Consensus of nonlinear multi-agent systems with self and communication time delays: A unified framework

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Abstract

In this paper, we study the consensus problem of multi-agent systems with parametric uncertainties on directed graph communication topologies containing a spanning tree. The challenge lies in that the input–output property of a strictly proper system transfer function involving self and communication delays and the extremum of a delay-involved transfer function are both unclear. By establishing a new input–output property of the delay-involved strictly proper transfer function and applying a constructive approach in frequency domain, we obtain the extremum of the delay-involved transfer function, upon which, we establish a unified framework to resolve the consensus problem of multi-agent systems with parametric uncertainties and time delays, and the terminal convergence point value is achieved. Based upon Lyapunov stability theory and frequency domain input–output analysis, we demonstrate that the proposed unified consensus control framework ensures scaled weighted average consensus with the integral action of the synchronization signal vector. Simulation results are provided to demonstrate the effectiveness of the proposed consensus scheme.

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1. Introduction

As an important topic in cooperative control, consensus problem for multi-agent systems has been extensively studied in recent years. Roughly speaking, the consensus objective is to reach some kind of agreement between some variables of interest of the systems. Clearly, the two main

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ingredients in this problem are the individual dynamics of the systems and the interconnection patterns among them.

The consensus problems of linear multi-agent systems have been extensively studied in [1–5], to name just a few. Since most of the real systems have nonlinear dynamics, recently, much attention has also been paid on consensus problem for nonlinear systems [6–16]. For example, [11] deals with the leader-follower consensus problem for a class of nonlinear multi-agent systems under a connected undirected information communication topology, which is extended by [12] to seek consensus in finite time. Similarly, [13] develops a rigorous framework for finite-time semi-stability, while [14] develops a thermodynamic framework to address consensus problems for nonlinear multi-agent systems with fixed and switching topologies. The second-order consensus of multiple nonlinear dynamical mobile agents with a virtual leader in a dynamic proximity network has also been studied in [15,16].

The communication time delay and the input (or self) time delay are very commonplace in networked multi-agent systems, which renders both the analysis and design problem challenging [17–26]. Under the assumption that the communication topology is undirected and fixed, [17] studies the first-order nonlinear systems’ consensus problem with constant communication time-delays. Refs. [18–20] extend this to time-varying communication delays and switching topologies. Based on the passivity theory, [21] presents a consensus framework for multiple variable-rate heterogeneous mechanical integrators, which considers a single mass–spring–damper type mechanical integrator. The discrete-time and continuous-time second-order multi-agent systems with communication delays are studied by [22] and [23] respectively, which is extended by [24] to switching topology cases. Unfortunately, they only focus on the double integrator systems with communication delays excluding input delays. Then, both time-domain (Lyapunov theorems) and frequency-domain (the Nyquist stability criterion) approaches are used to study the leaderless and leader-following consensus algorithms with communication and input delays under a directed network topology in [25]. However, it focuses on the linear single integrator systems. In [26,27], the adaptive controllers are proposed to deal with consensus problem for networked Euler–Lagrange system with unknown parameters, which allow for the existence of coupling communication delays under the directed communication topology. Ref. [28] extends this to the case with both communication delays and input delays. However, the final convergence point cannot be determined.

In this paper, we try to clarify the input–output property of a kind of delay-dependent strictly proper transfer function and also propose an approach to derive the limit value of it, upon which we construct a unified framework for the consensus of second-order nonlinear multi-agent systems with directed graphs, both communication and input delays, as well as parameter uncertainties. This unified consensus framework provides a design parameter to flexibly include two kinds of controllers into the same framework, and ensures scaled weighted average consensus by using an integral adaptive controller. The most interesting result is that the proposed integral adaptive controller allows us to derive the final consensus value of the agents.

Comparison with the existing work in the literature: In contrast with the consensus algorithms for first-order and second-order linear dynamics [1–5], we study the nonlinear second-order systems. In contrast with the consensus algorithms for nonlinear systems [11–16], we consider the consensus problem with time delays. In contrast with the consensus problem with only communication delays for multi-agent systems in [19–25], we deal with communication delays as well as input delays. Furthermore, distinguished from the conventional consensus problem with communication and input delays, we propose a unified framework which not only renders the multi-agents reach consensus or average consensus, but also derives the terminal
convergence point value, i.e., scaled weighted average consensus point, which is related to the communication delays as well as input delays. To the best of our knowledge, this result is not addressed in the current available papers.

This paper is organized as follows. In Section 2, we provide some backgrounds on the dynamics of interest, and the graph theory and some mathematical background information are presented. Main results on the uniformed framework to reach scaled weighted average consensus are provided in Sections 3. In Section 4, numerical examples are presented to illustrate the effectiveness of the proposed algorithm, and finally the results are summarized in Section 5.

Notation: \( \mathbb{R} := (-\infty, \infty) \), \( \mathbb{R}_0 := (0, \infty) \), \( \mathbb{R}_{\geq 0} := [0, \infty) \). The eigenvalues of matrix \( A \) are real. \( \lambda_m[A] \) and \( \lambda_M[A] \) represent the minimum and maximum eigenvalues of matrix \( A \), respectively. \( \| A \|_2 \) is the spectral norm of matrix \( A \). \( |x| \) stands for the standard Euclidean norm for the vector \( x \in \mathbb{R}^n \). For any function \( f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n \), the \( \mathbb{L}_\infty \)-norm is defined as \( \| f \|_\infty = \sup_{t \geq 0} |f(t)| \), and the \( \mathbb{L}_2 \)-norm as \( \| f \|_2 = \int_0^\infty |f(t)|^2 \, dt \). The \( \mathbb{L}_\infty \) and \( \mathbb{L}_2 \) spaces are defined as the sets \{ \( f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n : \| f \|_\infty < \infty \) \} and \{ \( f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n : \| f \|_2 < \infty \) \}, respectively. \( \text{Re}(\cdot) \) denotes the real part of a complex number.

2. Problem statement and background information

2.1. Dynamics of agent

We consider a multi-agent system consisting of \( n \) agents. The dynamics of the \( i \)th agent is described by

\[
\begin{aligned}
\dot{x}_i(t) &= v_i(t) \\
\dot{v}_i(t) &= f(v_i(t)) + u_i(t),
\end{aligned}
\]

where \( x_i(t) \in \mathbb{R}^m \) is the position state of \( i \)th agent, \( v_i(t) \in \mathbb{R}^m \) is the velocity state, \( u_i(t) \in \mathbb{R}^m \) is the control input, and \( f(v_i(t)) \) is the nonlinear dynamics of agent \( i \), which is assumed to be unknown. Reasonable assumptions for existence of unique solutions are made, i.e., \( f(v_i(t)) \) is continuous in \( t \) and Lipschitz in \( v_i(t) \), and can be parameterized as

\[
f(v_i(t)) = \phi_i^T(v_i(t))\theta_i, \quad i = 1, 2, \ldots, n,
\]

where \( \phi_i^T(v_i(t)) \in \mathbb{R}^{m \times m} \) is a basis function matrix of \( v_i(t) \), which can be referred to as a regressor, and \( \theta_i \in \mathbb{R}^m \) is an unknown constant parameter column vector to be estimated.

Remark 1. In this paper, the unknown nonlinear dynamics \( f(v_i(t)) \) of all agents can be linearly parameterized as \( \phi_i^T(v_i(t))\theta_i \), which have been studied widely in classical adaptive control [29,30]. The practical examples of linearly parameterized model of multi-agent systems can also be found in [31,32].

2.2. Graph theory

Graphs can be conveniently used to represent the information flow between agents. Let \( G = (\mathcal{V}, \varepsilon, \mathcal{W}) \) be an undirected graph or directed graph (digraph) of order \( n \) with the set of nodes \( \mathcal{V}(G) = \{v_1, v_2, \ldots, v_n\} \), the set of edges \( \varepsilon \subseteq \mathcal{V} \times \mathcal{V} \), and a weighted adjacency matrix \( \mathcal{W} = \{\omega_{ij}\} \) with non-negative adjacency elements \( \omega_{ij} \). The node indices belong to a finite index set \( \ell = \{1, 2, \ldots, n\} \). An edge of \( G \) is denoted by \( e_{ij} = (v_i, v_j) \) and it is said to be incoming with respect to \( v_j \) and outgoing with respect to \( v_i \). For an undirected graph, \( \forall i, j \in \ell \), if \( (v_i, v_j) \in \varepsilon(G) \), then
(v_j, v_i) \in \varepsilon(G)$, but this does not hold for a digraph. A directed path from node $i$ to node $j$ is a sequence of edges of the form $(i_1, i_2), (i_2, i_3), \ldots$, in a directed graph. A directed tree is a directed graph, where every node has exactly one parent except for one node, called the root, and the root has directed paths to every other node. A directed spanning tree of a directed graph is a directed tree that contains all nodes of the directed graph. A directed graph has a spanning tree if there exists a directed spanning tree as a subgraph of the directed graph. The set of neighbors of node $v_i$ is the set of all nodes which point (communicate) to $v_i$, denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_i, v_j) \in \varepsilon(G)\}$. The graph adjacency matrix $W = [w_{ij}], \forall w \in \mathbb{R}^{n \times n}$, is such that $w_{ij} > 0$ if $j \in \mathcal{N}_i$ and $w_{ij} = 0$ otherwise. The in-degree of vertex $v_i$ is denoted by $d_i = \sum_{j=1}^{n} w_{ji}$. Similarly, the out-degree of $v_i$ is denoted by $d_i = \sum_{j=1}^{n} w_{ij}$. If the in-degree equals to the out-degree for all $v_i \in \mathcal{V}(G)$, then the graph is said to be balanced. $D = \text{diag}[d_1, \ldots, d_n] \in \mathbb{R}^{n \times n}$ is called the degree matrix of $G$. The weighted Laplacian matrix of $G$ is $L_w = D - W$.

**Lemma 1 (Ren et al. [33]).** If $L_w$ is associated with a digraph containing a spanning tree, all the eigenvalues of $L_w$ excluding the simple zero eigenvalue have positive real parts. Furthermore, $L_w$ has a right eigenvector $\mathbf{1}_n = [1 \cdots 1]^T$ and a nonnegative left eigenvector $\mathbf{\eta} = [\eta_1 \cdots \eta_n]^T$ satisfying $\sum_{i=1}^{n} \eta_i = 1$ associated with the zero eigenvalue, i.e., $L_w \mathbf{1}_n = 0_n$ and $\mathbf{\eta}^T L_w = 0_n^T$.

### 3. A unified consensus framework

**Definition 1.** In this paper, the networked multi-agent system is said to reach consensus if $x_i(t) - x_j(t) \to 0$ and $\dot{x}_i(t) \to 0$ as $t \to \infty$, $\forall i, j \in \mathcal{V}$.

**Assumption 1.** Agent $i$ receives self-position state information $x_i(t)$ with delay $\tau_i \geq 0$, named self-time delay, and receives the neighbor's position state information $x_j(t)$ and $v_j(t)$ with delay $T_{ij}$, named communication time delay, where $T_{ij} > \tau_i \geq 0$.

**Assumption 2.** The Laplace transforms of $x_i(t)$ and $v_j(t)$ are well-defined, and the nonlinearities in the velocity dynamics do not affect the validity of their Laplace transforms.

#### 3.1. Design of the consensus framework

Inspired by [34,35], we define the first synchronization signal vector $s_i$ that relies on the delayed self-information and the neighbors' information

$$s_i = \dot{x}_i + \sum_{j \in \mathcal{N}_i} \omega_{ij} (x_i(t-\tau_i) - x_j(t-T_{ij})), \quad \text{(3)}$$

where $x_i$ is the position state, $\dot{x}_i$ is the velocity state, $\mathcal{N}_i$ stands for the set of neighbors of agent $i$, $\omega_{ij}$ is the communication weight, $T_{ij}$ and $\tau_i$ are the communication delay between agent $i$ and agent $j$ and the self-delay of agent $i$, respectively, which are both assumed to be bounded.

For the preparation of the second synchronization signal vector, we define a velocity-like vector, which is written as

$$\dot{x}_{r,i} = - \sum_{j \in \mathcal{N}_i} \omega_{ij} (x_i(t-\tau_i) - x_j(t-T_{ij})) - \alpha \int_0^t s_i(r) \, dr, \quad \text{(4)}$$

where the parameter $\alpha$ can be chosen to be either zero (i.e., $\alpha = 0$) or positive (i.e., $\alpha > 0$), and in this way, two different kinds of controllers are flexibly included.
With $\dot{x}_{r,i}$, we define another synchronization signal vector $\xi_i$ as

$$\dot{\xi}_i = \dot{x}_i - \dot{x}_{r,i} = s_i + a \int_{0}^{t} s_i(r) \, dr. \tag{5}$$

Note that when $\alpha = 0$, $\dot{\xi}_i$ is reduced to $s_i$, and when $\alpha > 0$, $\dot{\xi}_i$ includes the integration of $s_i$. Following [36], we design the control law for the $i$th agent as

$$u_i = -c\xi_i - \alpha s_i - \phi_i^T(v_i(t))\dot{\theta}_i - \sum_{j \in N_i} \omega_{ij}(v_j(t - \tau_i) - v_j(t - T_{ij})) \tag{6a}$$

$$\dot{\theta}_i = \phi_i(v_i(t))\xi_i, \tag{6b}$$

where $c$ is a positive constant, $v_i$ is the velocity state, $\phi_i^T(v_i(t))$ is defined as in Eq. (2), and $\dot{\theta}_i$ is the estimation of $\theta_i$, which is updated by $\ddot{\theta}_i$ in Eq. (6b).

By Eqs. (1), (3), and (6) and the fact that $\dot{x}_i(t) = v_i(t)$, the derivative of $\dot{\xi}_i$ in Eq. (5) can be written as

$$\ddot{\xi}_i = \ddot{s}_i + \alpha s_i$$

$$= \ddot{v}_i + \sum_{j \in N_i} \omega_{ij}(\ddot{x}_i(t - \tau_i) - \ddot{x}_j(t - T_{ij})) + \alpha s_i$$

$$= f_i(v_i(t)) + u_i + \sum_{j \in N_i} \omega_{ij}(\ddot{x}_i(t - \tau_i) - \ddot{x}_j(t - T_{ij})) + \alpha s_i$$

$$= \phi_i^T(v_i(t))\theta_i - \phi_i^T(v_i(t))\dot{\theta}_i - c\xi_i - \alpha s_i$$

$$- \sum_{j \in N_i} \omega_{ij}(v_i(t - \tau_i) - v_j(t - T_{ij})) + \sum_{j \in N_i} \omega_{ij}(\ddot{x}_i(t - \tau_i) - \ddot{x}_j(t - T_{ij})) + \alpha s_i$$

$$= \phi_i^T(v_i(t))\ddot{\theta}_i - c\xi_i \tag{7}$$

where $\ddot{\theta}_i = \dot{\theta}_i - \dot{\theta}_i$ is the estimation error of $\theta_i$.

### 3.2. Preparation for the seeking of the convergence point

In this subsection, we try to find some useful properties from the synchronization signal vector $s_i$, which is crucial to the main results of the whole paper.

By applying Laplace transformation to Eq. (3), we get

$$pX_i(p) - x_i(0) = -\sum_{j \in N_i} \omega_{ij}(e^{-\tau_{ij}p}X_i(p) - e^{-T_{ij}p}X_j(p)) + S_i(p) \tag{8}$$

where $p$ is the Laplace variable, $X_i(p)$, $X_j(p)$ and $S_i(p)$ denote the Laplace transforms of $x_i$, $x_j$ and $s_i$, respectively.

Obviously, we can rewrite Eq. (8) in a vector form as

$$pX(p) = -[(D_w - \mathcal{W}_T(p)) \otimes I_m]X(p) + x(0) + S(p) \tag{9}$$

where $D_w := \text{diag}[\sum_{j \in N_i} \omega_{ij}e^{-\tau_{ij}p}, \quad i = 1, 2, \ldots, n]$ is the degree matrix, $\mathcal{W}_T(p) := [\omega_{ij}e^{-T_{ij}p}]$ is the delay dependent adjacency matrix, $I_m$ denotes an $m \times m$ identity matrix, $X(p)$, $x(0)$ and $S(p)$ are the vector form of $X_i(p)$, $x_i(0)$ and $S_i(p)$, respectively.

Defining a function $G(p) = (pI_n + D_w - \mathcal{W}_T(p))^{-1}$, then Eq. (9) can be written as

$$X(p) = (G(p) \otimes I_m)(x(0) + S(p)), \tag{10}$$
or be written at the velocity level, i.e.,

\[ X_v(p) = (G(p) \otimes I_m)[-(D_w - W_T(p))x(0) + pS(p)], \]

where \( X_v(p) = pX(p) - x(0) \) denotes the Laplace transform of \( \dot{x} \).

**Remark 2.** Note that when \( T_{ij} > \tau_i \geq 0 \), \( (pI_n + D_w - W_T(p)) \) is a strictly diagonally dominant matrix, and it is invertible, which guarantees the validity of \( G(p) \).

Due to the inclusion of the delay \( \tau_i \) and \( T_{ij} \) in \( G(p) \), it is very difficult to analyze the systems (10) and (11) directly. Meanwhile, in order to ensure the convergent property, we have to examine the limit property of \( G(p) \), which is unknown up to now. Fortunately, we have the following input–output property.

**Lemma 2.** Suppose that the agents are interconnected on a digraph containing a spanning tree. For the system \( y = G(p)u \), if the integration of the input \( u(t) \) is bounded (i.e., \( u(t) \in L_\infty \)), the output \( y(t) \) will be bounded.

**Proof.** On one hand, following [34], using the Gershgorin Theorem, for the possible unstable poles with \( \text{Re}(p) > 0 \), \( e^{-\tau_i p} > e^{-T_{ij}p} \), we conclude that \( G(p) \) has no poles in the open half-plane \( \text{Re}(p) > 0 \). On the other hand, the Gershgorin disks intersect the origin of the complex plane and \( p = 0 \) may be a root – possibly with multiplicity greater than one. However, there is only one single root following immediately from the fact that \( G(0)^{-1} = L_w \), where \( L_w \) is the Laplacian matrix of the communication graph. From Lemma 1, \( L_w \) has a single zero-eigenvalue and the rest of the spectrum has positive real parts, thus all the poles of \( G(p) \) excluding the simple zero pole are in the open left half plane, and \( G(p) \) must be integral-bounded-input bounded-output stable in the sense of [37]. Therefore, the result of Lemma 2 follows. \( \square \)

**Proposition 1.** If the agents are interconnected on a digraph containing a spanning tree, it can be concluded that

\[ \lim_{p \to 0} pG(p) = \frac{1}{1 + \sum_{i=1}^{n} \eta_i \omega_{ij}(T_{ij} - \tau_i)} I_n \eta^T. \]

**Proof.** We rewrite \( G(p) \) as \( G(p) = (pI_n + D_w - W_T(p))^{-1} \). With the definition of \( D_w \) and \( W_T(p) \), \( G(p) \) can be written as the following form:

\[ G(p) = (pD^*(p) + L_w^*(p)))^{-1} \]

where

\[ D^*(p) = \text{diag} \left[ 1 + \sum_{j \in \mathcal{N}_i} \omega_{ij} e^{-\tau_j p} \frac{1 - e^{-(T_{ij} - \tau_i)p}}{p}, i = 1, 2, \ldots, n \right], \]

which can be further written as

\[ D^*(p) = \text{diag} \left[ 1 + \sum_{j \in \mathcal{N}_i} \omega_{ij} e^{-\tau_j p}(T_{ij} - \tau_i) + \frac{o(p)}{p}, i = 1, 2, \ldots, n \right] = \text{diag} \left[ d^*_i(p) \right]. \]
where \( o(p) \) denotes the polynomial terms with the order higher than \( p \), and
\[
\mathcal{L}_w^*(p) = \text{diag} \left[ \sum_{j \in N_i} \omega_{ij} e^{-T_{ij}p}, \ i = 1, 2, \ldots, n \right] - \mathcal{W}_T(p). \tag{15}
\]

The matrix \( \mathcal{L}_w^*(p) \) is quite like a Laplacian matrix in the sense that each of its diagonal entries is the sum of the off-diagonal entries at the same row. This results in the fact that \( \mathcal{L}_w^*(p) \mathbf{1}_n = 0_n \) and \( \eta^* \mathcal{L}_w^*(p) = 0_n^T \), where \( \eta^* \) is the left eigenvector of \( \mathcal{L}_w^*(p) \) associated with its zero eigenvalue. In addition, due to the fact that \( \lim_{p \to 0} \mathcal{L}_w^*(p) = \mathcal{L}_w \), we can appropriately choose \( \eta^*(p) \) such that \( \lim_{p \to 0} \eta^*(p) = \eta \).

Since \( p \mathcal{D}_w^*(p) = p + \sum_{j \in N_i} \omega_{ij} e^{-T_{ij}p} (1 - e^{-T_{ij}p}) \neq 0 \), \( \forall p \neq 0 \), the diagonal matrix \( \mathcal{D}_w^*(p) \) is always nonsingular \( \forall p \neq 0 \). Therefore, the inverse of the transfer function \( G(p) \) can be written as
\[
G^{-1}(p) = \mathcal{D}_w^*(p) H(p), \tag{16}
\]
where
\[
H(p) = p\mathbf{1}_n + \mathcal{D}_w^*{-1}(p) \mathcal{L}_w^*(p), \tag{17}
\]
which makes us capable of deriving the eigenvalues and eigenvectors of \( H(p) \) associated with the zero eigenvalue of \( \mathcal{L}_w^*(p) \). Since \( \mathcal{L}_w^*(p) \mathbf{1}_n = 0_n \), we get that \( H(p) \mathbf{1}_n = p\mathbf{1}_n \), which implies that \( H(p) \) has a right eigenvector \( \mathbf{1}_n \) associated with the eigenvalue \( p \). Furthermore, we have
\[
(\mathcal{D}_w^*(p) \eta^*(p))^T H(p) = p(\mathcal{D}_w^*(p) \eta^*(p))^T. \tag{18}
\]

Therefore, \( H(p) \) has a left eigenvector \( \mathcal{D}_w^*(p) \eta^* \) associated with the eigenvalue \( p \). As shown in Eq. (14), \( \mathcal{D}_w^*(p) \) is a diagonal matrix. In order to satisfy \( (\mathcal{D}_w^*(p))^T \mathbf{1}_n = 1 \), the left eigenvector can be normalized as
\[
\eta^*(p) = \frac{1}{\sum_{i = 1}^n \eta_i^*(p) \mathcal{D}_w^*(p) \eta^*(p)}, \tag{19}
\]
where \( \eta_i^*(p) \) is the \( i \)th element of \( \eta^*(p) \), and the form of \( \overline{\eta}^*(p) \) can be explicitly expressed as
\[
\overline{\eta}^*(p) = \frac{1}{\left[ 1 + \sum_{i = 1}^n \sum_{j \in N_i} \eta_i^*(p) \omega_{ij} e^{-T_{ij}p} (1 - e^{-T_{ij}p}) \right] + o(p)/p} \mathcal{D}_w^*(p) \eta^*(p). \tag{20}
\]

In the case that \( T_{ij} = 0 \), the eigenvalue \( p \) of \( H(p) \) is obviously simple since \( \text{rank}(\mathcal{L}_w) = n - 1 \). The presence of the delay renders it difficult to determine whether or not \( p \) is a simple eigenvalue. Fortunately, as \( p \) approaches zero, we have
\[
\lim_{p \to 0} \text{rank}(H(p) - p\mathbf{1}_n) = \text{rank}(\mathcal{D}_w^*{-1}(0) \mathcal{L}_w^*(0)) = \text{rank}(\mathcal{D}_w^*{-1}(0) \mathcal{L}_w) = n - 1. \tag{21}
\]

Since \( \mathcal{L}_w \) is associated with a digraph containing a spanning tree, the invertible diagonal matrix \( \mathcal{D}_w^*(0) \) is defined as
\[
\mathcal{D}_w^*(0) = \lim_{p \to 0} \mathcal{D}_w^*(p) = \text{diag} \left[ 1 + \sum_{j \in N_i} \omega_{ij} (T_{ij} - \tau_i), \ i = 1, 2, \ldots, n \right]. \tag{22}
\]
According to the continuous dependence of the eigenvalues on the parameter \( p \), we get that as \( p \) approaches zero or \( p \) is in the vicinity of 0, \( p \) is a simple eigenvalue of \( H(p) \), and the other eigenvalues of \( H(p) \) must approach \( p + \lambda_i \), where \( \lambda_i \) is the non-zero eigenvalue of \( \mathcal{D}_w^*{-1}(0) \mathcal{L}_w \).
Then, as \( p \to 0 \), \( H^{-1}(p) \) can be decomposed as
\[
H^{-1}(p) = \frac{1}{p} \mathbf{1}_n \eta^T(p) + \Delta(p),
\]
where \( \Delta(p) \) is exponentially stable and thus satisfies that \( \lim_{p \to 0} p\Delta(p) = 0 \).

Therefore, we can conclude that
\[
\lim_{p \to 0} pG(p) = \lim_{p \to 0} pH^{-1}(p)D^p = \lim_{p \to 0} (\frac{1}{p} \mathbf{1}_n \eta^T(p) + \Delta(p))D^p
\]
\[
= \frac{1}{1 + \sum_{i=1}^n \eta_i \omega_i (T_{ij} - \tau_i)} \mathbf{1}_n \eta^T.
\]

### 3.3. Stability analysis and main results

From the above analysis, we get the following theorem.

**Theorem 1.** The control law (6) with \( \alpha > 0 \) gives rise to scaled weighted average consensus among the agents, i.e., \( x_i(t) \to \sigma_s \sum_k^{p} \eta_k x_k(0) \) and \( \dot{x}_i(t) \to 0 \) as \( t \to \infty \), \( \forall i \in \mathcal{V} \), where the scaling factor

\[
\sigma_s = \frac{1}{1 + \sum_{i=1}^n \sum_{j \in \mathcal{N}_i} \eta_i \omega_i (T_{ij} - \tau_i)}.
\]

**Proof.** Define a positive semi-definite Lyapunov candidate function \( V : C \to \mathbb{R}^+ \) for the system as
\[
V(\xi, \tilde{\theta}) = \frac{1}{2} \xi^T \xi + \frac{1}{2} \tilde{\theta}^T \tilde{\theta}.
\]

The derivative of \( V(s_i, \tilde{\theta}_i) \) along the trajectory of Eq. (10) is given by
\[
\dot{V}(\xi, \tilde{\theta}) = \dot{\xi}^T \dot{\xi} + \dot{\tilde{\theta}}^T \dot{\tilde{\theta}} = \xi^T \phi_1 (v_i(t), t) \tilde{\theta} - \dot{\xi}^T \phi_1 (v_i(t), t) \dot{\xi} = -c \xi^T \xi
\]

Since \( V \geq 0 \) and \( \dot{V} \leq 0 \), we get that \( \xi \in L_2 \cap L_\infty \) and \( \tilde{\theta} \in L_\infty \).

As \( \alpha > 0 \), from the input–output property of exponentially stable and strictly proper linear systems [38], the fact that \( \xi_i \in L_2 \cap L_\infty \) results in \( s_i \in L_2 \cap L_\infty \), \( \int_0^t s_i(r) \, dr \in L_2 \cap L_\infty \) and \( \int_0^t s_i(r) \, dr \to 0 \) as \( t \to \infty \). Using the final value theorem, we get that \( \lim_{p \to 0} S(p) = \lim_{t \to \infty} \int_0^t s_i(r) \, dr = 0 \). By Eq. (12), we get that
\[
\lim_{t \to \infty} x(t) = \lim_{p \to 0} pX(p) = \lim_{p \to 0} pG(p) \otimes \mathbf{I}_m) x(0) = (\mathbf{1}_n \otimes \mathbf{I}_n) \sigma_s \sum_{k=1}^n \eta_k x_k(0),
\]

which means that the positions of the agents converge to the scaled weighted average value
\( \sigma_s \sum_{k=1}^n \eta_k x_k(0) \). This also implies that \( x_1(t \to \infty) = x_2(t \to \infty) = \cdots = x_n(t \to \infty) \). \( \square \)

**Corollary 1.** The control law (6) with \( \alpha = 0 \) results in consensus among the agents, i.e., \( x_i(t) - x_j(t) \to 0 \) and \( \dot{x}_i(t) \to 0 \) as \( t \to \infty \), \( \forall i, j \in \mathcal{V} \), but the convergence point is not constant.

**Proof.** If \( \alpha = 0 \), then \( s_i = \xi_i \). Choosing the same Lyapunov function as Eq. (25), we can get
\( s_i = \xi_i \in L_2 \cap L_\infty \).

Eq. (11) can be written as
\[
X_i(p) = (G(p) \otimes \mathbf{I}_m)[(-D_w + \mathcal{W}_T(p))x(0) + s(0)] + [pS(p) - s(0)]
\]
where \( pS(p) - s(0) \) can be considered to be the Laplace transform of \( \dot{s}(t) \), of which the integration is bounded, i.e., \( \int_0^t \dot{s}(r) \, dr = |s(t) - s(0)| \in L_\infty \), \( \forall t \geq 0 \). According to Lemma 2, the differential input
\[ pS(p) - s(0) \] gives a bounded output after passing through \( G(p) \otimes I_m \). At the same time, the initial input \(- (D_w - \mathcal{W}_T(p))x(0) + s(0)\) obviously gives a bounded output. Therefore, \( X_{i}(p) \) is bounded.

Furthermore, by Eq. (7), \( \xi_i = \hat{s}_i = \phi_i^T(v_i(t), t)\hat{\theta}_i - cs_i \), which implies that \( \xi_i \) and \( \hat{s}_i \) are bounded. Therefore, \( \xi_i = \hat{s}_i = 0 \) as \( t \to \infty \). Using the final value theorem, we get that \( \lim_{p \to 0} pS(p) = \lim_{t \to \infty} s(t) = 0 \), and

\[
\lim_{t \to \infty} x(t) = \lim_{p \to 0} X_{i}(p) = \lim_{p \to 0} (G(p) \otimes I_m)[-(D_w - \mathcal{W}_T(p))x(0) + pS(p)] \\
= \sigma_s[(1_n \eta^T L_w) \otimes I_m]x(0) = 0_n.
\] (29)

By Eq. (3), we can get that \( \sum_{i \in N_i} w_{ij}(x_i(t - \tau_i) - x_j(t - T_{ij})) \to 0 \), \( x_i(t - \tau_i) = x_i(t) - \int_0^{\tau_i} \dot{x}_i(t - r) dr \to x_i(t) \) and \( x_j(t - T_{ij}) = x_j(t) - \int_0^{T_{ij}} \dot{x}_j(t - r) dr \to x_j(t) \), as \( t \to \infty \). This leads us to obtain

\[
(L_w \otimes I_m)x(t \to \infty) = 0_n,
\] (30)

which implies that \( x_1(t \to \infty) = x_2(t \to \infty) = \cdots = x_n(t \to \infty) \).

However, we obviously cannot conclude that \( \int_0^t s_i(r) dr \to c_1 \) as \( t \to \infty \) from \( s_i = \xi_i \in L_2 \cap L_{\infty} \) (e. g., the signal \( 1/(t + 1) \in L_2 \cap L_{\infty} \), however, its integration \( \int_0^t (1/(r + 1)) dr = \ln(t + 1) \to \infty \) as \( t \to \infty \), where \( c_1 \) is a constant vector. Thus, we cannot get the conclusion that the convergence point is constant. □

**Remark 3.** No matter the interconnection graph is directed or undirected, balanced or unbalanced, as long as it contains a spanning tree, the scaled weighted average consensus can be achieved by using the control law (6). Meanwhile, we define agent \( k \) with \( \eta_k > 0 \) as the root of the interconnection graph. When all \( \omega_{ij} > 0 \) are with the same value, \( \eta_k \) decides the weight that root \( k \) will take in the consensus value.

**Remark 4.** The proposed framework can not only be used to the systems in Eq. (1), but also other kinds of systems. For example, the first-order linear integrator systems \( \dot{x}_i = u_i \) using the protocol \( u_t = -\sum_{j \in N_i} \omega_{ij}(x_i(t - \tau) - x_j(t - T_{ij})) \) is a special case of Eq. (6), i.e., \( s_i(t) \equiv 0 \). Therefore, the convergence point of the systems is the scaled weighted average initial position of the agents, i.e., \( \sigma_s \sum_{k=1}^n \eta_k x_k(0) \).

**Remark 5.** If \( \tau_i = T_{ij} \), the convergence point will be \( \sum_{k=1}^n \eta_k x_k(0) \), which is as the same as the case with no time delay, i.e., \( \tau_i = 0 \) and \( T_{ij} = 0 \). It is worthwhile nothing that [39] gives out a similar result, which considers only undirected topology and equal time delay cases rather than the directed topology and different time delays. We can see that the result in [39] is a special case of this paper.

**Remark 6.** Ref. [18] considers the consensus problems for discrete-time multi-agent systems with time-varying delays (including sensing delays and controller–actuator delays) and switching interaction topologies and provides a class of effective consensus protocols that are built on repeatedly using the same state information at two time-steps. The difference between our research and [18] is that, our framework not only achieves consensus, but also gains the convergence value of the multi-agent systems.

**Remark 7.** To gain further insight in the relation between the time delays and the convergence properties of consensus protocol (6), we consider the scenario satisfying the following conditions:

1. The communication topology is a directed graph containing a spanning tree, and every agent has a neighbor, which implies that \( \sum_{j=1}^n \omega_{ij} \neq 0 \), \( \forall i \) and the degree matrix \( D = I_n \). The communication weight \( w_{ij} = 1 \) if \( j \in N_i \) and \( w_{ij} = 0 \) otherwise.
2. The self-delays of all agents are the same, i.e., \( \tau_i = \tau_j = \tau \), and so as the communication delays, i.e., \( T_{ij} = T \).

3. The parameter uncertainties can be compensated perfectly.

If the above conditions are satisfied, the control law can be written as

\[
    u_i = -c\xi_i - ax_i - \phi_i^T(v_i(t))\hat{\theta}_i - \sum_{j \in \mathcal{N}_i} \omega_{ij}(v_i(t) - v_j(t - T)).
\]

(31)

Defining \( y_i(t) = \int_0^t x_i(r) \, dr \) and by Eqs. (1), (2), (3), (5) and (31), we get that

\[
    \begin{align*}
    \dot{y}_i(t) &= x_i(t) \\
    \dot{x}_i(t) &= v_i(t) \\
    \dot{v}_i(t) &= -c[\dot{x}_i(t) + \sum_{j \in \mathcal{N}_i} w_{ij}(x_i(t - \tau) - x_j(t - T)) + ax_i(t)] \\
    &\quad + \alpha \int_0^t \sum_{j \in \mathcal{N}_i} w_{ij}(x_i(r - \tau) - x_j(r - T)) \, dr - \alpha \dot{x}_i(t) \\
    &\quad - \alpha \sum_{j \in \mathcal{N}_i} w_{ij}(x_i(r - \tau) - x_j(r - T)) - \sum_{j \in \mathcal{N}_i} w_{ij}(v_i(r - \tau) - v_j(r - T))
    \end{align*}
\]

(32)

which can be written in the matrix form as

\[
    \begin{bmatrix}
    \dot{y}(t) \\
    \dot{x}(t) \\
    \dot{v}(t)
    \end{bmatrix}
    =
    \begin{bmatrix}
    0 & I_n & 0_{n \times n} \\
    0 & 0_{n \times n} & I_n \\
    0 & -caI_n & -(c + \alpha)I_n
    \end{bmatrix}
    \begin{bmatrix}
    y(t) \\
    x(t) \\
    v(t)
    \end{bmatrix}
    +
    \begin{bmatrix}
    0 & 0_{n \times n} & 0_{n \times n} \\
    0 & 0_{n \times n} & 0_{n \times n} \\
    0_{n \times n} & -caI_n & -(c + \alpha)I_n \\
    0_{n \times n} & 0_{n \times n} & 0_{n \times n}
    \end{bmatrix}
    \begin{bmatrix}
    y(t - \tau) \\
    x(t - \tau) \\
    v(t - \tau)
    \end{bmatrix}
    +
    \begin{bmatrix}
    0_{n \times n} & 0_{n \times n} & 0_{n \times n} \\
    0_{n \times n} & 0_{n \times n} & 0_{n \times n} \\
    ca\mathcal{W} & (c + \alpha)\mathcal{W} & \mathcal{W}
    \end{bmatrix}
    \begin{bmatrix}
    y(t - T) \\
    x(t - T) \\
    v(t - T)
    \end{bmatrix},
\]

(33)

where \( y(t) = [y_1(t), \ldots, y_n(t)]^T \), \( x(t) = [x_1(t), \ldots, x_n(t)]^T \), \( v(t) = [v_1(t), \ldots, v_n(t)]^T \), \( \mathcal{W} \) is the adjacency matrix. Define \( \mathcal{L}_w = \mathcal{D} - \mathcal{W} = I_n - \mathcal{W} \), from Lemma 1, the following singular vector decomposition is valid:

\[
    Q^{-1} \mathcal{L}_w Q = \begin{bmatrix} \tilde{\mathcal{L}}_w & 0_{n \times 1} \\ 0_{n \times 1}^T & 0 \end{bmatrix}.
\]

Among the infinite options of \( Q \), we choose the one that the last column of \( Q \) is the vector \( 1_n \), which makes the following analysis convenient. Define \( \tilde{\mathcal{V}} = Q^{-1}\mathcal{V} \), \( \tilde{\mathcal{V}} = Q^{-1}y \), \( \tilde{x} = Q^{-1}x \), \( \tilde{v} = Q^{-1}v \), and denote \( \tilde{y}_{n-1}, \tilde{x}_{n-1}, \tilde{v}_{n-1} \) as the first \( n - 1 \) rows of \( \tilde{y}, \tilde{x}, \tilde{v} \) respectively, \( \tilde{y}_{\text{last}}, \tilde{x}_{\text{last}}, \tilde{v}_{\text{last}} \) as the last row of \( \tilde{y}, \tilde{x}, \tilde{v} \) respectively. By multiplying \( Q^{-1} \) on both sides of Eq. (33), we get two decoupled parts as

\[
    \begin{bmatrix}
    \tilde{y}_{n-1}(t) \\
    \tilde{x}_{n-1}(t) \\
    \tilde{v}_{n-1}(t)
    \end{bmatrix}
    =
    \begin{bmatrix}
    0_{(n-1) \times (n-1)} & I_{n-1} & 0_{(n-1) \times (n-1)} \\
    0_{(n-1) \times (n-1)} & 0_{(n-1) \times (n-1)} & I_{n-1} \\
    0_{(n-1) \times (n-1)} & -caI_{n-1} & -(c + \alpha)I_{n-1}
    \end{bmatrix}
    \begin{bmatrix}
    \tilde{y}_{n-1}(t) \\
    \tilde{x}_{n-1}(t) \\
    \tilde{v}_{n-1}(t)
    \end{bmatrix}
\]
De Laplace transformation, the third row of Eq. (35) can be written as

\[
\begin{bmatrix}
0_{(n-1)\times(n-1)} & 0_{(n-1)\times(n-1)} & 0_{(n-1)\times(n-1)} \\
-c\alpha I_{n-1} & -(c+\alpha)I_{n-1} & -I_{n-1} \\
0_{(n-1)\times(n-1)} & 0_{(n-1)\times(n-1)} & 0_{(n-1)\times(n-1)} \\
c\alpha \tilde{\mathcal{N}} & (c+\alpha)\tilde{\mathcal{N}} & \tilde{\mathcal{N}} \\
\end{bmatrix}
\begin{bmatrix}
\tilde{y}_{n-1}(t-T) \\
\tilde{x}_{n-1}(t-T) \\
\tilde{v}_{n-1}(t-T) \\
\end{bmatrix},
\]  

(34)

\[
\begin{bmatrix}
\tilde{y}_{\text{last}}(t) \\
\tilde{x}_{\text{last}}(t) \\
\tilde{v}_{\text{last}}(t) \\
\end{bmatrix}
= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -c\alpha & -(c+\alpha) \end{bmatrix}
\begin{bmatrix}
\tilde{y}_{\text{last}}(t) \\
\tilde{x}_{\text{last}}(t) \\
\tilde{v}_{\text{last}}(t) \\
\end{bmatrix}
+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -c\alpha & -(c+\alpha) & -1 \end{bmatrix}
\begin{bmatrix}
\tilde{y}_{\text{last}}(t-\tau) \\
\tilde{x}_{\text{last}}(t-\tau) \\
\tilde{v}_{\text{last}}(t-\tau) \\
\end{bmatrix}
+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c\alpha & (c+\alpha) & 1 \end{bmatrix}
\begin{bmatrix}
\tilde{y}_{\text{last}}(t-T) \\
\tilde{x}_{\text{last}}(t-T) \\
\tilde{v}_{\text{last}}(t-T) \\
\end{bmatrix},
\]  

(35)

For system (35), we apply Nyquist stability criterion to find its stability condition. After Laplace transformation, the third row of Eq. (35) can be written as

\[
p^2 \tilde{x}_{\text{last}}(p) = -c\alpha \tilde{x}_{\text{last}}(p) - (c+\alpha)\tilde{x}_{\text{last}}(p)p - c\alpha \frac{\tilde{x}_{\text{last}}(p)}{p} e^{-\tau p} - (c+\alpha)\tilde{x}_{\text{last}}(p)e^{-\tau p}
\]

\[
-\tilde{x}_{\text{last}}(p)e^{-\tau p}p + c\alpha \frac{\tilde{x}_{\text{last}}(p)}{p} e^{-\tau p} + (c+\alpha)\tilde{x}_{\text{last}}(p)e^{-\tau p} + \tilde{x}_{\text{last}}(p)e^{-\tau p}p,
\]

(36)

Thus, the stability is determined by the roots distribution of the following:

\[
p^2 + c\alpha + (c+\alpha)p = -\left[ \frac{c\alpha}{p} + (c+\alpha) + p \right] (e^{-\tau p} - e^{-\tau p}),
\]

(37)

which can be further simplified as

\[
p = -(e^{-\tau p} - e^{-\tau p}),
\]

(38)

Define \( g(p) = -(e^{-\tau p} - e^{-\tau p})\), and if the trajectory of \( g(j\omega) \), \( \forall \omega \in (-\infty, \infty) \) does not enclose the point \((-1, j0)\), then Eq. (38) is stable. As \( \text{Re}(g(j\omega)) = -(\sin(T\omega)/\omega) + (\sin(\tau\omega)/\omega) \), we get that \( \text{Re}(g(j\omega)) \geq -T - \tau \). As one sufficient condition for the stability of Eq. (38) is that \( \text{Re}(g(j\omega)) \geq -1 \), and \( T > \tau \), we get that \( T + \tau < 1 \) and \( \tau < 0.5 \). Noting that \( T + \tau < 1 \), \( \tau < 0.5 \) is only a sufficient condition for convergence, but not an essential one, which can be proved by the simulation results. We see that the upper ground of communication delay in Theorem 10 of [39] is different from ours. The reason is that [39] considers the integrator dynamics, undirected topology and no self-delay, however, we consider second-order nonlinear systems, directed topology containing a spanning tree as well as self-delay. Moreover, the protocols used are different from each other. All in all, communication topology (especially directed graphs), communication weight, time delays and parameter uncertainties will all affect the convergence of the whole system.
4. Simulation results

In this section, we present a numerical example to illustrate the effectiveness of our algorithms. Consider a group of six agents whose dynamics can be expressed by Eq. (1), in which the position state is denoted by $x_i(t) = [x_i^1(t), x_i^2(t)]^T$. The matrix $\phi_i$ is denoted by

$$\phi_i = \begin{bmatrix} \phi_i^{11} & \phi_i^{12} \\ \phi_i^{21} & \phi_i^{22} \end{bmatrix},$$

where

$$\phi_i^{11}(t) = v_i^1(t) \sin(t),$$
$$\phi_i^{12}(t) = v_i^1(t) \cos(t),$$
$$\phi_i^{21}(t) = v_i^2(t) \sin(t),$$
$$\phi_i^{22}(t) = v_i^2(t) \cos(t).$$

The unknown constant parameters are chosen here as $\theta_1 = [1 2]^T$, $\theta_2 = [2 1]^T$, $\theta_3 = [1 1]^T$, $\theta_4 = [2 2]^T$, $\theta_5 = [1 2]^T$ and $\theta_6 = [2 1]^T$, respectively. All these parameters are set to be 10–80% of accuracy of their real values. We set the communication topology to be a complicated one, as shown in Fig. 1, and the communication weight $\omega_{ij} = 1$ if $e_{ij} \in \varepsilon$, $\omega_{ij} = 0$ otherwise.

Simulations regarding the case $\alpha = 0$ and the case $\alpha > 0$ are both implemented. We set $c = 10$ in all cases. When $\alpha > 0$, we set $\alpha = 40$.

In the first case, we demonstrate the performance of the adaptive control (6) with the integral action, i.e., setting $\alpha = 40$, $\tau_i = 0.3$ and $T_{ij} = 1$. In such a case, the configurations of the agents are shown in Figs. 2 and 3, and we see that the configurations of the multi-agent systems, indeed, converge to the scaled weighted average value, which can be calculated by Eq. (27) as

$$\begin{align*}
&1 + \sum_{i=1}^{\eta} \sum_{j \in \mathcal{N}_i} w_{ij}(T_{ij} - \tau_i) x_i(0) \\
&= \frac{1}{1 + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6}\right) \times 0.7} \left(\frac{1}{2} x_6(0) + \frac{1}{3} x_5(0) + \frac{1}{6} x_4(0)\right) \\
&= [1.8699 0.8130].
\end{align*}$$

Next, we examine the performance of the adaptive controller (6) without the integral action, i.e., setting $\alpha = 0$, and the other controller parameters are selected to be the same as the first case. The initial parameter estimates are set to be 10–80% of accuracy of their real values. The positions of the agents are shown in Figs. 4 and 5. Under the adaptive controller without integral action, the configuration variables of the agents are examined in the first case, yet, converge to a point $[1.7832 0.7116]$ in simulation, which is obviously different from the scaled weighted average value, just as Corollary 1 shows. Moreover, though $\tau + T = 0.3 + 1 = 1.3 > 1$, the configuration variables of the agents are examined in the first case, yet, converge to a point $[1.7832 0.7116]$ in simulation, which is obviously different from the scaled weighted average value, just as Corollary 1 shows.

![Fig. 1. The communication topology among the multi-agents.](image-url)
Fig. 2. Configuration variables of the multi-agent systems (first configuration variables with integral action).

Fig. 3. Configuration variables of the multi-agent systems (second configuration variables with integral action).

Fig. 4. Configuration variables of the multi-agent systems (first configuration variables without integral action).
which does not satisfy the sufficient condition, the whole system convergents to the expectant point. That is because the communication topology and parameter uncertainties affect the convergence.

5. Conclusion

We have investigated the consensus problem of nonlinear multi-agent systems with parametric uncertainties, possible time delays, and directed interconnection graph containing a spanning tree. Under the assumption that some parameters in the system dynamics are unknown, we establish a unified consensus framework for multiple networked systems with self and communication delays. Furthermore, by presenting a novel input–output property of a strictly proper delay-involved transfer function and deriving the extremum of a delay-involved transfer function, we demonstrate that the adaptive controller with integral action gives rise to the scaled weighted average consensus of the multi-agent systems with nonlinear dynamics. Simulation results are presented to show the effectiveness of the proposed consensus schemes. In the future, we will study the case with switching topology together with time delays.

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