On the linear algebra of local complementation

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Abstract
We survey the connections between the linear algebra of symmetric matrices over \( GF(2) \) and the circuit theory of 4-regular graphs. In particular, we show that the equivalence relation on simple graphs generated by local complementation can also be generated by an operation defined using inverse matrices.

Keywords. Euler circuit, interlacement, inverse matrix, local complement, pivot

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1 Introduction

This paper is about the connection between the circuit theory of 4-regular multigraphs and the elementary linear algebra of symmetric matrices over the two-element field \( GF(2) \).

Definition 1 A square matrix \( S = (s_{ij}) \) with entries in \( GF(2) \) is symmetric if \( s_{ij} = s_{ji} \) for all \( i \neq j \). \( S \) is skew-symmetric if it is symmetric and also has \( s_{ii} = 0 \) for all \( i \).

As \( GF(2) \) is the only field that concerns us, we will often omit the phrase “with entries in \( GF(2) \).”

Symmetric matrices are important to us because they arise as adjacency matrices of graphs, and for our purposes it is not important if the vertices of a graph are listed in any particular order. Consequently we regard the indices of the rows and columns of a symmetric matrix as elements of a set that is not ordered in any particular way.

Many skew-symmetric matrices are singular, of course, and consequently do not have inverses in the usual sense. Nevertheless matrix inversion gives rise to an interesting relation among skew-symmetric matrices.

Definition 2 Let \( S \) be a skew-symmetric \( n \times n \) matrix. A modified inverse of \( S \) is a skew-symmetric matrix obtained as follows: first toggle some diagonal
enters of \( S \) to obtain an invertible symmetric matrix \( S' \), and then change every nonzero diagonal entry of \((S')^{-1}\) to 0.

Here toggling refers to the function \( x \mapsto x + 1 \), which interchanges the elements of \( GF(2) \).

The relation defined by modified inversion is obviously symmetric, but examples indicate that it is not reflexive or transitive.

**Example 3** For each of these four matrices, the set of modified inverses consists of the other three.

\[
\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}
\quad \begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}
\quad \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\quad \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}
\]

**Example 4** The modified inverses of the first of these three matrices include the other two, but not itself. The modified inverses of the second include the first, but not itself or the third. The modified inverses of the third include the first and itself, but not the second.

\[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{pmatrix}
\quad \begin{pmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0
\end{pmatrix}
\quad \begin{pmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0
\end{pmatrix}
\]

Definition 5 yields an equivalence relation in the usual way.

**Definition 5** Let \( S \) and \( T \) be skew-symmetric \( GF(2) \)-matrices. Then \( S \sim_{mi} T \) if \( T \) can be obtained from \( S \) through a finite (possibly empty) sequence of modified inversions.

We recall some relevant definitions from graph theory. In a 4-regular multigraph every vertex is of degree 4. Loops and parallel edges are allowed; a loop contributes twice to the degree of the incident vertex. In order to distinguish between the two orientations of a loop it is technically necessary to consider half-edges rather than edges; we will often leave it to the reader to refine statements regarding edges accordingly. A walk in a 4-regular graph is a sequence \( v_1, h_1, h'_2, v_2, h_2, h'_3, \ldots, h_{k-1}, h'_k, v_k \) such that for each \( i \), \( h_i \) and \( h'_i \) are half-edges incident on \( v_i \), and \( h_i \) and \( h'_{i+1} \) are half-edges of a single edge. The walk is closed if \( v_1 = v_k \). A walk in which no edge is repeated is a trail, and a closed trail is a circuit. An Euler circuit is a circuit that contains every edge of the graph. Every connected 4-regular multigraph has Euler circuits, and every 4-regular multigraph has Euler systems, each of which contains one Euler circuit for every connected component of the graph.

For example, Figure 4 illustrates two Euler circuits in a connected 4-regular multigraph. These two Euler circuits illustrate the following.

**Definition 6** If \( C \) is an Euler system of a 4-regular multigraph \( F \) and \( v \in V(F) \) then the \( v \)-transform \( C \ast v \) is the Euler system obtained by reversing one of the two \( v \)-to-\( v \) trails within the circuit of \( C \) incident at \( v \).
The $\kappa$-transformations were introduced by Kotzig \cite{Kotzig}, who proved the fundamental fact of the circuit theory of $4$-regular multigraphs.

**Theorem 7** (Kotzig’s theorem) All the Euler systems of a $4$-regular multigraph can be obtained from any one by applying finite sequences of $\kappa$-transformations.

The alternance or interlacement graph $I(F,C)$ associated to an Euler system $C$ of a $4$-regular multigraph $F$ was introduced shortly after Kotzig’s work became known \cite{Bouchet1, Bouchet2, Read}. Two vertices $v \neq w$ of $F$ are interlaced with respect to $C$ if and only if they appear in the order $v...w...v...w$ on one of the circuits included in $C$.

**Definition 8** The interlacement graph of a $4$-regular graph $F$ with respect to an Euler system $C$ is the simple graph $I(F,C)$ with $V(I(F,C)) = V(F)$ and $E(I(F,C)) = \{vw \mid v$ and $w$ are interlaced with respect to $C\}$. The interlacement matrix of $F$ with respect to $C$ is the adjacency matrix of this graph; we use $I(F,C)$ to denote both the graph and the matrix.

A simple graph that can be realized as an interlacement graph is called a circle graph, and Kotzig’s $\kappa$-transformations give rise to the fundamental operation of the theory of circle graphs, which we call simple local complementation. This operation has been studied by Bouchet \cite{Bouchet1, Bouchet2, Bouchet3}, de Fraysseix \cite{Fraysseix}, and Read and Rosenstiehl \cite{Read}, among others.

**Definition 9** Let $S$ be a skew-symmetric $n \times n$ matrix, and suppose $1 \leq i \leq n$. Then the simple local complement of $S$ at $i$ is the skew-symmetric $GF(2)$-matrix $S'$ obtained from $S$ as follows: whenever $i \neq j \neq k \neq i$ and $s_{ij} \neq 0 \neq s_{ik}$, replace $s_{jk}$ with $s_{jk} + 1$.

We call this operation simple local complementation to distinguish it from the similar operation that Arratia, Bollobás and Sorkin called local complementation in \cite{Arratia}. That operation applies to looped graphs, or equivalently, symmetric matrices. It differs from simple local complementation in that it also includes loop-toggling (reversal of diagonal entries).
For instance, the first $3 \times 3$ matrix of Example 3 has three distinct simple local complements, which are the same as its three modified inverses. Each of the three other $3 \times 3$ matrices of Example 3 has only two distinct simple local complements, itself and the first matrix.

**Definition 10** Let $S$ and $T$ be skew-symmetric $GF(2)$-matrices. Then $S \sim_{lc} T$ if $T$ can be obtained from $S$ through a finite (possibly empty) sequence of simple local complementations.

**Proposition 11** This defines an equivalence relation.

**Proof.** As $(S^i)^i = S$, $\sim_{lc}$ is symmetric. The reflexive and transitive properties are obvious.

In Section 3 we prove a surprising result:

**Theorem 12** Let $S$ and $T$ be skew-symmetric $GF(2)$-matrices. Then $S \sim_{lc} T$ if and only if $S \sim_{mi} T$.

We might say that modified inversion constitutes a kind of global complementation of a skew-symmetric matrix (or equivalently, a simple graph). Theorem 12 shows that even though individual global complementations do not generally have the same effect as individual local complementations, the two operations generate the same equivalence relation.

Theorem 12 developed as we tried to understand some recent results regarding a restricted form of local complementation. Several authors have written about the equivalence relation on looped graphs (or equivalently, symmetric $GF(2)$-matrices) generated by pivots on unlooped edges and (non-simple) local complementations at looped vertices; we denote this relation $\sim_{piv}$. Genest [22] called the equivalence classes under $\sim_{piv}$ Sabidussi orbits. Glantz and Pelillo [23] and Brijder and Hoogeboom [16, 17, 18] observed that another way to generate $\sim_{piv}$ is to use a matrix operation related to inversion, the principal pivot transform [45, 46]. In particular, Theorem 24 of [17] shows that the combination of loop-toggling with the principal pivot transform yields a new description of $\sim_{lc}$, different from Definition 9 (and also different from Definition 2). Ilyutko [25, 26] has also studied inverse matrices and the equivalence relation $\sim_{piv}$; he used them to compare the adjacency matrices of certain kinds of chord diagrams that arise from knot diagrams. Ilyutko’s account includes an analysis of the effect of the Reidemeister moves of knot theory, and also includes the idea of generating an equivalence relation on nonsingular symmetric matrices by toggling diagonal entries. Considering the themes shared by these results, it seemed natural to look for fundamental properties of $\sim_{lc}$ that underlie them.

Theorem 12 is part of a very pretty theory tying the elementary linear algebra of symmetric $GF(2)$-matrices to the circuit theory of 4-regular multigraphs. This theory has been explored by several authors over the last forty years, but the relevant literature is fragmented and it does not seem that the generality and simplicity of the theory are fully appreciated. We proceed to give an account.
At each vertex of a 4-regular multigraph there are three transitions – three distinct ways to sort the four incident half-edges into two disjoint pairs. Kotzig [31] introduced this notion, and observed that each of the $3^n$ ways to choose one transition at each vertex yields a partition of $E(F)$ into edge-disjoint circuits; such partitions are called circuit partitions [1, 2, 3] or Eulerian partitions [32, 35]. (Kotzig actually used the term transition in a slightly different way, to refer to only one pair of half-edges. As we have no reason to ever consider a single pair of half-edges without also considering the complementary pair, we follow the usage of Ellis-Monaghan and Sarmiento [20] and Jaeger [28] rather than Kotzig’s.)

In [44] we introduced the following way to label the three transitions at $v$ with respect to a given Euler system $C$. Choose either of the two orientations of the circuit of $C$ incident at $v$, and use $\phi$ to label the transition followed by this circuit; use $\chi$ to label the other transition in which in-directed edges are paired with out-directed edges; and use $\psi$ to label the transition in which the two in-directed edges are paired with each other, and the two out-directed edges are paired with each other. See Figure 2, where the pairings of half-edges are indicated by using solid line segments for one pair and dashed line segments for the other.

**Definition 13** Let $C$ be an Euler system of a 4-regular multigraph $F$, and let $P$ be a circuit partition of $F$. The relative interlacement matrix $\mathcal{I}_P(F, C)$ of $P$ with respect to $C$ is obtained from $\mathcal{I}(F, C)$ by modifying the row and column corresponding to each vertex at which $P$ does not involve the transition labeled $\chi$ with respect to $C$:

(i) If $P$ involves the $\phi$ transition at $v$, then modify the row and column of $\mathcal{I}(F, C)$ corresponding to $v$ by changing every nonzero entry to 0, and changing the diagonal entry from 0 to 1.

(ii) If $P$ involves the $\psi$ transition at $v$, then modify the row and column of $\mathcal{I}(F, C)$ corresponding to $v$ by changing the diagonal entry from 0 to 1.

The relative interlacement matrix determines the number of circuits in $P$:

$$\nu(\mathcal{I}_P(F, C)) + c(F) = |P|,$$

where $\nu(\mathcal{I}_P(F, C))$ denotes the $GF(2)$-nullity of $\mathcal{I}_P(F, C)$ and $c(F)$ denotes the number of connected components in $F$. We refer to this equation as the circuit-
nullity formula or the extended Cohn-Lempel equality; many special cases and reformulations have appeared over the years [5, 6, 8, 15, 19, 27, 29, 30, 33, 34, 36, 37, 40, 41, 42, 47]. A detailed account is given in 43.

A sharper form of the circuit-nullity formula includes a precise description of the nullspace of $I_P(F,C)$:

**Definition 14** Let $P$ be a circuit partition of the 4-regular multigraph $F$, and let $C$ be an Euler system of $F$. For each circuit $\gamma \in P$, let the relative core vector of $\gamma$ with respect to $C$ be the vector $\rho(\gamma, C) \in GF(2)^{V(F)}$ whose nonzero entries correspond to the vertices of $F$ at which $P$ involves either the $\chi$ or the $\psi$ transition, and $\gamma$ is singly incident.

**Theorem 15** Let $P$ be a circuit partition of the 4-regular multigraph $F$, and let $C$ be an Euler system of $F$.

(i) The nullspace of the relative interlacement matrix $I_P(F,C)$ is spanned by the relative core vectors of the circuits of $P$.

(ii) For each connected component of $F$, the relative core vectors of the incident circuits of $P$ sum to 0.

(iii) If $Q \subseteq P$ and there is no connected component of $F$ for which $Q$ contains every incident circuit of $P$, then the relative core vectors of the circuits of $Q$ are linearly independent.

Theorem 15 is an example of a comment made above, that the generality and simplicity of the relationship between linear algebra and the circuit theory of 4-regular multigraphs have not been fully appreciated.

On the one hand, Theorem 15 is general: it applies to every 4-regular multigraph $F$, every Euler system $C$ and every circuit partition $P$. In Proposition 4 of [27], Jaeger proved an equivalent version of the special case of Theorem 15 involving the additional assumptions that $F$ is connected and $C$ and $P$ involve different transitions at every vertex. (The latter assumption is implicit in Jaeger’s use of left-right walks on chord diagrams.) It is Jaeger who introduced the term core vector; we use relative core vector to reflect the fact that in Definition 14 the vector is adjusted according to the Euler system with respect to which it is defined. Bouchet [8] gave a different proof with the additional restriction that $C$ and $P$ respect a fixed choice of edge-direction in $F$; or equivalently, that $P$ involve only transitions that are labeled $\chi$ with respect to $C$. (Bouchet’s result is also presented in [24].) These results may give the erroneous impression that an Euler system $C$ provides information only about those circuit partitions that disagree with $C$ at every vertex. In fact, every Euler system gives rise to a mapping

$$\{\text{circuit partitions of } F\} \rightarrow \{\text{subspaces of } GF(2)^{V(F)}\}$$

under which the image of an arbitrary circuit partition $P$ is the $(|P| - c(F))$-dimensional subspace spanned by the relative core vectors of the circuits in $P$. 

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On the other hand, Theorem 15 is simple: Definition 13 gives an explicit description of the relative interlacement matrix associated to a circuit partition, and Definition 14 gives an explicit description of its nullspace. Although some version of Theorem 15 may be implicit in the theory of delta-matroids, isotropic systems and multimatroids associated with 4-regular graphs (cf. for instance [11, 12, 13, 14]), these structures are sufficiently abstract that it is difficult to extract explicit descriptions like Definition 13 and Definition 14 from them.

Theorem 15 implies that if $C$ is an Euler system of $F$, then we can find every other Euler system $C'$ of $F$ by finding every way to obtain a nonsingular matrix from $I(F,C)$ using the modifications given in Definition 13. Moreover, there is a striking symmetry tying together the relative interlacement matrices of two Euler systems:

**Theorem 16** Let $F$ be a 4-regular graph with Euler systems $C$ and $C'$. Then

$$I_{C'}(F,C)^{-1} = I_C(F,C').$$

Like Theorem 15, Theorem 16 generalizes results of Bouchet and Jaeger discussed in [8, 24, 27], which include the additional assumption that $C$ and $C'$ are incompatible (i.e., they do not involve the same transition at any vertex). Bouchet’s version requires also that $C$ and $C'$ respect the same edge-directions.

Greater generality is always desirable, of course, but it is important to observe that in fact, Theorem 16 is particularly valuable when $C$ and $C'$ are incompatible. The equation $I_{C'}(F,C)^{-1} = I_C(F,C')$ does not allow us to construct the full interlacement matrix $I(F,C')$ directly from $I(F,C)$ if $C$ and $C'$ share a transition at any vertex, because there is no way to recover the information that is lost when off-diagonal entries are set to 0 in part (i) of Definition 13 (For instance, if $C$ is any Euler system and $v$ is any vertex then we learn nothing from the fact that $I_C(F,C) = I_C(F,C \ast v) = I_{C \ast v}(F,C)$ is the identity matrix.) This observation motivates our last definition.

**Definition 17** Let $F$ be a 4-regular multigraph with an Euler system $C$, and suppose $W \subseteq V(F)$ has the property that a nonsingular matrix $M$ is obtained from $I(F,C)$ by changing the diagonal entries corresponding to elements of $W$ from 0 to 1. The $\iota$-transform of $C$ with respect to $W$ is the Euler system $C\#W$ with $I_{C\#W}(F,C) = M$.

That is, $C\#W$ is the Euler system obtained from $C$ by using the $\psi$ transition at every vertex in $W$, and the $\chi$ transition at every vertex not in $W$.

Theorem 12 directly implies the following analogue of Kotzig’s theorem:

**Theorem 18** All the Euler systems of a 4-regular multigraph can be obtained from any one by applying finite sequences of $\iota$-transformations.

The fact that the full interlacement matrix $I(F,C\#W)$ is determined by $I(F,C)$ implies that the transition labels of $C\#W$ are also determined. Suppose $v \in V(F)$; we use $\phi, \chi, \psi$ to label the three transitions at $v$ with respect to $C$, $C'$, and $C\#W$. **
Figure 3: An Euler circuit of $C$ incident at a vertex $v$, along with the four different ways $C \# W$ might be configured at $v$.

and $\phi', \chi', \psi'$ to label the three transitions with respect to $C \# W$. Let $M$ denote the nonsingular matrix obtained from $I(F,C)$ by changing the diagonal entries corresponding to elements of $W$ from 0 to 1, and let $X \subseteq V(F)$ denote the set of vertices corresponding to nonzero diagonal entries of $M^{-1}$. Then $\phi' = \psi$ if $v \in W$, $\phi' = \chi$ if $v \notin W$, $\phi = \psi'$ if $v \in X$ and $\phi = \chi'$ if $v \notin X$. Consequently $v \in W \cap X$ implies $\phi' = \psi$, $\phi = \psi'$ and $\chi = \chi'$; $v \in W - X$ implies $\phi' = \psi$, $\phi = \chi'$ and $\chi = \psi'$; $v \in X - W$ implies $\phi' = \chi$, $\phi = \psi'$ and $\psi = \chi'$; and $v \in V(F) - W - X$ implies $\phi' = \chi$, $\phi = \chi'$ and $\psi = \psi'$. See Figure 3 where the four cases are indexed by first listing $\phi'$ with respect to $C$ and then listing $\phi$ with respect to $C'$, so that (for instance) $v \in W - X$ is indexed $\psi \chi'$.

Theorem 16 also implies that interlacement matrices satisfy a limited kind of multiplicative functoriality, which we have not seen mentioned elsewhere.

**Corollary 19** Let $C$ and $C'$ be Euler systems of a 4-regular multigraph $F$, and let $P$ be the circuit partition described by: (a) at every vertex where $C$ and $C'$ involve the same transition, $P$ involves the same transition; and (b) at every vertex where $C$ and $C'$ involve two different transitions, $P$ involves the third transition. Then

$$I_P(F,C) = I_{C'}(F,C) \cdot I_P(F,C').$$

**Proof.** Let $I$ be the identity matrix, and let $I'$ be the diagonal matrix whose $vv$ entry is 1 if and only if $C$ and $C'$ involve different transitions at $v$. Then

$$I_P(F,C) = I' + I_{C'}(F,C) = I + I_{C'}(F,C) \cdot I' = I_{C'}(F,C) \cdot (I_{C}(F,C') + I') = I_{C'}(F,C) \cdot I_P(F,C').$$

**Corollary 20** Suppose a 4-regular multigraph $F$ has three pairwise incompatible Euler systems $C$, $C'$ and $C''$. (That is, no two of $C, C', C''$ involve the same
transition at any vertex). Then
\[ I_{\mathcal{C}}(F, C') \cdot I_{\mathcal{C}''}(F, C) \cdot I_{\mathcal{C}'}(F, C'') \]
is the identity matrix.

Proof. By Corollary \[19\] \[ I_{\mathcal{C}''}(F, C) = I_{\mathcal{C}'}(F, C) \cdot I_{\mathcal{C}''}(F, C'). \] ■

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2 The \( \nu, \nu, \nu + 1 \) lemma

Lemma 2 of Balister, Bollobás, Cutler and Pebody \[4\] is a very useful result about the nullities of three related matrices. In this section we prove a sharpened form of the lemma, involving the nullspaces of the three matrices rather than only their nullities.

Lemma 21 Suppose \( M \) is a symmetric \( \text{GF}(2) \)-matrix. Let \( \rho \) be an arbitrary row vector, and let \( \mathbf{0} \) be the row vector with all entries 0; denote their transposes \( \kappa \) and \( \mathbf{0} \) respectively. Let \( M_1, M_2 \) and \( M_3 \) denote the indicated symmetric matrices.

\[
M_1 = \begin{pmatrix} M & \kappa \\ \rho & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} M & \kappa \\ \rho & 0 \end{pmatrix} \quad M_3 = \begin{pmatrix} M & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}
\]

Then two of \( M_1, M_2, M_3 \) have the same nullspace, say of dimension \( \nu \). The nullspace of the remaining matrix has dimension \( \nu + 1 \), and it contains the nullspace shared by the other two.

Proof. We use \( \text{rk}(N) \) and \( \nu(N) \) to denote the rank and nullity of a matrix \( N \), \( \text{row}(N) \) and \( \text{col}(N) \) to denote the spaces spanned by the rows and columns of \( N \), and \( \ker N \) to denote the nullspace of \( N \), i.e., the space of row vectors \( x \) with \( x \cdot N = \mathbf{0} \).

Case 1. Suppose \( \text{rk}(M_1) \leq \text{rk}(M_2) \) and \( \text{rk}(M_1) \leq \text{rk}(M_3) \).

If \( \rho \notin \text{row}(M) \), then \( \text{rk}(M_1) = \text{rk}(M) + 2 > \text{rk}(M) + 1 = \text{rk}(M_3) \), contradicting the hypothesis that \( \text{rk}(M_1) \leq \text{rk}(M_3) \). It follows that in this case \( \rho \in \text{row}(M) \). That is, there is a row vector \( \rho_0 \) with \( \rho_0 \cdot M = \rho \).

If \( \rho_0 \cdot (M \ \kappa) = (\rho \ \mathbf{0}) \), then \( (\rho \ \mathbf{0}) \in \text{row}(M \ \kappa) \), so \( \text{rk}(M_2) = \text{rk}(M \ \kappa) \). Note that \( \rho \in \text{row}(M) \) implies \( \kappa \in \text{col}(M) \), so we conclude that \( \text{rk}(M_2) = \text{rk}(M) \). As \( M_1 \) is of smallest rank, it follows that \( \text{rk}(M_1) = \text{rk}(M) \) also. But this implies that \( (\rho \ \mathbf{0}), (\rho \ 1) \in \text{row}(M \ \kappa) \), and hence

\[
\text{rk}(M) = \text{rk} \begin{pmatrix} M & \kappa \\ \rho & 0 \\ \rho & 1 \end{pmatrix} = \text{rk} \begin{pmatrix} M & \kappa \\ \rho & 0 \\ 0 & 1 \end{pmatrix} = \text{rk} \begin{pmatrix} M & \mathbf{0} \\ \rho & 0 \\ 0 & 1 \end{pmatrix}.
\]
which is ridiculous.

Consequently
\[ \rho_0 \cdot (M_\kappa) = (\rho_1), \]
and hence \((\rho_0 \ 1) \in \text{row} (M_\kappa)\) and \(\text{rk}(M_1) = \text{rk} (M_\kappa)\). Note that \(\rho \in \text{row}(M)\) implies \(\kappa \in \text{col}(M)\), so we conclude that \(\text{rk}(M_1) = \text{rk}(M)\). Then
\[
\text{rk}(M_2) = \text{rk} \begin{pmatrix} M & \kappa \\ \rho & 0 \end{pmatrix} = \text{rk} \begin{pmatrix} M & \kappa \\ \rho & 1 \end{pmatrix} = \text{rk} \begin{pmatrix} M & 0 \\ \rho & 0 \end{pmatrix} = \text{rk} \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} = \text{rk}(M_3),
\]
so \(\text{rk}(M_2) = \text{rk}(M_3) = \text{rk}(M) + 1 = \text{rk}(M_1) + 1\).

We claim again that \(\text{ker} (M_1) \subseteq \ker (M_3) \cap \ker (M_2)\). Suppose \(x \in \ker (M_3)\); clearly then \(x = (y \ 0)\) for some \(y \in \ker M\). As \(y \in \ker M\), \(y \cdot z = 0 \ \forall z \in \text{col}(M)\). It follows that \(y \cdot \kappa = 0\), and hence \(x \in (\ker M_1) \cap (\ker M_2)\).

As \(\text{rk}(M_3) = \text{rk}(M_2) = \text{rk}(M_1) + 1\), the claim implies \(\ker M_3 = \ker M_2 \subseteq \ker M_1\) and \(\nu(M_3) = \nu(M_2) = \nu(M_1) - 1\).

**Case 2.** Suppose \(\text{rk}(M_1) > \text{rk}(M_2)\) and \(\text{rk}(M_2) \leq \text{rk}(M_3)\).

As in Case 1, \(\rho \notin \text{row}(M) \Rightarrow \text{rk}(M_2) = \text{rk}(M) + 2 > \text{rk}(M) + 1 = \text{rk}(M_3)\), so by contradiction we conclude that \(\rho \in \text{row}(M)\) and hence at least one of \((\rho \ 0)\), \((\rho \ 1)\) is an element of row \((M_\kappa)\). The hypothesis \(\text{rk}(M_2) < \text{rk}(M_1)\) implies that \((\rho \ 0) \in \text{row} (M_\kappa)\) and \((\rho \ 1) \notin \text{row} (M_\kappa)\), so
\[
\text{rk}(M_1) = \text{rk} \begin{pmatrix} M & \kappa \\ \rho & 1 \end{pmatrix} = \text{rk} \begin{pmatrix} M & \kappa \\ \rho & 0 \end{pmatrix} = \text{rk} \begin{pmatrix} M & \kappa \\ 0 & 1 \end{pmatrix} = \text{rk}(M_3),
\]
so \(\text{rk}(M_1) = \text{rk}(M_3) = \text{rk}(M) + 1 = \text{rk}(M_2) + 1\) in this case.

We claim again that \(\ker M_3 \subseteq (\ker M_1) \cap (\ker M_2)\); the proof is the same as it was in the first case.

As \(\text{rk}(M_3) = \text{rk}(M_1) = \text{rk}(M_2) + 1\), the claim implies \(\ker M_3 = \ker M_1 \subseteq \ker M_2\) and \(\nu(M_3) = \nu(M_2) = \nu(M_1) - 1\).

**Case 3.** Suppose \(\text{rk}(M_1) > \text{rk}(M_3)\) and \(\text{rk}(M_2) > \text{rk}(M_3)\).

As \(\text{rk}(M_3) = \text{rk}(M) + 1\) and \(\text{rk}(M_1), \text{rk}(M_2) \leq \text{rk}(M) + 2\), it must be that \(\text{rk}(M_1) = \text{rk}(M_2) = \text{rk}(M) + 2\) in this case.

We claim that \(\ker M_3 \supseteq (\ker M_1) \cap (\ker M_2)\). Note first that no row vector \((y \ 1)\) can be an element of both \(\ker M_1\) and \(\ker M_2\), because \((y \ 1) \cdot M_1 = 0 \Rightarrow y \cdot \kappa = 1\) and \((y \ 1) \cdot M_2 = 0 \Rightarrow y \cdot \kappa = 0\). Consequently every \(x \in (\ker M_1) \cap (\ker M_2)\) is of the form \((y \ 0)\) for some \(y \in \ker M\). Obviously
\[
\ker M_3 = \left\{ (y \ 0) \mid y \in \ker M \right\},
\]
so the claim is verified.

The inclusion
\[
(\ker M_1) \cap (\ker M_2) \supseteq \left\{ (y \ 0) \mid y \in \ker M \text{ and } y \cdot \kappa = 0 \right\}
\]
is obvious, but note that the dimension of the right hand side is at least $\nu(M) - 1$. As $rk(M_1) = rk(M_2) = rk(M) + 2$, it must be that $\nu(M) - 1 = \nu(M_1) = \nu(M_2)$. We conclude that

$$\ker M_1 = \ker M_2 = (\ker M_1) \cap (\ker M_2) = \{(y, 0) \mid y \in \ker M \text{ and } y \cdot \kappa = 0\}.$$  

3 Theorem 12

3.1 Modified inverses from local complementation

Theorem 12 concerns the equivalence relation $\sim_{lc}$ on skew-symmetric matrices generated by simultaneous re-indexings of rows and columns along with simple local complementations $S \mapsto S_i$, where $i$ is an arbitrary index. As noted in the introduction, Theorem 12 is related to results of Brijder and Hoogeboom \[16, 17, 18\], Glantz and Pelillo \[23\] and Ilyutko \[25, 26\] regarding an equivalence relation $\sim_{piv}$ on symmetric matrices that is generated by two kinds of operations: (non-simple) local complementations $S \mapsto S_i$ where the $i^{th}$ diagonal entry is 1, and edge pivots $S \mapsto S_{ij}$ where the $i^{th}$ and $j^{th}$ diagonal entries are 0 and the $ij$ and $ji$ entries are 1. (The reader familiar with the work of Arratia, Bollobás and Sorkin \[1, 2, 3\] should be advised that the operation they denote $S_{ij}$ differs from this one by interchanging the $i^{th}$ and $j^{th}$ rows and columns.) It is well known that an edge pivot coincides with two triple local complements, $((S_i)^j)^i = ((S_j)^i)^j = S_{ij}$. (See \[17\] for an extension of this familiar equality to set systems.) This equality is not very useful in connection with $\sim_{piv}$, as the individual local complementations $S_i$ and $S_j$ are “illegal” for $\sim_{piv}$ when the $i^{th}$ and $j^{th}$ diagonal entries are 0. It is useful for us, though: non-simple local complementation coincides with simple local complementation off the diagonal, so the equality implies that if $S$ and $T$ are skew-symmetric matrices, and $S'$ and $T'$ are symmetric matrices that are (respectively) equal to $S$ and $T$ except for diagonal entries, then $S' \sim_{piv} T' \Rightarrow S \sim_{lc} T$.

This implication allows us to prove half of Theorem 12 using a result noted in \[16, 17, 18, 23\], namely that symmetric matrices related by the principal pivot transform are also related by $\sim_{piv}$. Suppose $S_1$ is skew-symmetric and $S_2$ is a modified inverse of $S_1$. Then there are nonsingular symmetric matrices $S'_1$ and $S'_2$ that are (respectively) equal to $S_1$ and $S_2$ except for diagonal entries, and have $(S'_1)^{-1} = S'_2$. As inverses, $S'_1$ and $S'_2$ are obviously related by the principal pivot transform, so $S'_1 \sim_{piv} S'_2$; consequently the implication of the last paragraph tells us that $S_1 \sim_{lc} S_2$. Iterating this argument, we conclude that $S \sim_{mi} T \Rightarrow S \sim_{lc} T$.

3.2 A brief discussion of the principal pivot transform

The principal pivot transform (ppt) was introduced by Tucker \[46\]; a survey of its properties was given by Tsatsomeros \[45\]. The reader who would like to
learn about the ppt and its relationship with graph theory should consult these papers, the work of Brijder and Hoogeboom [10, 17] and Glantz and Pelillo [23], and the references given there. For the convenience of the reader who simply wants to understand the argument above, we sketch the details briefly.

The principal pivot transform is valuable for us because it provides a way to obtain the inverse of a nonsingular symmetric $GF(2)$-matrix incrementally, using local complementation. This property is not immediately apparent from the definition:

**Definition 22** Suppose

$$M = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

is an $n \times n$ matrix with entries in a field $F$, and suppose $P$ is a nonsingular principal submatrix involving the rows and columns whose indices lie in a set $X$. Then

$$M \ast X = \begin{pmatrix} P^{-1} & -P^{-1}Q \\ RP^{-1} & S - RP^{-1}Q \end{pmatrix}.$$ 

In particular, if $M$ is nonsingular then $M \ast \{1, \ldots, n\} = M^{-1}$.

The matrix $M$ is displayed in the given form only for convenience; the definition may be applied to any set of indices $X$ whose corresponding principal submatrix $P$ is nonsingular.

A direct calculation (which we leave to the reader) yields an alternative characterization: if we think of $F^n$ as the direct sum of a subspace corresponding to indices from $X$ and a subspace corresponding to the rest of the indices, then the linear endomorphism of $F^n$ corresponding to $M \ast X$ is related to the linear endomorphism corresponding to $M$ by the relation

$$M \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \text{ if and only if } (M \ast X) \begin{pmatrix} y_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_2 \end{pmatrix}.$$ 

This characterization directly implies two useful properties.

1. If $X_1$ and $X_2$ are disjoint subsets of $\{1, \ldots, n\}$ then $M \ast (X_1 \cup X_2)$ is defined if and only if $(M \ast X_1) \ast X_2$ is defined; and if they are defined, then they are the same.

2. If $M$ is nonsingular then the submatrix $S - RP^{-1}Q$ of $M \ast X$ must be nonsingular too. For

$$(M \ast X) \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \text{ implies } M \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which requires $x_1 = 0$ and $x_2 = 0$.

Suppose now that $M$ is a nonsingular symmetric $GF(2)$-matrix. We begin a recursive calculation by performing principal pivot transforms $\ast \{i_1\}, \ast \{i_2\}, \ldots, \ast \{i_k\}$ for as long as we can, subject to the proviso that $i_1, \ldots, i_k$ are pairwise distinct. It is a direct consequence of Definition 22 that off the diagonal, each
transformation \(\ast \{i,j\}\) is the same as local complementation with respect to \(i,j\). For convenience we display the special case \(\{i_1,\ldots,i_k\}\): 

\[
M \ast \{i_1,\ldots,i_k\} = M \ast \{i_1\} \ast \cdots \ast \{i_k\} = M' = \begin{pmatrix} P' & Q' \\ R' & S' \end{pmatrix},
\]

where the diagonal entries of \(P'\) (resp. \(S'\)) are all 1 (resp. all 0). Note that by property (2), \(S'\) is nonsingular; hence each row of \(S'\) must certainly contain a nonzero entry. Consequently we may (again for convenience) display the matrix in a different way:

\[
M \ast \{i_1,\ldots,i_k\} = M' = \begin{pmatrix} P'' & Q'' \\ R'' & S'' \end{pmatrix} \quad \text{where} \quad P'' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

The next step in the calculation is to perform the principal pivot transform \(\ast \{i_{k+1},i_{k+2}\}\), where \(i_{k+1}\) and \(i_{k+2}\) are the indices involved in \(P''\). A direct consequence of Definition 22 is that off the diagonal, \(M' \ast \{i_{k+1},i_{k+2}\}\) is the same as the triple simple local complement \(((M')^{i_{k+1}})^{i_{k+2}})^{i_{k+1}}\).

If the resulting matrix has a new nonzero diagonal entry (i.e., one whose index has not yet been used), then repeat the first step of the calculation; if not, repeat the second step. At no point do we re-use an index that has already been involved in a principal pivot transform, and at no point is the principal submatrix determined by the as-yet unused indices singular. Consequently the calculation proceeds until every index has been involved in precisely one principal pivot transform. By property (1), at the end of the calculation we have obtained \(M \ast \{1,\ldots,n\} = M^{-1}\) using individual steps each of which is the same (off the diagonal) as either one simple local complementation or the composition of three simple local complementations.

### 3.3 Local complements from modified inversion

The implication \(S \sim_{lc} T \Rightarrow S \sim_{mi} T\) does not follow directly from results regarding \(\sim_{piv}\), but our argument uses two lemmas very similar to ones used by Ilyutko [25].

**Lemma 23** Let \(S\) be an \(n \times n\) skew-symmetric GF(2)-matrix. Suppose \(1 \leq i \leq n\) and the \(i^{th}\) row of \(S\) has at least one nonzero entry. Then there is a nonsingular symmetric GF(2)-matrix \(M\) that differs from \(S\) only in diagonal entries other than the \(i^{th}\).

**Proof.** We may as well presume that \(i = 1\), and that \(s_{12} \neq 0\).

For \(k \geq 2\) let \(S_k\) be the submatrix of \(S\) obtained by deleting all rows and columns with indices \(> k\). We claim that for every \(k \geq 2\), there is a subset \(T_k \subseteq \{3,\ldots,k\}\) with the property that we obtain a nonsingular matrix \(S_k'\) by toggling the diagonal entries of \(S_k\) with indices in \(T_k\). When \(k = 2\) the claim is satisfied by \(T_2 = \emptyset\).

Proceeding inductively, suppose \(k \geq 2\) and \(T_k \subseteq \{3,\ldots,k\}\) satisfies the claim. If we obtain a nonsingular matrix by toggling the diagonal entries of \(S_{k+1}\) with
indices in \( T_k \), then \( T_{k+1} = T_k \) satisfies the claim for \( k + 1 \), If not, then the \( \nu, \nu, \nu + 1 \) lemma implies that \( T_{k+1} = T_k \cup \{ k + 1 \} \) satisfies the claim.

The required matrix \( M \) is obtained from \( S = S_n \) by toggling the diagonal entries with indices in \( T_n \). ■

**Lemma 24** Suppose

\[
M = \begin{pmatrix}
0 & 1 & 0 \\
1 & M_{11} & M_{12} \\
0 & M_{12} & M_{22}
\end{pmatrix}
\]

is a nonsingular symmetric matrix, with

\[
M^{-1} = \begin{pmatrix}
a & \rho \\
\kappa & N
\end{pmatrix}.
\]

Then the matrix

\[
M' = \begin{pmatrix}
0 & 1 & 0 \\
1 & M'_{11} & M_{12} \\
0 & M_{12} & M_{22}
\end{pmatrix}
\]

obtained by toggling all entries within the block \( M_{11} \) is also nonsingular, and

\[
(M')^{-1} = \begin{pmatrix}
a + 1 & \rho \\
\kappa & N
\end{pmatrix}.
\]

**Proof.** \( M^{-1} \cdot M \) is the identity matrix, so

\[
\begin{pmatrix}
a & \rho \\
\kappa & N
\end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 1;
\]

consequently the row vector \( \rho \) must have an odd number of nonzero entries in the columns corresponding to the \( 1 \) in the first row of \( M \). Similarly, each row of \( N \) must have an even number of nonzero entries in these columns, because

\[
\begin{pmatrix} \kappa & N \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0.
\]

It follows that the products

\[
\begin{pmatrix} a & \rho \\ \kappa & N \end{pmatrix} \cdot M \text{ and } \begin{pmatrix} a + 1 & \rho \\ \kappa & N \end{pmatrix} \cdot M'
\]

are equal. ■

As stated, Lemma 24 requires that the first row of \( M \) be in the form \( \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \). This is done only for convenience; the general version of the lemma applies to any row in which the diagonal entry is 0.

Suppose now that \( S \) is a skew-symmetric \( n \times n \) matrix and \( 1 \leq i \leq n \). If every entry of the \( i^\text{th} \) row of \( S \) is 0, then the local complement \( S^i \) is the same as \( S \).
If the $i^{th}$ row of $S$ includes some nonzero entry, then Lemma 23 tells us that there is a nonsingular matrix $M$ such that (a) $M$ is obtained from $S$ by toggling some diagonal entries other than the $i^{th}$. The general version of Lemma 24 then tells us that there is a nonsingular matrix $M'$ such that (b) $(M')^{-1}$ equals $M^{-1}$ except for the $i^{th}$ diagonal entry and (c) $M'$ equals the local complement $S'$ except for diagonal entries. Condition (a) implies that a modified inverse $T$ of $S$ is obtained by changing all diagonal entries of $M^{-1}$ to 0; condition (b) implies that $T$ is also equal to $(M')^{-1}$ except for diagonal entries; and condition (c) implies that the local complement $S'$ is a modified inverse of $T$.

It follows that every local complement $S'$ can be obtained from $S$ using no more than two modified inversions. Applying this repeatedly, we conclude that $S \sim_{lc} T \Rightarrow S \sim_{mi} T$.

4 Theorem 15

Let $F$ be a 4-regular multigraph with an Euler system $C$, and suppose $v \in V(F)$. Let the circuit of $C$ incident at $v$ be $vC_1vC_2v$, where $v$ does not appear within $C_1$ or $C_2$. Every edge of the connected component of $F$ that contains $v$ lies on precisely one of $C_1, C_2$. If $w$ is a vertex of this component which is not a neighbor of $v$ in $I(F, C)$, i.e., $v$ and $w$ are not interlaced with respect to $C$, then all four half-edges incident at $w$ appear on the same $C_i$. On the other hand, if $v$ and $w$ are neighbors in $I(F, C)$ then two of the four half-edges incident at $w$ appear on $C_1$, and the other two appear on $C_2$. Moreover the only transition at $w$ that pairs together the half-edges from the same $C_i$ is the $\phi$ transition; the $\chi$ and $\psi$ transitions at $w$ pair each half-edge from $C_1$ with a half-edge from $C_2$. At $v$, instead, only the $\chi$ transition pairs together half-edges from the same $C_i$; the $\phi$ and $\psi$ transitions pair each half-edge from $C_1$ with a half-edge from $C_2$.

These observations are summarized in the table below, where $N(v)$ denotes the set of neighbors of $v$ in $I(F, C)$, (1)(2) indicates a transition that pairs half-edges from the same $C_i$ and (12) indicates a transition that pairs half-edges from $C_1$ and $C_2$.

<table>
<thead>
<tr>
<th>transition</th>
<th>vertex</th>
<th>$v$</th>
<th>$w \in N(v)$</th>
<th>$w \notin N(v)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$</td>
<td>(12)</td>
<td>(1)(2)</td>
<td>(1)(2)</td>
<td></td>
</tr>
<tr>
<td>$\chi$</td>
<td>(1)(2)</td>
<td>(12)</td>
<td>(1)(2)</td>
<td></td>
</tr>
<tr>
<td>$\psi$</td>
<td>(12)</td>
<td>(12)</td>
<td>(1)(2)</td>
<td></td>
</tr>
</tbody>
</table>

Suppose $\gamma$ is a circuit that is singly incident at $v$, and involves the $\phi$ transition at $v$. We start following $\gamma$ at $v$, on a half-edge that belongs to $C_i$. We return to $v$ on a half-edge that belongs to $C_j$, $j \neq i$, so while following $\gamma$ we must have switched between $C_i$ and $C_j$ an odd number of times. (See Figure 4.) That is, $|\{w \in N(v) \mid \gamma \text{ is singly incident at } w \text{ and involves the } \chi \text{ or } \psi \text{ transition at } w\}|$ must be odd.
Figure 4: An Euler circuit and three circuits that are singly incident at a vertex.

Suppose instead that \( \gamma \) is a circuit that is singly incident at \( v \) and involves the \( \chi \) transition at \( v \). If we start following \( \gamma \) along a half-edge that belongs to \( C_i \), we must return to \( v \) on the other half-edge that belongs to \( C_i \), so we must have switched between \( C_i \) and \( C_j \) an even number of times. Consequently

\[
|\{ w \in N(v) \mid \gamma \text{ is singly incident at } w \text{ and involves the } \chi \text{ or } \psi \text{ transition at } w \}|\]

must be even.

Similarly, if \( \gamma \) is a circuit that is singly incident at \( v \) and involves the \( \psi \) transition at \( v \) then

\[
|\{ w \in N(v) \mid \gamma \text{ is singly incident at } w \text{ and involves the } \chi \text{ or } \psi \text{ transition at } w \}|\]

must be odd.

Suppose \( \gamma \) is doubly incident at \( v \) and we follow \( \gamma \) after leaving \( v \) on a half-edge that belongs to \( C_1 \). We cannot be sure on which half-edge we will first return to \( v \). But after leaving again, we must return on the one remaining half-edge. If the first return is from \( C_1 \), then on the way we must traverse an even number of vertices \( w \in N(v) \) such that \( \gamma \) is singly incident at \( w \) and involves the \( \chi \) or \( \psi \) transition at \( w \); moreover when we follow the second part of \( \gamma \) we will leave \( v \) in \( C_2 \) and also return in \( C_2 \), so again we must encounter an even number of such vertices. If the first return is from \( C_2 \) instead, then on the way we must traverse an odd number of vertices \( w \in N(v) \) such that \( \gamma \) is singly incident at \( w \) and involves the \( \chi \) or \( \psi \) transition at \( w \); the second part of the circuit will begin in \( C_i \) and end in \( C_j \) (\( j \neq i \)), so it will also include an odd number of such vertices. In any case,

\[
|\{ w \in N(v) \mid \gamma \text{ is singly incident at } w \text{ and involves the } \chi \text{ or } \psi \text{ transition at } w \}|\]

must be even.

**Proposition 25** Let \( F \) be a 4-regular multigraph with an Euler system \( C \), and suppose \( P \) is a circuit partition of \( F \). Then \( \rho(\gamma, C) \in \ker I_P(F, C) \forall \gamma \in P \).

**Proof.** Recall that \( \rho(\gamma, C) \in GF(2)^{V(F)} \) has nonzero entries corresponding to the vertices where \( \gamma \) is singly incident and does not involve the \( \phi \) transition; we consider \( \rho(\gamma, C) \) as a row vector. Let \( v \in V(F) \), and let \( \kappa(v) \) be the column of \( I_P(F, C) \) corresponding to \( v \).
If \( \gamma \) lies in a different connected component than \( v \), or if \( P \) involves the \( \phi \) transition at \( v \), then for every \( w \in V(F) \) at least one of \( \rho(\gamma, C) \) and \( \kappa(v) \) has its entry corresponding to \( w \) equal to 0; consequently \( \rho(\gamma, C) \cdot \kappa(v) = 0 \).

If \( \gamma \) lies in the same component as \( v \) and \( P \) involves the \( \chi \) transition at \( v \), considering the definitions of \( \rho(\gamma, C) \) and \( I_P(F, C) \), we see that \( \rho(\gamma, C) \cdot \kappa(v) \) is the mod 2 parity of \( \{ \{ w \in N(v) \mid \gamma \text{ is singly incident at } w \text{ and involves the } \chi \text{ or } \psi \text{ transition at } w \} \} \).

Whether \( \gamma \) is singly or doubly incident at \( v \), this number is even as observed above.

Proposition 26  Let \( F \) be a 4-regular multigraph with an Euler system \( C \), and let \( P \) be a circuit partition of \( F \). Suppose \( Q \subset P \) and there is at least one connected component of \( F \) for which \( Q \) contains some but not all of the incident circuits of \( P \). Then there is at least one vertex \( v \in V(F) \) such that \( P \) involves the \( \chi \) or \( \psi \) transition at \( v \), precisely one circuit of \( Q \) is incident at \( v \), and this incident circuit of \( Q \) is only singly incident.

Proof. Suppose first that \( P \) does not involve the \( \phi \) transition at any vertex. Let \( F_0 \) be a connected component of \( F \) in which \( Q \) includes some but not all of the incident circuits of \( P \). Then there must be an edge of this component not included in any circuit of \( Q \).

Choose such an edge, \( e_1 \). If an end-vertex of \( e_1 \) is incident on a circuit of \( Q \), then that vertex satisfies the proposition. If not, choose an edge \( e_2 \) that connects an end-vertex of \( e_1 \) to a vertex that is not incident on \( e_1 \). Continuing this process, we must ultimately find a vertex that satisfies the proposition.

In order to relax the requirement that \( P \) not involve any \( \phi \) transition, we use Nash-Williams’ idea of detachment [35]. Replace \( F \) by the graph \( F' \) obtained by eliminating every vertex at which \( P \) involves the \( \phi \) transition, as indicated in Figure 6. Then \( C \) and \( P \) give rise to an Euler system \( C' \) and a circuit partition \( P' \) of \( F' \) in the obvious way, and \( P' \) does not involve any transition labeled \( \phi \) with respect to \( C' \). The first part of the argument applies in \( F' \).

Corollary 27  Let \( F \) be a 4-regular multigraph with an Euler system \( C \), and suppose \( P \) is a circuit partition of \( F \). Suppose \( Q \subset P \) and there is no connected component of \( F \) for which \( Q \) contains every incident circuit of \( P \). Then \( \{ \text{relative core vectors of the circuits of } Q \} \) is linearly independent.
Proof. If $Q$ is empty then $Q$ is independent by convention. Otherwise, let $Q'$ be any nonempty subset of $Q$. Proposition 26 tells us that there is a vertex $v$ of $F$ at which $P$ involves the $\chi$ or $\psi$ transition, precisely one circuit of $Q'$ is incident, and this circuit is singly incident. It follows that the $v$ coordinate of

$$\sum_{\gamma \in Q'} \rho(\gamma, C)$$

is 1, and hence

$$\sum_{\gamma \in Q'} \rho(\gamma, C) \neq 0.$$  

Proposition 25 and Corollary 27 tell us that the relative core vectors of the circuits of a circuit partition $P$ span a $(|P| - c(F))$-dimensional subspace of $\ker I_P(F, C)$. The circuit-nullity formula [43] tells us that $|P| - c(F)$ is the nullity of $I_P(F, C)$, so we conclude that the relative core vectors span the nullspace of $I_P(F, C)$.

This completes our proof of Theorem 15. Before proceeding we should recall the work of Jaeger [27], whose Proposition 4 is equivalent to the special case of Theorem 15 in which $P$ involves no $\phi$ transitions. A different way to prove Theorem 15 is to use detachment to reduce to the special case, and then cite Jaeger's result. We prefer the argument above because it avoids the conceptual complications introduced by Jaeger's use of chord diagrams and surface imbeddings.

5 Theorem 16

Suppose $C$ and $C'$ are Euler systems of $F$, and $v \in V(F)$. We use transition labels $\phi, \chi, \psi$ with respect to $C$, and $\phi', \chi', \psi'$ with respect to $C'$.

If $C$ and $C'$ involve the same transition at $v$, then the row and column of both $I_C(F, C')$ and $I_{C'}(F, C)$ corresponding to $v$ are the same as those of the identity matrix, so the row and column of $I_C(F, C') \cdot I_{C'}(F, C)$ corresponding to $v$ are the same as those of the identity matrix.

Suppose instead that $C$ and $C'$ involve different transitions at $v$. Then $C'$ involves the $\chi$ or the $\psi$ transition, and $C$ involves the $\chi'$ or $\psi'$ transition; the
Let \( v(C_1vC_2v) \) be the circuit of \( C \) incident at \( v \); then \( vC_1v \) and \( vC_2v \) are the two circuits obtained by “short-circuiting” \( C \) at \( v \). We claim that the relative core vector \( \rho(vC_1v, C') \) is the same as the row of \( I_{C'}(F, C) \) corresponding to \( v \). The entry of \( \rho(vC_1v, C') \) corresponding to a vertex \( w \neq v \) is 1 if and only if \( vC_1v \) is singly incident at \( w \), and \( vC_1v \) does not involve the \( \phi' \) transition at \( w \). Also, the \( vw \) entry of \( I_{C'}(F, C) \) is 1 if and only if \( v \) and \( w \) are interlaced with respect to \( C \), and \( C' \) does not involve the \( \phi \) transition at \( w \). As \( vC_1v \) and \( C \) both involve the \( \phi \) transition at every \( w \neq v \), the two entries are the same.

The caption of Figure 3 mentions \( C \#W \), but the figure is valid for any Euler circuits that do not involve the same transition at \( v \).

It follows that \( \det \text{ detachment along } s \) of \( \psi \), and the general case may be reduced to the special case by Theorem 16 involving incompatible Euler systems follows from Jaeger’s Proposition 5 of [27], and the general case may be reduced to the special case by detachment along \( \phi \) transitions.
References


