Note
Series and parallel reductions for the Tutte polynomial
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Abstract

We discuss reducing the number of steps involved in computing the Tutte polynomial of a matroid by using series and parallel reductions in conjunction with the usual deletion and contraction operations. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The Tutte polynomial or dichromate is an invariant of matroids which has been found to be associated with an astounding variety of important and seemingly unrelated matters — vertex colorings, acyclic orientations and flows in graphs, reliability of communication networks, statistical mechanics and knot theory. We refer the interested reader to [2,12,13,15] for accounts of the Tutte polynomial and its many applications, and to [7,14] for general accounts of matroid theory. The Tutte polynomial may be defined in several equivalent ways; we will mention two here.

Definition 1. If \( M \) is a matroid on a finite set \( E \) then the Tutte polynomial of \( M \) has the subset expansion

\[
\tau(M) = \sum_{S \subseteq E} (x - 1)^{\gamma(E) - \gamma(S)} (y - 1)^{|S| - \rho(S)},
\]

where \( (x - 1)^0 = 1 = (y - 1)^0 \).

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Simply because there are \(2^{|E|}\) terms in the sum, this definition suggests the theorem of [5] that calculating the Tutte polynomial of an arbitrary matroid is \#P-hard. (The Tutte polynomials of some kinds of matroids can be calculated more easily however [8].)

In the second definition of \(t(M)\) we adopt the convention that there is a (unique) matroid \(\emptyset\) on \(E = \emptyset\).

**Definition 2.** The Tutte polynomial of \(M\) may be calculated recursively using these three reduction properties: if \(e \in E\) is a loop of \(M\) then \(t(M) = yt(M - e)\); if \(e \in E\) is an isthmus of \(M\) then \(t(M) = xt(M/e)\); and if \(e \in E\) is neither a loop nor an isthmus of \(M\) then \(t(M) = t(M - e) + t(M/e)\). The initial condition of the recursion is that \(t(\emptyset) = 1\).

To prove that these two definitions are equivalent one need only observe that the three reduction properties of Definition 2 follow from Definition 1. If we use Definition 2 to calculate \(t(M)\) recursively, the resulting sum is called a basis activities expansion of \(t(M)\). The name is explained by the fact that if we associate with each term of the sum the subset of \(E\) consisting of those elements which were contracted in obtaining that term, then these subsets turn out to be the bases of \(M\). (Note: It is traditional to follow a fixed order of \(E\) in obtaining a basis activities expansion, but it is not necessary to do so.)

Analyzing the complexity of a computation of \(t(M)\) is complicated by the fact that there are so many different ways to present a matroid: one may give its bases, circuits, rank function, hyperplanes, etc. These different ways of presenting a matroid have genuinely different algorithmic personalities: the complexity of even very simple-seeming problems (identifying loops, for instance) can vary considerably from one method of matroid presentation to another. See [9] for a detailed discussion of these matters. For our purposes it is sufficient to say that it seems reasonable to presume that for many methods of presenting matroids a basis activities expansion will be more efficient than the subset expansion for the simple reason that it involves fewer terms. In this paper we discuss modified basis activities expansions which involve even fewer terms.

It is convenient for us to use the notion of a doubly weighted matroid, a matroid given with a pair of functions \(p\) and \(q\) mapping \(E\) into some commutative ring with unity; an ordinary unweighted matroid will be considered to be doubly weighted with \(p(e) = q(e) = 1\ \forall e \in E\). Generalizing the definitions mentioned above, the Tutte polynomial of a doubly weighted matroid may be defined as follows.

**Definition 1.** If \(M\) is a doubly weighted matroid then its Tutte polynomial has the subset expansion

\[
t(M) = \sum_{S \subseteq E} \left( \prod_{e \in S} p(e) \right) \left( \prod_{e \notin S} q(e) \right) (x - 1)^{r(E) - r(S)} (y - 1)^{|S| - r(S)},
\]

where \(r(S)\) is the rank of \(S\).
Definition 2. The Tutte polynomial of a doubly weighted matroid $M$ may be calculated recursively using these three reduction properties: if $e \in E$ is a loop of $M$ then $t(M) = (q(e) + (y - 1)p(e))t(M - e)$; if $e \in E$ is an isthmus of $M$ then $t(M) = (p(e) + (x - 1)q(e))t(M/e)$; and if $e \in E$ is neither a loop nor an isthmus of $M$ then $t(M) = q(e)t(M - e) + p(e)t(M/e)$. The initial condition of the recursion is that $t(\emptyset) = 1$.

Here $M/e$ and $M - e$ are to be made into doubly weighted matroids by restricting $p$ and $q$ to $E - f_{e}g$. We leave it to the reader to verify that the three reduction properties mentioned in Definition 2 follow from Definition 1. An expansion that results from Definition 2 is still referred to as a basis activities expansion of $t(M)$.

The following proposition is easily deduced from the reduction properties of $t(M)$.

Proposition 1. Suppose $e_1, \ldots, e_k$ are parallel non-loops in $M$. Construct a new matroid $M'$ from $M - \{e_k, \ldots, e_2\}$ by replacing $e_1$ with a single element $e$ such that

$$p(e) = \sum_{j=1}^{k} \left( \prod_{i=j+1}^{k} q(e_i) \right) p(e_j) \left( \prod_{i=1}^{j-1} (q(e_i) + (y - 1)p(e_i)) \right)$$

and

$$q(e) = \prod_{i=1}^{k} q(e_i).$$

Then $t(M) = t(M')$.

We stress that the only difference between $M'$ and $M - \{e_k, \ldots, e_2\}$ is in their weight functions; the underlying unweighted matroids are identical. We refer to instances of Proposition 1 as parallel reductions. The dual of a parallel reduction is a series reduction: if $e_1, \ldots, e_k$ are non-isthmuses which are in series in $M$ then the reduction involves replacing $e_1$ in $M - \{e_k, \ldots, e_2\}$ with an element $e$ whose weights are given by the ‘duals’ of the formulas in Proposition 1, i.e., the same formulas, with $p$ and $q$ interchanged throughout.

Our first modification of the recursion of Definition 2 is motivated by the observation that if $e_1, \ldots, e_k$ are parallel non-loops in a matroid $M$ then removing them in the usual way will give rise to $k + 1$ branches in the resulting computation of $t(M)$ but a parallel reduction will give rise to only two branches, $M' - e$ and $M'/e$. Hence, we may create an expansion of $t(M)$ with fewer terms by modifying the recursion of Definition 2 so that at each stage, after the loops and isthmuses are dealt with but before any use of $t(M) = q(e)t(M - e) + p(e)t(M/e)$, we search for sets of parallels and reduce each set to a single edge using Proposition 1.

Theorem 1. Suppose the recursion of Definition 2 is modified so that it involves parallel reductions as just indicated. Then the resulting expansions of $t(M)$ have at least $\beta(M^+)$ terms, where $M^+$ is the free one-point extension of $M$, and if $\beta(M^+) > 0$ then this lower bound will be realized by some implementation.
We may also modify the recursion of Definition 2 so that it involves series reductions in addition to parallel reductions.

**Theorem 2.** Suppose the recursion of Definition 2 is modified to include both parallel and series reductions. Then the resulting expansions of $t(M)$ have at least $\beta(M)$ terms, and if $\beta(M) > 0$ then this lower bound will be realized by some implementation.

The appearance of Crapo's $\beta$-invariant [3] in Theorems 1 and 2 will not be a surprise to the reader familiar with the theory of reliability domination, an important concept in the analysis of network reliability [2,10]. Indeed if $M$ is the cycle matroid of a graph $G$ then the statements obtained from Theorems 1 and 2 by replacing $t(M)$ with the reliability of $G$ follow from results of Huseby [4] and Johnson [6] relating $\beta(M^+)$ and $\beta(M)$ to the all-terminal domination and minimum domination of $G$, respectively.

Theorem 1 usually produces an expansion of $t(M)$ which has considerably more terms than the expansion produced by Theorem 2, so the reader may wonder why we have chosen to state both rather than only Theorem 2. There are two features which may complicate implementations of Theorem 2 enough that it is sometimes less efficient than Theorem 1. The first feature is that some ways of presenting matroids make it significantly easier to find sets of parallel elements than sets of elements in series [9]. (Of course if one can refer to dual matroids throughout the recursion then elements in series can be detected as easily as parallels, but keeping track of duals would complicate an implementation of Theorem 2 in a way that is unnecessary in an implementation of Theorem 1.) The second feature is visible in the last paragraph of the proof of Theorem 2, where it is necessary to find an $e \in E$ such that $\beta(M - e) \neq 0 \neq \beta(M/e)$ to use in $t(M) = q(e)t(M - e) + p(e)t(M/e)$. At the corresponding point in the proof of Theorem 1, any $e \in E$ can be used.

If $M$ is the cycle matroid of a series-parallel graph then it is possible to completely determine its structure in polynomial time through a simple procedure: search for elements which are parallel or in series, perform a parallel or series reduction, and repeat as many times as is necessary. Together with the recursion of Theorem 2, this simple procedure provides a polynomial-time algorithm for calculating the Tutte polynomial of a series-parallel matroid. Oxley and Welsh [8] provide a different algorithm which does not require the use of weights; their algorithm is considerably more sophisticated than the one we have just described and generalizes to other families of matroids.

2. Proofs

Recall that if $M$ is a matroid on $E$ with rank function $r$, then the free one-point extension $M^+ = (M \cap \{-\}) = M \cup \{-\}$ whose circuits include the circuits of $M$ and, in addition, the sets $B \cup \{-\}$ with $B$ a basis of $M$ [14]. We leave proofs of the following elementary properties of $M^+$ to the reader.
Lemma 2.1. If $e \in E$ is not an isthmus of $M$ then $(M - e)^+ = M^+ - e$. If $e \in E$ is not a loop of $M$ then $(M/e)^+ = M^+/e$. If $M$ has no loops then $M^+$ is connected.

We deduce Theorem 1 inductively from Proposition 1, Lemma 2.1 and the theory of Crapo’s $\beta$-invariant [1,3]. If $M$ has a loop then $\beta(M^+) = 0$ and Theorem 1 is satisfied vacuously, so we may presume $M$ has no loops. If $|E| \leq 1$ then $\beta(M^+) = 1$ and Theorem 1 is satisfied, as the recursion of Definition 2 produces a formula for $t(M)$ with just one term.

We proceed by induction on $|E| \geq 2$. If $M$ has an isthmus $e$ then $e$ and $*$ are in series in $M^+$, so $\beta(M^+) = \beta(M^+/e)$; by Lemma 2.1 it follows that $\beta(M^+) = \beta((M/e)^+)$. As Theorem 1 applies to $M/e$ by the inductive hypothesis, we conclude that it also holds for $M$. If $M$ has parallel non-loop elements $e_1, \ldots, e_k$ then they are certainly parallel in $M^+$ too, so $\beta(M^+) = \beta(M^+ - \{e_k, \ldots, e_2\})$. Lemma 2.1 implies that $\beta(M^+ - \{e_k, \ldots, e_2\}) = \beta((M-e)^+)$; the validity of Theorem 1 for $M$ follows from its validity for $M'$. If $M$ has no isthmuses or sets of parallel elements then choose any $e \in E$; $e$ is not an isthmus or a loop of $M$ or $M^+$. Using Lemma 2.1, it follows that $\beta(M^+) = \beta(M^+ - e) + \beta(M^+/e) = \beta(M - e)^+ + \beta((M/e)^+)$; Lemma 2.1 also implies that $\beta((M - e)^+) \neq 0 \neq \beta((M/e)^+)$, so the validity of Theorem 1 for $M$ follows from its validity for $M - e$ and $M/e$.

The proof of Theorem 2 is quite similar to the proof of Theorem 1 in outline; of course the argument focuses on $M$ rather than $M^+$ and includes series reductions. The only other difference occurs in the last paragraph of the proof, where we observed that every $e \in E$ has $\beta((M - e)^+) \neq 0 \neq \beta((M/e)^+)$. In proving Theorem 2 we observe instead that if $\beta(M) = 1$ then $M$ is series–parallel and hence the recursion of Theorem 2 provides an expansion of $t(M)$ as a single term, while if $\beta(M) > 1$ and $M$ has no sets of elements that are parallel or in series then it is possible to find an $e \in E$ with $\beta(M - e) \neq 0 \neq \beta(M/e)$; the existence of such an $e$ follows from Seymour’s theorem [11] that $M$ is either 3-connected or a 2-sum of 3-connected matroids.

3. An example

Suppose we wish to calculate the Tutte polynomial of $M = M(K_4)$, the cycle matroid of the complete graph on four vertices. $E = E(K_4)$ has six elements, so the subset expansion of $t(M)$ involves 64 terms. In Fig. 1 we picture a computation of an activities expansion of $t(M)$. This expansion has 16 terms; each of the 16 terms is represented by one of the vertices at the bottom of the figure. The term represented by a vertex may be identified by counting the numbers of isthmuses and loops in the graphs pictured above that vertex; these numbers give the powers to which $x$ and $y$ are raised in that term. Consequently,

$$t(M) = x^3 + x^2 + xy + x^2 + x + y + xy + y^2 + x^2 + x + y + xy + y^2 + xy + y^2$$

$$+ xy + y^2 + y^3.$$
An implementation of Theorem 1 will express $t(M)$ using only six terms, corresponding to the vertices which are circled in the figure. These six terms are $x^3$, $x^2 + xy$, $x^2 + y$, $x + y + y^2$, $(x + y)^2$, and $x + y + y^2 + y^3$. An implementation of Theorem 2 will express $t(M)$ using only two terms, corresponding to the vertices which are circled twice. These two terms are the sums of the first and second eight of the original 16, respectively.

References