774 \end{bmatrix}, \begin{bmatrix} 1.7321 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0.7071 & -0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}$$

$$\mathtt{PseudoInv} := \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

6 Discussion

The pseudo-inverse of an operator is useful for best approximation problems of the form $Af = g$ in a Hilbert space $\mathcal{H}$. When the system A is over-complete over-determined, there are infinitely many ways to reconstruct f from g. The pseudo-inverse $A^\dagger$ helps in determining the optimal way to reconstruct f from g: $\widehat{f} = A^\dagger g$. This notion plays a crucial role in the construction of frames duals which are used in the reconstruction of finite frames, that find applications in digital reconstruction, analysis and transmission of a signal $f \in H$ from analysis operator A.

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