A poroelastic model of the lung

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November 6, 2014

Abstract

This work is motivated by the modelling of ventilation and deformation in the lung for understanding the biomechanics of respiratory diseases. The main contribution is the derivation and implementation of a lung model that tightly couples a poroelastic model of lung parenchyma to an airway fluid network. The poroelastic model approximates the porous structure of lung parenchyma using a continuum model that allows us to naturally model changes in physiology by spatially varying material parameters, whilst conserving mass and momentum within the tissue. The proposed model will also take advantage of realistic deformation boundary conditions obtained from image registration, to drive the simulation. A finite element method is presented to discretize the equations in a monolithic way to ensure convergence of the nonlinear problem. To demonstrate the coupling between the poroelastic medium and the network flow model numerical simulations on a realistic lung geometry are presented. These numerical simulations are able to reproduce global physiological realistic measurements. We also investigate the effect of airway constriction and tissue weakening on the ventilation, tissue stress and alveolar pressure distribution and highlight the interdependence of ventilation and deformation.
1 Introduction

The main function of the lungs is to exchange gas between air and blood, supplying oxygen during inspiration and removing carbon dioxide by subsequent expiration. Gas exchange is optimised by ensuring efficient matching between ventilation and blood flow, the distributions of which are largely governed by branching structure of the airway and vascular trees, tissue deformation and gravity. In this work, we focus on the link between tissue deformation and ventilation. Previous work of whole organ lung models has typically focused on modelling either ventilation or tissue deformation \cite{1, 2, 3, 4, 5, 6, 7, 8}. However evaluation of each component (i.e. tissue deformation or ventilation) separately does not necessarily give accurate predictions or provide a good indication of how the integrated organ works. This is because both components are interdependent. To gain a better understanding of the biomechanics in the lung it is therefore necessary to fully couple the tissue deformation with the ventilation. To achieve this tight coupling between the tissue deformation and the ventilation we propose a novel multiscale model that approximates the lung parenchyma by a biphasic (tissue and air) poroelastic model, that is then coupled to an airway fluid network model.

Some early work on a mechanical model of lung parenchyma as a poroelastic medium has already been proposed in \cite{9}. This work developed a similar poroelastic model to the one we propose, however it has only been applied to a very simple 2D geometry. Also in \cite{10} homogenisation theory has been used to derive macroscopic poroelastic equations for average air flows and tissue displacements in lung parenchyma during high frequency ventilation.

The proposed fully coupled poroelastic fluid-network model will enable research for understanding the biomechanics of respiratory diseases, take advantage of realistic deformation boundary conditions extracted from imaging data to drive the simulation, and be able to conserve mass locally. These points will now be discussed in more detail.

An integrated model of ventilation and tissue mechanics will be particularly important for understanding respiratory diseases since nearly all pulmonary diseases lead to some abnormality of lung tissue mechanics \cite{11}. For example, chronic obstructive pulmonary disease (COPD) encompasses emphysema (destruction of alveolar tissue) and chronic bronchitis which can cause severe, airway remodeling and bronchoconstriction and air trapping. This in turn affects tissue deformation since sections of lung are not able to expand or compress to release air. The impact of alterations during disease,
such as airway narrowing or changes in tissue properties, on regional ventilation and tissue stresses are not well understood. For example, one hypothesis is that airway disease may precede emphysema [12]. The computational lung model could be applied to investigate the impact of airway narrowing and tissue stiffness during obstructive lung diseases on tissue stresses, alveoli pressure and ventilation.

Ventilation models that incorporate tissue mechanics often model the tissue using many independent elastic alveolar units [1, 2], these are often known as balloon models. There is no clear way to conserve mass locally, so alveolar units can expand irrespectively of the size and position of neighbouring units. In reality acini do not function as independent elastic balloons. They are physically coupled through fibrous scaffolding and shared alveolar septa. In Figure 1 we illustrate this issue by comparing two simplistic balloon models. In our proposed model the tissue is modelled as one continuum, thus making it trivial to conserve volume and couple neighbouring units.

Considering the issue of boundary conditions: these lung models [1, 2] give information about the distribution of flow within the lung as a result of a pleural pressure boundary condition. However it is not possible to experimentally measure the pleural pressure in vivo using imaging or other apparatus. As part of the simulation protocol, the pleural pressure is therefore often tuned until physiological realistic flow rates are achieved. To overcome this issue, [13, 14] proposed to estimate the flow boundary conditions for full organ ventilation models by means of image registration. However by relying on image registration to determine the ventilation distribution within the tissue one is not able to model the change in ventilation distribution due to progression of disease. We build on [14] by integrating image registration based boundary conditions within the proposed poroelastic model of lung deformation. In particular, we propose to register expiratory images to the inspiratory images, to yield an estimate of the deformation boundary condition for the lung surface, and drive the simulation through this deformation boundary condition. Thus the tissue deformation and subsequent flow boundary condition for tree branches inside the lung and ventilation distribution is not pre-determined, but calculated from the coupled poroelastic-airway-tree model.

The rest of this paper is organized as follows: in section 2 we present the model assumptions for the proposed lung model, and outline the mathematical lung model and its implementation; in section 3 we detail the generation of the computational lung geometry and boundary conditions; in section 4 we present numerical simulations of tidal breathing, and investigate the effect of
airway constriction and tissue weakening. Finally in section 5, we conclude and outline future work to improve the current lung model.

Figure 1: Sketch of two balloon models where the right alveolar unit is more compliant, thus being able to expand more easily. (a) Balloon model with independent alveolar units. The overlap in the alveolar units illustrates that mass is not conserved spatially. (b) Balloon model where the alveolar units are coupled. Here the inflation of each alveolar unit is compromised by the expansion of its neighbor.

2 Poroelastic lung model

2.1 Model assumptions
2.1.1 Approximating lung parenchyma using a poroelastic medium.

Averaging over the tissue: One of the major assumptions is that we can approximate the lung parenchyma using a poroelastic continuum description. This makes our model computationally tractable and allows us to use the well studied theory of poroelasticity to couple the air with the tissue. In publications such as [10] homogenisation techniques have already been applied to approximate the lung parenchyma as a poroelastic continuum model.

The use of a continuum model can be further supported by looking at the different length scales and structures of the tissue. For the microscopic length scale denoted by $l$ of the parenchyma we will use the diameter of an alveolus that can be approximated to be 0.02 cm [15]. The macroscopic length scale $L$ can be taken to be the diameter of a segment which measures
around 4 cm of tissue. So the ratio of the different length scales is small i.e
\[ \epsilon := \frac{l}{L} \approx 0.005 \ll 1. \] This along with the assumption that the structure of an acinus is porous (see Figures 2a and 2b) and periodic supports the use of averaging techniques over the tissue to obtain a continuum description in the form of a poroelastic medium. To further simplify the poroelastic equations we assume that the poroelastic continuum can be described by a solid phase (blood and tissue) and a fluid phase (air), where both phases are assumed to be incompressible. The interaction between the fluid pressure and the deformation of the solid skeleton is assumed to obey the effective stress principle. Note that by averaging over the tissue we are not able to model individual alveoli and instead have to rely on macroscopic parameters such as the permeability and elasticity coefficient. In general, lung diseases usually affect whole regions of alveoli (lung tissue), thus, by changing the macro-scale parameters over the affected area of tissue we are still able to model changes in the tissue due to disease.

**Ignoring blood flow:** Apart from collagen, fibers and air the other major component in the lung is blood. The volume taken up by collagen and elastin fibers is similar to the volume occupied by the capillaries filled with blood (illustrated in Figure 2a). In fact, the space not occupied by air is about 7% of the parenchymal volume and is made up of 50% capillary blood and 50% of collagen and elastin fibers [16]. Also the density of blood is similar to the density of tissue and much larger than that of air (1060 kg m\(^{-3}\) \(\gg\) 1.18 kg m\(^{-3}\)). Due to the large accelerations present in blood, the inertia and gravity forces of blood acting on the surrounding tissue could therefore be of importance when predicting deformation and ventilation in the lung. Since the capillaries are constantly filled with blood and the density of blood is similar to that of alveolar tissue we will make the simplifying assumption that the blood is simply part of the tissue (solid phase) and thus ignore accelerations and any redistribution of blood during breathing.

**Assuming incompressibility of the solid and the fluid:** Blood and tissue are often assumed to be incompressible. Under physiological conditions, air is also often assumed to be incompressible [1].

**Ignoring solid inertia forces:** Simple calculations considering the sinusoidal motion of tissue near the diaphragm during normal breathing yield an estimate of 0.02 ms\(^{-2}\) for the maximum acceleration of lung parenchyma.
Compared to the acceleration of gravity this is negligible, and it is therefore reasonable to ignore the inertia forces in the tissue.

**Ignoring fluid inertia forces:** The fluid’s Reynolds number in the lower airways, that form part of the lung parenchyma, has been estimated to be around 1 to 0.01 [17]. Due to this relatively low Reynolds we choose to ignore fluid inertia forces in the poroelastic medium.

**Ignoring viscous forces in the fluid:** By applying the procedure of dimensional analysis, it is possible to show that the viscous stress in the fluid is small compared to the drag forces between the fluid and the porous structure, when the ratio of the different length scales is small [18]. We will therefore neglect the fluid viscous stress implying that the fluid behaves more or less inviscid within the porous structure.

![Figure 2](image-url)

*Figure 2:* (a) Alveoli in an alveolar duct. The dark round openings are pores between alveoli. The alveolar wall is quite thin and contains a network of capillaries. The average diameter of one alveoli is 0.2 mm. (b) Transition from terminal bronchiole to alveolar duct, from conducting airway to oxygen transfer area, diameter of terminal bronchiole is 0.5 mm. Images are reproduced from [19].
2.1.2 Approximating the airways using a fluid network model.

In order to make the coupled model computationally feasible we assume that a simple laminar flow model can describe the air flow in the airways. We will make the common Poiseuille flow assumption. This flow assumption is also made in [2, 20] where the air flow in a whole airway tree, from trachea down to the final bronchioles was assumed to be governed by Poiseuille flow. Diseases affecting the airway tree can be modelled effectively by changing resistance (airway radius) parameters in the network flow model.

2.2 Mathematical model

2.2.1 A poroelastic model for lung parenchyma.

Before presenting the final balance equations of poroelasticity, we give a short review of the kinematics required to model a poroelastic medium undergoing large deformations. A more detailed description and full derivation of the poroelastic equations can be found in [21] and [22].

**Kinematics:** The kinematic quantities described here will be associated with the solid skeleton, since this also describes the motion of the fluid domain. Let the volume \( \Omega_0 \) be the undeformed Lagrangian (material) reference configuration and let \( X = \{Xe_1 + Ye_2 + Ze_3\} \) indicate the position of a solid particle in \( \Omega_0 \) at \( t = 0 \), where \( X, Y \) and \( Z \) are the components of the position with respect to the standard orthonormal basis \( \{e_1, e_2, e_3\} \) for \( \mathbb{R}^3 \). The position of a solid particle in the current Eulerian (spatial) configuration \( \Omega_t \) is given by \( x = \{xe_1 + ye_2 + ze_3\} \), with \( x = \chi(X, t) \). The deformation map, \( \chi(X, t) \), is a continuously differentiable, invertible mapping from \( \Omega_0 \) to \( \Omega_t \). The displacement field is given by

\[
\mathbf{u}(X, t) = \chi(X, t) - X.
\]

The deformation gradient tensor,

\[
\mathbf{F} = \frac{\partial \chi(X, t)}{\partial X} = \mathbf{I} + \frac{\partial \mathbf{u}(X, t)}{\partial X},
\]

maps a material line element in the reference configuration \( dX \), to a line element \( dx \) in the current configuration, i.e. \( dx = \mathbf{F} dX \). The symmetric right Cauchy-Green deformation tensor is given by \( \mathbf{C} = \mathbf{F}^T \mathbf{F} \). The jacobian is defined as \( J = \text{det}(\mathbf{F}) \), and represents the change in an infinitesimal small volume from the reference to the current configuration.
**Volume fractions:** The porous medium is assumed to be saturated consisting of a solid part and a fluid part. The fluid part accounts for volume fractions $\phi_0(X, t = 0)$ and $\phi(x, t)$ of the total volume in the reference and the current (deformed) configuration, respectively. The fractions for the solid are thus given by $1 - \phi_0$ and $1 - \phi$ in the reference and the current configuration, respectively. The volume fraction $\phi$ is also known as the porosity. The density of the mixture in the current configuration, $\rho$, is defined as

$$\rho := \rho^s(1 - \phi) + \rho^f \phi \quad \text{in } \Omega_t,$$

where $\rho^s$ and $\rho^f$ are the densities of the fluid and solid, respectively. We will also assume that both the solid and the fluid are incompressible, such that $\rho^s = \rho^s_0$ and $\rho^f = \rho^f_0$. Due to mass conservation and the incompressibility of both the solid and the fluid phase we have

$$J = \frac{1 - \phi_0}{1 - \phi}.$$  

Here $J$ represents the change in volume of the solid skeleton, the solid skeleton includes the tissue and the voids occupied by the fluid (air).

### 2.2.2 Poroelastic equations.

Having made the assumptions in section 2.1 for the tissue we are left with the following poroelastic system of equations. We define the boundary $\partial \Omega_t = \Gamma_d \cup \Gamma_t$ for the displacement and stress boundary conditions of the poroelastic mixture, and $\partial \Omega_t = \Gamma_p \cup \Gamma_f$ for the pressure and flux boundary conditions of the fluid, with an outward pointing unit normal $n$. The strong problem is to find $\chi(X, t), z(x, t)$ and $p(x, t)$ such that

$$-\nabla \cdot (\sigma_e - pI) = \rho f \quad \text{in } \Omega_t,$$  

$$k^{-1}z + \nabla p = \rho^f f \quad \text{in } \Omega_t,$$  

$$\nabla \cdot (v^s + z) = g \quad \text{in } \Omega_t,$$  

$$\chi = X + u_D \quad \text{on } \Gamma_d,$$  

$$(\sigma_e - pI)n = t_N \quad \text{on } \Gamma_t,$$  

$$z \cdot n = q_D \quad \text{on } \Gamma_f,$$  

$$p = p_D \quad \text{on } \Gamma_p,$$  

$$\chi = X + u_0^0 \quad \text{in } \Omega_0,$$
Equation (5a) is the balance of linear momentum for the poroelastic mixture. The elastic stress is denoted by $\sigma_e$ (defined in section 2.2.3); $p$ denotes the fluid pressure, and $f$ is a body force. This equation is governed by the effective stress concept, which says that the deformations of the skeleton are governed by the so-called ‘effective stress’, also known as the ‘elastic stress’. The elastic stress is given by $\sigma_e = \sigma + pI$, where $\sigma$ denotes the total stress, and is the quantity we wish to evaluate in the momentum balance equation (5a). This principle states that the deformation of the porous solid is only driven by the excess of stress over the pore pressure [23, 24]. Equation (5b) is the balance of linear momentum for the fluid, and is also commonly known as Darcy’s law. Here $z$ denotes the fluid flux and $k$ is the permeability tensor, defined in section 2.2.3. The balance of mass is modelled by equation (5c). Here $v^s = \frac{\partial}{\partial t} \chi(X,t)$ denotes the velocity of the solid skeleton. Equations (5h)-(5h) are the associated boundary and initial conditions. A detailed derivation of the above equations is given in [21].

2.2.3 Constitutive laws.

To close the poroelastic model (5) we need to choose constitutive laws for the permeability and strain energy. The permeability tensor in the current configuration is given by

$$k = J^{-1} F k_0(\chi) F^T,$$

(6)

where $k_0(\chi)$ is the permeability in the reference configuration, which may be chosen to be some (nonlinear) function dependent on the deformation. We will use the same law that has already been proposed in [25] to model lung parenchyma,

$$k_0 = k_0 \left( J \frac{\phi}{\phi_0} \right)^{2/3} I.$$

(7)

The elastic (effective) stress tensor given by

$$\sigma_e = \frac{1}{f} F \cdot 2 \frac{\partial W(\chi)}{\partial C} \cdot F^T.$$

(8)

Here $W(\chi)$ denotes a strain-energy law (hyperelastic Helmholtz energy functional) dependent on the deformation of the solid. Exponential strain energy laws for lung parenchyma exist, for example the popular law by [26]. However little is known about how the constants in these laws should be interpreted.
and altered to model weakening of the tissue in an diseased state. Further, the constants in these laws are thought to have no physical meaning [27]. To make the interpretation of the elasticity constants and dynamics of the model as simple as possible we chose a Neo-Hookean law taken from [28], with the penalty term chosen such that $0 \leq \phi < 1$,

$$W(C) = \frac{\mu}{2}(\text{tr}(C) - 3) + \frac{\Lambda}{4}(J^2 - 1) - (\mu + \frac{\Lambda}{2})\ln(J - 1 + \phi_0).$$

The material parameters $\mu$ and $\Lambda$ can be related to the more familiar Young’s modulus $E$ and the Poisson ratio $\nu$ by $\mu = \frac{E}{2(1+\nu)}$ and $\Lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$. The values of these constants for modeling lung tissue have been investigated [29, 5, 30] and are shown in Table 1.

2.2.4 A network flow model for the airway tree.

The flow rate $Q_i$ through the $i$th pipe segment in the fluid network is given by the pressure-flow relationship

$$P_{i,1} - P_{i,2} = R_i Q_i,$$

where $R_i = \frac{8\mu_f}{\pi r^4}$ is the Poiseuille flow resistance of a pipe segment ($r$ is the radius, $l$ is the length of the pipe, $\mu_f$ is the dynamic viscosity) and $P_{i,1}$ and $P_{i,2}$ are the pressures of the two nodes at each end of the pipe segment. We also have conservation of flow at branches such that

$$Q_i = \sum_{Q_{i,j}\in Q_i} Q_{i,j},$$

where $Q_{i,j}$ are the flow rates of the children branches of the $i$th flow segment. The outlet pressure of the fluid network is set using the boundary condition $P_0 = \hat{P}$.

2.2.5 Coupling the fluid network to the poroelastic model.

We split our domain into $N$ separate subdomains such that $\Omega_t = \sum_i^N \Omega_t^i$. The introduction of subdomains allows each endpoint of the fluid network to supply and remove fluid from the poroelastic medium at different spatial
locations. The $i$th subdomain $\Omega_i^t$ is defined as the volume closest to the position of the $i$th inlet, denoted by $\text{pos}(P_{di})$.

$$\Omega_i^t := \{ \mathbf{x} \in \Omega_t : \| \mathbf{x} - \text{pos}(P_{di}) \| < \| \mathbf{x} - \text{pos}(P_{dj}) \|, \ j = 1, 2, ..., N, j \neq i \}. \quad (12)$$

For notational purposes we have added subscript $di$ to the most distal branches that have no further conducting branches coming from them but instead enter a group of acinar units (approximated by the poroelastic model). In Figure 3a we demonstrate how the domain is coupled to the distal branches using a simple 2D example.

We couple the airway network to the poroelastic domain by adding the flow contribution from each distal airway to the poroelastic domain as a source term in the poroelastic mass conservation equation (5c), such that

$$\nabla \cdot (\mathbf{v}^s + \mathbf{z}) = Q_{di} \text{ in } \Omega_i^t. \quad (13)$$

We also couple the airway network to the poroelastic domain by setting the average pressure in the poroelastic domain within $\Omega_i^t$ to be the same as the corresponding distal pressure node $P_{di}$ of the flow segment $Q_{di}$,

$$\frac{1}{|\Omega_i^t|} \int_{\Omega_i^t} p = P_{di}, \quad (14)$$

where $|\Omega_i^t|$ denotes the volume of the segment $\Omega_i^t$. Equation (14) enforces the assumption that the end pressure in a terminal bronchiole is the same.
as the alveolar pressure in the surrounding tissue. If we discretize the space using triangles and employ a piecewise constant pressure approximation (one node at the center of each element), the resulting coupling for a simple 2D example is shown in Figure 3b. Once we refine the mesh (Figure 3c), the discretized division of subdomains tends to the subdivision of the original problem (Figure 3a).

2.2.6 Summary of the coupled lung model.

To solve the coupled poroelastic-fluid-network lung model we need to find $\chi(X, t), z(x, t), p(x, t), P_i$ and $Q_i$ such that

\begin{align*}
-\nabla \cdot (\sigma_e - pI) &= \rho f \quad \text{in } \Omega_t, \quad (15a) \\
K^{-1}z + \nabla p &= \rho^f f \quad \text{in } \Omega_t, \quad (15b) \\
\nabla \cdot (\nu_s + z) &= Q_{di} \quad \text{in } \Omega_i^t, \quad (15c) \\
\chi &= X + u_D \quad \text{on } \Gamma_d, \quad (15d) \\
(\sigma_e - pI)n &= t_N \quad \text{on } \Gamma_t, \quad (15e) \\
z \cdot n &= q_D \quad \text{on } \Gamma_f, \quad (15f) \\
p &= p_D \quad \text{on } \Gamma_p, \quad (15g) \\
\chi(0) &= X + u_0, \quad \text{in } \Omega_0, \quad (15h) \\
P_0 &= \hat{P}, \quad (15i) \\
P_{i,1} - P_{i,2} &= R_i Q_i, \quad (15j) \\
Q_i &= \sum_{Q_{i,j} \in Q_i} Q_{i,j}, \quad (15k) \\
\frac{1}{|\Omega_i|} \int_{\Omega_i^t} p &= P_{di}. \quad (15l)
\end{align*}

2.2.7 Implementation.

Since the system of equations (15) is highly nonlinear, its solution requires a scheme such as Newton’s method. In [21] a finite element scheme using Newton’s method for the solution of the poroelastic equations valid in large deformations (5) has already been presented. In this work we adopt the same finite element scheme as presented in [21] for solving the poroelastic equations and expand the linear system (discretized linearization) to include
additional matrices required for solving the fluid network and its coupling to the poroelastic medium (equations (15c,15j,15k,15l)). This results in a monolithic coupling scheme that ensures good convergence even for problems with strong coupling interactions between the poroelastic medium and the fluid network (see section 4.2.1). Let \( u^n_i \) denote the fully discrete solution of all the poroelastic and fluid network variables at the \( i \)th step within the Newton method at time \( t^n \), and let \( \xi u^n_{i+1} \) denote the corresponding solution increment vector used to update the current solution within the Newton method. Let \( K \) denote the stiffness matrix (discretized linearization of the full lung model (15)), and \( R \) the residual vector. For details on how these are built, see section A.1. To solve the nonlinear poroelastic problem using Newton’s method at a particular time step, we perform the following steps:

**Algorithm 1 Fully discrete Newton’s method (for a particular time step \( t^n \))**

1. Initialize \( u^n_0 = u^{n-1} \)
2. Iteration loop \( i = 0, 1, \ldots \) until convergence.
   1. Assemble \( R(u^n_i, u^{n-1}) \) and \( K(u^n_i) \).
   2. Compute the solution increment from the linear system: \( K(u^n_i)\xi u^n_{i+1} = -R(u^n_i, u^{n-1}) \).
   3. Compute the new solution: \( u^n_{i+1} = u^n_i + \xi u^n_{i+1} \).
   4. Update the mesh, \( \Omega_t = X + u^n_i \).
5. Test for convergence: \( ||R(u^n_i, u^{n-1})|| > TOL \rightarrow \text{Set } i = i + 1 \text{ go to 1}, \)
   \( ||R(u^n_i, u^{n-1})|| \leq TOL \rightarrow \text{Stop and move to the next time step.} \)

For the implementation we use the C++ library libmesh [31], and the multi-frontal direct solver mumps [32] to solve the resulting linear system. We set the relative tolerance to be \( TOL = 10^{-4} \). For the subsequent numerical results shown in section 4, a maximum of 5 Newton iterations were required to solve each time step.
3 Model generation

3.1 Mesh generation

We derive a whole organ lung model, of the right lung, from a high-resolution CT image taken at total lung capacity (TLC) and functional residual capacity (FRC). The bulk lung is first segmented from the CT data (slice thickness and pixel size 0.73 mm) using the commercially available segmentation software Mimics\(^1\). We then use the open-source image processing toolbox iso2mesh [33] to generate a Tetrahedral mesh containing 38369 elements. The conducting airways are also segmented from the CT data taken at TLC level, and a centerline with radial information is calculated. To approximate the remaining airways up to generation 8-13 we use a volume filling airway generation algorithm to generate a mesh of the airway tree containing 13696 nodes, with 2140 terminal branches [Oxford ref].

3.2 Reference state, boundary conditions and initial conditions

A reference state is typically chosen to associate with a stress-free state. Biological tissues do not possess a ‘reference state’ in space where the material is free of both stress and strain. The cells that make up tissues are born into stressed states and live out their lives in these stressed states [34].

In this work, we uniformly deflate the lung from FRC, to a configuration, the reference state, in which the internal stresses and strains are zero. The geometry of the reference state is then used as the initial configuration of the lung model. The lung model is then uniformly from the reference state to a pre-stressed FRC configuration, which is known to typically have a mean elastic recoil of about \(0.49 \times 10^3\) Pa [35]. From there we simulate tidal breathing. A similar approach has also been used in [8].

We register the expiratory (FRC) segmentation to the segmentation at TLC using a simple registration procedure that uses three scalings in the x,y and z direction to map between the bounding boxes of the segmentations at FRC and TLC. This yields a rough estimate of the deformation field for the lung surface from expiration to inspiration. To simulate tidal breathing we assume a sinusoidal breathing cycle and expand the lung surface from FRC.

\(^1\)http://biomedical.materialise.com/mimics
to 40% of the deformation from FRC to TLC,

\[ \mathbf{u}_D(t) = 0.2 \left( 1 + \sin \left( \frac{t\pi}{2} + \frac{3\pi}{2} \right) \right) \mathbf{u}_{D,TLC} \quad \text{on } \Gamma_d. \]  

(16)

Here \( \mathbf{u}_{D,TLC} \) is the deformation of the lung surface from FRC to TLC, obtained using the registration procedure. For our mesh and registration this results in a physiologically realistic tidal volume of 0.59 liters. We simulate breathing for a total of 8 seconds (2 breathing cycles) resulting in a breathing frequency of 15 breaths per minute. Due to the incompressibility of the poroelastic tissue this also determines the total volume of air inspired/expired and the flowrate at the trachea, see Figure 5a and 5b respectively. For the fluid boundary condition we have that the whole lung is sealed so that no fluid can escape through the lung surface, with \( \mathbf{z} \cdot \mathbf{n} = 0 \) along the whole boundary. For the airway network boundary condition we set the outlet pressure of the airway network to zero atmospheric pressure, \( P_0 = 0 \).

3.3 Simulation parameters

Several parameters for lung tissue elasticity and poroelasticity have been proposed [29, 5, 36, 10, 30]. There is no consensus in the values in the literature. In this study we have chosen parameters from the literature, as shown in Table 1. These parameters are within range of existing models, and result in physiologically realistic simulation results (see section 4).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Reference</th>
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<tbody>
<tr>
<td>( \phi_0 )</td>
<td>0.99</td>
<td>[36]</td>
</tr>
<tr>
<td>( \kappa_0 )</td>
<td>( 10^{-5} ) m(^3)s(^{-1})kg(^{-1} )</td>
<td>[36]</td>
</tr>
<tr>
<td>( E )</td>
<td>( 0.73 \times 10^3 ) Pa</td>
<td>[30]</td>
</tr>
<tr>
<td>( \nu )</td>
<td>0.3</td>
<td>[30]</td>
</tr>
<tr>
<td>( \mu_f )</td>
<td>( 1.92 \times 10^{-5} ) kg m(^{-1})s(^{-1} )</td>
<td>[2]</td>
</tr>
<tr>
<td>( T )</td>
<td>8s</td>
<td>-</td>
</tr>
<tr>
<td>( \Delta t )</td>
<td>0.2s</td>
<td>-</td>
</tr>
<tr>
<td>( \delta )</td>
<td>( 10^{-5} )</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 1: Simulation parameters.

Here \( \delta \) is a stabilization parameter required by the finite element scheme, see [21, 37] for details.
4 Model exploration

We will now explore the behavior of the proposed model using a series of simulations to investigate the coupling between the airways and the tissue, hysteresis effects and how mass is conserved within the tissue. This will give us an understanding of the dynamics of the model and how the various variables are dependent on each other.

In the subsequent analysis the total and elastic stress is calculated as $\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}$, where $\lambda_1, \lambda_2, \lambda_3$ are the three eigenvalues of the stress tensor, respectively. We also use the relative Jacobian, denoted by $J_V$, as a measure for ventilation, which is calculated to be the volume ratio between the current state and FRC, and is a direct measure of tissue expansion. Due to the quasi-static nature of the equations differences between the results during the second breath and subsequent breaths were negligible. Therefore only results from the second breath, $t = 4s$ to $t = 8s$ are presented. As part of the tissue we will take a sagital slice as shown in Figure 4a. This slice gives a good representation of the general dynamics within the tissue. All subsequent figures that do not show time courses are taken at $t = 5.8s$, just before reaching peak inhalation for the second time during the simulation.

![Figure 4: (a) The blue sagital slice indicates the position of subsequent slices used for the data analysis of the tissue. (b) The red ball represents the structurally modified region, used to prescribe airway constriction and tissue weakening.](image-url)
4.1 Normal breathing

To simulate tidal breathing we apply the boundary conditions and simulation parameters previously discussed in sections 3.2 and 3.3, respectively.

4.1.1 Lung volume, flow and pressure drop

Figure 5 details the lung tidal volume, flow rate and pressure drop information obtained from simulations of tidal breathing. Due to the incompressibility of the poroelastic medium and the steady state nature of the airway network, the lung tidal volume (Figure 5a) and flow rate (Figure 5b) follow a sinusoidal pattern that matches the form of the deformation boundary condition prescribed by equation (16). The mean pressure drop of the airways, is shown in Figure 5c, and agrees with previous simulation studies on full airway trees [1, 2].

Figure 5: Simulated natural tidal breathing: (a) lung tidal volume (volume increase from FRC), (b) flow rate at the inlet, (c) mean pressure drop from the inlet to the most distal branches.

4.1.2 Pathway resistance

For the subsequent analysis we have calculated the pathway resistance (Poiseuille flow resistance) from the inlet (right bronchi) to each terminal airway. The pathway resistance for the whole tree is shown in Figure 6a. In Figure 6b we show the pathway resistance of the terminal airways mapped onto the tissue.
4.1.3 Airway tree-tissue coupling

In order to quantify the contribution of airway resistance to tissue expansion (ventilation), measured by $J_V$, the correlations between pathway resistance and $J_V$ were calculated in Figure 7a. There is a clear correlation between pathway resistance and ventilation ($J_V$). This is expected since the elastic coefficients have been parameterized homogeneously. The Pearson correlation coefficients is $-0.55$, with a p-value $< 0.0001$. Figure 7b also shows a strong correlation between the pathway resistance and pressure in the poroelastic tissue. Here the Pearson correlation coefficients is also $-0.55$, with a p-value $< 0.0001$. Also note that for regions that are coupled to terminal branches with a low pathway resistance, positive pressures are possible to enable a pressure gradient that pushes fluid from these well ventilated regions to neighbouring less ventilated regions (collateral ventilation).

Figure 6: (a) Pathway resistance (Pa mm$^{-3}$s) from the inlet to the terminal branches in the airway tree. (b) Pathway resistance mapped onto a slice of tissue. The deformation of both the tree and the tissue in this figure correspond to the reference configuration.

shows the distribution of pressure in the airway tree. As previously discussed the pressure in the airway tree correlates with airway tree resistance, shown in Figure 6b. The pressure inside the poroelastic tissue is shown in Figure 8b. Again the pressure distribution correlates well with the pathway resistance in the airways, shown in Figure 6b. Figure 8c shows the pressure on the lung surface. The patchy pressure field is well approximated by the piecewise constant pressure elements employed by the finite element method used to
Figure 7: (a) Correlation between tissue expansion (ventilation) and resistance of the pathways from the inlet to the terminal branch. (b) Correlation between pressure in the poroelastic medium (alveolar pressure) and pathway resistance.

solve the poroelastic equations (see [21]). Figure 8d shows the distribution of tissue expansion. Although the distribution is heterogeneous the variations are quite small, since the elastic coefficients have been parameterized homogeneously.

4.1.4 Hysteresis

Figure 9 shows the change in elastic recoil (total stress) with volume throughout the breathing cycle for three different breathing rates. This curve is also known as a dynamic pressure-volume (PV) curve. The increase of hysteresis in the PV curve and its shift to the right as the breathing rate increases agrees with findings in the literature [38, 39]. In our model, this shift and widening of the curve can be attributed to the resistance in the airway tree which causes a larger and more heterogeneous pressure drop and flow distribution within the airways at increased flow rates. For tissue regions coupled to airways with high resistance this results in delayed, out of phase, alveolar recruitment (tissue expansion). This delayed filling of tissue during inspiration and emptying during expiration causes delayed pressure gradients that
Figure 8: (a) Pressure in the airway tree. (b) Sagital slice showing pressure in the tissue using a linear interpolation. (c) Pressure on the lung surface. (d) Sagital slice showing tissue expansion from FRC.

give rise to the hysteresis seen in Figure 9. In the literature, hysteresis associated with dynamic pressure volume (PV) curves is mostly hypothesized to be caused by flow-dependent resistances, pendelluft effects, chest wall rearrangement, and recruitment and derecruitment of lung units [40, 41, 39].

4.2 Breathing with airway constriction

In addition to the previously applied boundary conditions and simulation parameters, we now alter the simulation setup to simulate localize constriction of the airways. We implement this by reducing the radius of the lower
Figure 9: Pressure-volume curve: mean elastic recoil (total stress) against lung tidal volume during one full breathing cycle, for three different breathing rates. The arrows indicate the direction of time during the breathing cycle.

Airways (with radius less than 4mm) within a ball near the right middle lobe, represented by a red ball in Figure 4b, and referred to as the structurally modified region. We reduce the radius of the aforementioned lower airways by 0%, 40%, 50%, 60% and 65%. This corresponds to a mean pathway resistance within the ball of 0.0507, 0.112, 0.188, 0.399 and 0.651 Pa·mm$^{-3}$s, respectively. We will refer to this region as the structurally modified region. Figure 10 shows the changes in variables of physiological interest within the ball as the pathway resistance increases. The amount of tissue expansion during inspiration decreases as the airways become constricted (airway radius decreases and pathway resistance increases), shown in Figure 10a. This is due to the reduced amount of flow in these airways. Also the standard deviation increases because the pathway resistance of each branch increases by a different amount, depending on its original length and radius. Long and narrow branches will be affected most by the constriction. In Figure 10b the pressure decreases with increasing pathway resistance, since a larger pressure drop is needed to force the air down the constricted branches. Figure 10c shows the elastic stress in the tissue decrease as pathway resistance increases. This is due to the decrease in tissue deformation (strain), previously mentioned in Figure 10a. The largest elastic stress appears around the constricted region, where the tissue is made to expand more, see Figure 11c.
The simulations results shown in Figure 11 were performed with 65% airway constriction in the lower airways, applied within the structurally modified region.

### 4.2.1 Mass conservation and collateral ventilation

The volume conserving property (mass conservation) of the method is illustrated in Figure 11b where the tissue surrounding the constricted area is expanding to compensate for the reduction of tissue expansion due to the constriction within the structurally modified region. Figure 11a shows an increase in pressure near the boundary of this region. This facilitates a pressure gradient that allows for air to flow into the constricted region (collateral
ventilation) to partially compensate for the reduced amount of ventilation, as is shown in Figure 11d. The magnitude of the maximum flow within the tissue is $8 \times 10^{-4} \text{ ms}^{-1}$, this is quite small and is due to the low permeability applied homogeneously within the model.

![Figure 11: (a) Sagital slices showing the elastic stress, (b) Relative Jacobian, (c) pressure and (d) direction of the fluid flux near the structurally modified (constricted) region.](image)

4.3 Breathing with tissue weakening

We now alter the simulation setup to simulate localized weakening of the tissue. We implement this by reducing the Young’s modulus of the tissue within the structurally modified region represented by the red ball in Figure 4b. We reduce the Young’s modulus by 0%, 50%, 75% and 90%. This corresponds to a modified Young’s modulus of 730, 365, 182.5 and 73 Pa, respectively.
The amount of tissue expansion during inspiration increases as the tissue becomes weaker, shown in Figure 10d. This results in more fluid flowing along the branches into the weakened region, thus causing an increase in the magnitude of pressures in the tissue to drive the flow, shown in Figure 10e. The reduction in the Young’s modulus within the modified region causes the calculated elastic stress to decrease, since the marginal increase in strain is outweighed by the reduction in the strain energy, see Figure 10f.

4.3.1 Mass conservation

Also, due to the large amount of tissue expansion within the structurally modified region, the tissue immediately surrounding this region expands the least, seen in Figure 12. This is again due to mass conservation enforced by equation (15c).

![Figure 12: Slice showing the amount of tissue expansion ($J_V$) from FRC during inspiration with 90% localized tissue weakening.](image)

5 Discussion

We have presented a mathematical model of the lung that tightly couples tissue deformation with ventilation using a poroelastic model coupled to a fluid network model. We have highlighted the assumptions necessary to arrive at such a model, and outlined its limitations. In comparison with previous ventilation models, the proposed model is mass conserving by modelling the tissue as a continuum, able to model collateral ventilation, and driven by deformation boundary conditions extracted from imaging data to avoid having
to prescribe a pleural pressure which is impractical to be measured experimentally. In simulations of normal breathing, the model is able to produce physiologically realistic global measurements and dynamics. In simulations with altered airway resistance and tissue stiffness, the model illustrates the interdependence of the tissue and airway mechanics and thus the importance of a fully coupled model.

5.1 Contributors of airway resistance and tissue mechanics to lung function

We have found that there is a strong correlation between airway resistance and ventilation, see Figure 7a. Also, due to the airway resistance, hysteresis effects appear during breathing (Figure 9) and result in a complex ventilation distribution, caused by delayed filling and emptying of the tissue. Due to the Poiseuille law that governs the flow through the airways, small changes in airway radii can result in large changes in pathway resistance, which in turn can significantly affect the results of the coupled model. Thus, parametrizing the airways correctly is very important. However this is notoriously difficult since CT data is only available down to the 5-6th generation, and small errors and biases in the segmentation, that get propagated by the airway generation algorithm, can have large influences in determining the simulation results. Changes in tissue elasticity coefficients also play an important role in determining the function of the lung model. This has been demonstrated in section 4.3 where a reduction in the Young’s modulus within a specified region causes significant changes in ventilation, pressure and stress.

5.2 Limitations and future work

In order to move towards a more realistic model of the lung breathing, many steps need to be taken. We will list the main limitations that exist in the airway tree model, the poroelastic model, the boundary conditions and the geometry, and give indications on how these could be addressed in a future model.

Airway tree limitations: 1. The airway tree flow model currently implemented makes the Poiseuille flow assumption for the whole tree. The Poiseuille flow assumption requires flow to be fully developed and laminar which is often true for long pipes and fluid flowing at low Reynolds numbers.
This may also be true for the smaller airways but is certainly false for the larger upper airways where high Reynolds number flows can be measured. Such a model will therefore not be able to capture the high Reynolds number flows and turbulent effects that are known to exist in the upper airways. This could be improved by modifying the airway resistance at different generations according to the Reynolds number [2, 42]. Further improvements could be made by using a more sophisticated flow model for the airways, such as the 3D-0D model presented in [1].

2. The coupling of each terminal branch to the tissue currently assumes that there is no added resistance to air flowing from the terminal branch to each alveolar unit within the tissue. This could be improved by adding a simple resistive (impedance) model considering the volume of tissue that the terminal branch is feeding. This would also slightly increase the mean pressure drop of the lung model.

3. At the moment the airway tree is assumed to be static, and its configuration is not influenced by the deformation and stresses in the tissue. This could be improved by modelling the interaction of stresses and strains on the airway wall, opening up the airways during inspiration.

**Poroelastic tissue limitations:**

1. We have assumed a Neo-Hookean law for strain energy law to make the interpretation of the elasticity constants and dynamics of the model as simple as possible. However lung parenchyma is known to follow an exponential stress-strain relation, especially past tidal volume, where a law such as the one proposed by [26] might be more appropriate. Also little is known about the form of the strain-energy law during disease (e.g. fibrosis or emphysema). Similarly, for the permeability law little is known about its form for healthy or diseased tissue. Further experiments and modelling investigation would be needed to develop these.

2. Currently the tissue has been parameterized homogeneously to simplify the analysis of the results. Density information from CT images could be used to parameterize the initial porosity and elasticity coefficients, and diffusion tensor MRI be used to parameterize the permeability.

3. We have ignored the effect of blood in the tissue. The inertia and gravity forces of blood acting on the tissue could be of importance when predicting deformation and ventilation in the lung. Due to the modular framework of the poroelastic theory it should be possible to include blood as a separate phase in a future version of the model. A vascular tree could also be generated from CT images and coupled to the poroelastic medium.

4. The airflow within the poroelastic tissue has been assumed to be inviscid. However, if we were to consider diseased states such as emphysema, where large areas of lung tissue completely break down
leaving big holes, it could be argued that viscous forces could well play an important role, making it important to include them in our model. In a future version of the model the Darcy flow model could be replaced with a Brinkman, or even a Stokes flow model for big holes, where the homogenization assumption of a porous continuum is not valid.

**Boundary condition limitations:** The current registration should be updated to a more sophisticated non-linear registration algorithm (e.g. [43, 13, 44]) that is able to account for the complicated deformation of the lung surface during breathing.

**Geometry limitations:** 1. To model the complete organ and give a more accurate pressure drop, both the right and left lung, and the trachea and mouth should be included. 2. The airway tree generated in this work goes down to generations 8-13. More generations should be added to result in a fuller and more realistic tree. This would also require a finer mesh to approximate the lung tissue to resolve the coupling between each terminal branch and a subregion of lung tissue. 3. Cavities in the lung parenchyma due to large airways are currently not accounted for, i.e. it is assumed that the volume occupied by the airways is zero. To improve on this, a mesh of the lung with the larger upper airways removed would need to be generated. This new mesh could also incorporate a model of the cartilage found in the upper airways. 4. Additional no-flux boundaries should be introduced to represent the well defined and thought to be impermeable boundaries, between fissures and lung segments.

**Validation:** For this model to be of practical use it is crucial that it is properly validated. Computed tomography and 4D (dynamic) Magnetic resonance imaging (MRI) can be used to track displacements and calculate volume changes of lung structures. MRI of gases such as Hyperpolarized Xenon [45] and Helium 3 can be used to infer the flow and diffusion of gases.

**Other organs:** The proposed methodology could be adapted to model other biological tissues where blood vessels flow through and interact with a deforming tissue. For example, when modelling perfusion of blood flow in the beating myocardium [46, 47], modelling brain oedema [48] or hydrocephalus [49], or microcirculation of blood and interstitial fluid in the liver lobule [50].

5.3 Summary

The model presented in this paper is a valid tool for solving the mechanical problem of tightly coupling lung deformation and ventilation during normal
breathing and breathing with disease. The numerical simulations are shown to be able to reproduce global physiological realistic measurements. A fully nonlinear formulation permits the inclusion of various constitutive models, allowing investigation into different diseased states during various breathing conditions. A finite element method has been used to discretize the equations in a monolithic way to ensure convergence of the nonlinear problem, even under strong poroelastic-fluid-network coupling conditions. Due to the flexibility of the model, further improvements in its physiological accuracy are possible.

A Appendix

A.1 Finite element matrices

For the fully-coupled large deformation poroelastic fluid network model we need to solve the linear system $K(u^n_i)\xi_{n+1} = -R(u^n_i, u^{n-1})$ at each Newton iteration. This can be expanded as

$$
\begin{bmatrix}
K^e & 0 & B^T & 0 & 0 & 0 & 0 & 0 \\
0 & M & B^T & L^T & 0 & 0 & 0 & 0 \\
-B & -\Delta B & J & 0 & 0 & 0 & 0 & -\Delta tG^T \\
0 & L & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & T_{11} & \cdots & \cdots & T_{14} \\
0 & 0 & 0 & 0 & : & \cdots & \cdots & : \\
0 & 0 & 0 & 0 & T_{31} & \cdots & \cdots & T_{34} \\
0 & 0 & G & 0 & 0 & -X & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\xi u^n_i \\
\xi z^n_i \\
\xi p^n_i \\
\xi \lambda^n_i \\
\xi \rho^n_i \\
\xi \lambda^n_p \\
\xi \rho^n_p \\
\xi Q^n_d \\
\xi Q^n_d \\
\xi P^n_d \\
\xi P^n_d \\
G P^n_d - X P^n_d
\end{bmatrix}
= 
\begin{bmatrix}
r_1 \\
r_2 \\
r_3 - \Delta tG^T Q^n_d \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix},
$$

where we have defined the following matrices and vectors:

$$
K^e = [a_{kl}], \quad k^e_{kl} = \int_{\Omega_t} B^T_k D(u^n_i) B_l + (\nabla \phi_k)^T \sigma_e(u^n_i) \nabla \phi_l \, dv,
$$

$$
M = [m_{kl}], \quad m_{kl} = \int_{\Omega_t} k^{-1}(u^n_i) \phi_k \cdot \phi_l \, dv,
$$

$$
B = [b_{kl}], \quad b_{kl} = -\int_{\Omega_t} \psi_k \nabla \cdot \phi_l \, dv,
$$

$$
J = [j_{kl}], \quad j_{kl} = \delta \sum_K \int_{\partial K \cap \Omega_t} h_{\partial K}[\psi_k][\psi_k] \, ds.
$$
\[ r_1 = [r_{1i}], \quad r_{1i} = \int_{\Omega_t} (\sigma_e(u_i^n) - p_i^n I) : \nabla \phi_i - \rho(u_i^n) \phi_i \cdot f \ dv - \int_{\Gamma_t} \phi_i \cdot t_N \ ds, \]
\[ r_2 = [r_{2i}], \quad r_{2i} = \int_{\Omega_t} k^{-1}(u_i^n) \phi_i \cdot z_i^n - p_i^n \nabla \cdot \phi_i - \rho'(u_i^n) \phi_i \cdot f \ dv, \]
\[ r_3 = [r_{3i}], \quad r_{3i} = \int_{\Omega_t} \psi_i \nabla \cdot (u_i^n - u^{n-1}) + \Delta t \psi_i \nabla \cdot z_i^n - \Delta t \psi_i g \ dv + \delta \sum_K \int_{\partial K \cap \partial \Omega_t} h_{\partial K}[\psi_i] [p_i^n - p^{n-1}] \ ds, \]
\[ L = [l_{ij}], \quad l_{ij} = \int_{\Omega} \epsilon_i \phi_j \cdot n, \]
\[ X = [x_{ij}], \quad x_{ij} := \begin{cases} 1 & \text{if } ||Q_{di} - \text{cent}(E_j)|| < ||Q_{dk} - \text{cent}(E_j)||, \ k = 1, 2, ..., N, k \neq i, \\ 0 & \text{otherwise}, \end{cases} \]
\[ G = [g_{ij}], \quad g_{ij} = \int_{\Omega} x_{ij} \frac{\phi_j}{|E_j|}, \]
\[ T \text{ represents the matrix entries required for the fluid network.} \]

Here \( \phi_k \) are vector valued linear basis functions such that the displacement vector at the \( i \)th iteration can be written as \( u_i^n = \sum_{k=1}^{n_a} u_{i,k}^n \phi_k \), with \( \sum_{k=1}^{n_a} u_{i,k}^n \phi_k \in W_h^E \). Similarly for the fluid flux vector we have \( z_i^n = \sum_{k=1}^{n_a} z_{i,k}^n \phi_k \), with \( \sum_{k=1}^{n_a} z_{i,k}^n \phi_k \in W_h^D \). The scalar valued constant basis functions \( \psi_i \) are used to approximate the pressure, such that \( p_i^n = \sum_{k=1}^{n_p} p_{i,k}^n \psi_k \), with \( \sum_{k=1}^{n_p} p_{i,k}^n \psi_k \in Q_h \). Similarly, \( \epsilon_k \) are scalar valued linear basis functions such that the Lagrangian multiplier vector at the \( i \)th iteration can be written as \( \lambda_i^n = \sum_{k=1}^{n_a} \lambda_{i,k}^n \epsilon_k \), and \( \text{cent}(E_j) \) denotes the centroid of the \( j \)th element. Also to aid the assembly of the fourth order tensor we have adopted the matrix voigt notation. In particular \( D \) is the matrix form of \( c \), and \( B_k \) is the matrix version of \( \nabla S \phi_k \), see (19) and (20) for details.

### A.2 Matrix voigt notation

The spatial tangent modulus, fourth-order tensor, can be written as (see [51, section 5.3.2] and [52, section 6.6])
\[
\epsilon_{ijkl} = \frac{1}{J} F_{ii} F_{jj} F_{kk} F_{ll} C_{ijkl}, \tag{17}
\]
where $C$ is the associated tangent modulus tensor in the reference configuration, given by (in component form)

$$C_{IJKL} = 4\frac{\partial^2 W}{\partial C_{IJ} \partial C_{KL}} + pJ \frac{\partial C_{IJ}^{-1}}{\partial C_{KL}}.$$  \hspace{1cm} (18)

To ease the implementation of the spatial tangent modulus we make use of matrix Voigt notation. The matrix form of $c$ is given by $D$, which can be written as (see [51, section 7.4.2])

$$D = \frac{1}{2} \begin{pmatrix} 2c_{1111} & 2c_{1122} & c_{1112} + c_{1121} & c_{1113} + c_{1131} & c_{1123} + c_{1132} \\ 2c_{2222} & 2c_{2233} & c_{2212} + c_{2221} & c_{2213} + c_{2231} & c_{2223} + c_{2232} \\ 2c_{3333} & c_{3312} + c_{3321} & c_{3313} + c_{3331} & c_{3323} + c_{3332} \\ c_{1212} + c_{1221} & c_{1213} + c_{1231} & c_{1223} + c_{1232} \\ c_{1313} + c_{1331} & c_{1323} + c_{1332} & c_{2323} + c_{2332} \end{pmatrix}. \hspace{1cm} (19)$$

We also make use of the following implementation friendly notation

$$\nabla^S \phi_k = \begin{bmatrix} \phi_{k,1} & 0 & 0 \\ 0 & \phi_{k,2} & 0 \\ 0 & 0 & \phi_{k,3} \\ \phi_{k,2} & \phi_{k,1} & 0 \\ 0 & \phi_{k,3} & \phi_{k,2} \\ \phi_{k,3} & 0 & \phi_{k,1} \end{bmatrix} = B_k. \hspace{1cm} (20)$$

### A.3 Dyadic product notation

The dyadic product of two vectors $a$ and $b$ is the second-order tensor $a \otimes b$ defined by

$$(a \otimes b)v = (b \cdot v)a, \quad \forall v \in \mathbb{V}. \hspace{1cm} (21)$$

This implies

$$[a \otimes b]_{ij} = a_i b_j. \hspace{1cm} (22)$$

We can extend this to build fourth order tensors from the two second order tensors $A$ and $B$ such that

$$[A \otimes B]_{ijkl} = A_{ij} B_{kl}. \hspace{1cm} (23)$$

See [53] for details.
A.4 Neo-Hookean strain energy

For the numerical examples we have used the following Neo-Hookean strain-energy law

\[ W(C) = \frac{\mu}{2}(\text{tr}(C) - 3) + \frac{\Lambda}{4}(J^2 - 1) - (\mu + \frac{\Lambda}{2})\ln(J - 1 + \phi_0). \]  (24)

Thus, the resulting effective stress tensor is given by

\[ \sigma_e = \frac{\Lambda}{2J} \left( J^2 - \frac{J}{J - 1 + \phi_0} \right) I + \frac{\mu}{J} \left( C^T - \frac{JI}{J - 1 + \phi_0} \right), \]  (25)

and the spatial tangent modulus tensor is given as

\[ c = c_e + p(I \otimes I - 2Z), \]  (26)

where

\[ c_e = \left[ \Lambda J^2 - 2\mu J \left( \frac{1}{2(J - 1 + \phi_0)} - \frac{J}{2(J - 1 + \phi_0)^2} \right) \right] I \otimes I \]

\[ + \left[ \frac{2\mu J}{J - 1 + \phi_0} - \Lambda \left( J^2 - \frac{J}{J - 1 + \phi_0} \right) \right] B, \]  (27)

and

\[ B_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad Z_{ijkl} = \delta_{ik}\delta_{jl}, \quad I \otimes I = \delta_{ij}\delta_{kl}. \]  (28)

See [51, chapter 5] and [28, chapter 3] for further details.

References


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