Distributed observer-based consensus protocol for descriptor multi-agent systems

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Abstract

In this paper, the consensus problem of descriptor multi-agent systems is considered. Each agent’s dynamics is given in general form of continuous-time linear descriptor system, and the communication topology among the agents is assumed to be directed under fixed topology case and balanced under switching topology case. To solve the multi-agent consensus problem, three kinds of distributed observer-based consensus protocols are proposed to solve the descriptor multi-agent consensus problem. Multi-step algorithms are provided to design the gain matrices and coupling constants involved in the protocols. The design approach is based on the generalized Riccati equation and Lyapunov stability theory. Some sufficient consensus conditions are established to guarantee the descriptor multi-agent system achieves consensus under directed fixed topology and balanced switching topology respectively. Furthermore, the state variable feedback consensus problem and the observer-based consensus problem with desired decay rate are discussed. Finally, a simulation example is provided to illustrate the obtained results.

Keywords: Multi-agent system, descriptor system, distributed control, consensus, state-observer

1 Introduction

During the past decade, many researchers have focused on the coordination control problem, owing to its useful application in many areas such as formation control, flocking and swarming, cooperative unmanned air vehicles [1]. Among them, consensus control as one of the most important coordination control problem has been widely studied by many researchers from different academic backgrounds. Many topics, such as agreement, flocking, formation, swarm, consensus filtering, time-synchronization, are related with consensus problem [2, 3, 4, 5, 6, 7].

The dynamical model of agents and their interaction topology are two key factors to solve the consensus problems. Till now, numerous interesting results have been obtained for the multi-agent consensus problems with different agent dynamics, including first-order systems [8, 9], second-order systems [10, 11], general linear system [12], descriptor system [13], fractional-order systems [14], and non-linear system [15, 16].
However, most of consensus protocols are based on the state disagreements. Unfortunately, due to economic cost/or constrains on measurement, it is often difficult or even unavailable to get the full state information. It is very interesting to construct the consensus protocols via the output information. To achieve control aim in this case, a common method is to adopt an observer for each agent to estimate those unmeasurable state variables. Till now, observer-based consensus design has become an important topic of multi-agent networks. In [17], the authors proposed a distributed consensus protocol for each first-order following agent based on an observer to estimate the leaders unmeasurable velocity. To tack the active leader with unmeasurable velocity, observer-based consensus protocol was proposed for the second-order following agents in [18]. To track the accelerated motion leader, [19] provided a distributed observer-based consensus protocol for the second-order following agents. A unified framework of the observer-based consensus protocol for multi-agent with general linear dynamics was introduced in [20]. Leader-following observer-based consensus framework was proposed by [21]. Reduced-order observer-based consensus protocols were proposed by [22]. In [23], another design approach for reduced-order observed-based protocol was proposed.

The descriptor system is also referred to as singular state-space system, generalized system, or implicit system [24]. Recently, the descriptor system has been received considerable attention, because its simultaneous description of dynamic and algebraic relationships between state variables makes such system especially suitable for modeling many engineering systems, such as power system [25], robotic system [26], mechanical system [27], and so on. Some fundamental concepts, such as controllability and observability, pole assignment, stability, have been successfully extended to descriptor systems, and a greater number of useful results have been obtained [28, 29, 30, 31]. Among these properties of descriptor systems, the observability problem is an important topic. Some sufficient observability conditions were established in [32, 33]. In [32], the authors presented a LMI iterative designing method for $H_2$ observer of linear continuous-time descriptor systems. The PD observer design problem for descriptor systems was discussed in [33], and a LMI necessary and sufficient condition was obtained. The stability problem for switched linear descriptor systems was probed by [34, 35]. Recently, the coordination control problems for multi-agent with descriptor dynamics have been drawing increasing attention. The necessary and sufficient conditions of consensusability with respect to a set of admissible consensus protocols were investigated in [13]. The admissible consensus analysis and consensualizing controller design problems for high-order linear time-invariant descriptor swarm systems were investigated in [36]. The state and output containment analysis and design problems for high-order linear time-invariant descriptor systems under directed topology were investigated in [37] and [38] respectively. The swarm problems for high-order linear time-invariant descriptor multi-agent systems under directed topology were investigated in [39, 40]. Most existed references related with descriptor consensus problem are based on state information. It is very necessary to investigate the descriptor consensus problem via the output information.

Motivated by the above works, we investigate the consensus problem with descriptor linear systems dynamics by assumption that only the output information can be available. The main contribution of this paper is that three kinds of distributed observer-consensus protocols are proposed to solve the descriptor consensus problem. In virtue of the generalized Riccati equation and generalized Lyapunov equation,
some multi-step algorithms are provided to design the parameters and gain matrices involved in the protocols, and then some consensus conditions are established. For balanced switching topologies, we also prove that the proposed three kinds of protocols can solve the descriptor multi-agent consensus problem. In addition, the problems of state feedback consensus and consensus with desire decay rate are discussed. For the special case that $E = I$, our considered problem degenerates to the consensus problem with general linear dynamics. The related leader-following consensus problem with general linear dynamics has been investigated by [21], which only considered directed fixed interaction topology case. This paper not only generalizes the result of [21] to descriptor system case but also more cases such as balanced switching topology and desire decay rate are discussed.

The rest of the paper is organized as follows. In section 2, the consensus problem is formulated with the help of graph theory, and some useful notations and preliminaries are introduced. Then in section 3, three kinds of distributed observer-based consensus protocol are provided. The protocol design approach is discussed in the section 4. Some special and general cases are discussed in 5. Following that, a simulation example is provided in section 6, and finally, section 7 contains some concluding remarks.

The notations of this paper are standard. Let $R^{m \times n}$ and $C^{m \times n}$ be the set of $m \times n$ real matrices and complex matrices, respectively. The real part of $s \in C$ is denoted by $Re(s)$. $A^T$ and $A^*$ represent transpose and conjugate transpose of matrix $A \in C^{m \times n}$ respectively. $1_n = [1, \ldots, 1]^T \in R^n$. For symmetric matrices $A$ and $B$, $A \geq B$ means $A - B$ is positive (semi-) definite. $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ represent the minimum and maximum eigenvalue of $A$ respectively. $\text{Rank}(A)$ represents the rank of matrix $A$. $\otimes$ denotes the Kronecker product, which satisfies: (1) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$; (2) If $A \geq 0$ and $B \geq 0$, then $A \otimes B \geq 0$.

2 Preliminaries and problem formulation

2.1 Descriptor system

In this subsection, we introduce some basic concepts and results for the continuous-time descriptor system

$$Ex(t) = Ax(t).$$

Definition 1 [24]. Let $E, A \in R^{m \times m}$.

(i). The pair $(E, A)$ is said to be regular if $\det(sE - A)$ is not identically zero;

(ii). The pair $(E, A)$ is said to be impulse free if $(E, A)$ is regular and $\deg(\det(sE - A)) = \text{rank}(E)$;

(iii). The pair $(E, A)$ is said to be stable if all finite generalized eigenvalues of the pair $(E, A)$ lie in the left half plane, i.e., $\sigma(E, A) \subset C^-$;

(iv). The pair $(E, A)$ is said to be admissible if $(E, A)$ is impulse free and stable.

The concepts of $R$-controllability and $R$-observability of descriptor system (1) can be referenced to [41]. For the regular descriptor system (1), $(E, A, B)$ is $R$-controllable if and only if $\text{Rank}[sE - A, B] = m$ for any finite $s \in C$, and $(E, A, C)$ is $R$-observable if and only if $\text{Rank}\begin{bmatrix} sE - A \\ C \end{bmatrix} = m$ for any finite $s \in C$. 

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From a practical standpoint, a descriptor system should be stable and impulse-free. The admissibility of descriptor system has been probed by many researchers. Before establishing our main results, some preliminary results for descriptor system are introduced, which will be used later.

**Lemma 1.** If all pairs \((E_i, A_i), i = 1, 2, \cdots, M\), are admissible, then the pair \((E_{add}, A_{add})\) is admissible, where

\[
E_{add} = \begin{pmatrix} E_1 & & \\ E_2 & & \\ & \ddots & \\ E_M & & \\ \end{pmatrix}, \quad A_{add} = \begin{pmatrix} A_1 & A_{12} & \cdots & A_{1M} \\ A_2 & \cdots & \ddots & \\ \vdots & \ddots & \ddots & \\ A_M & & & \\ \end{pmatrix},
\]  

(2)

**Proof.** Obviously, we have

\[
\det(sE_{add} - A_{add}) = \det(sE_1 - A_1)\det(sE_2 - A_2)\cdots\det(sE_M - A_M).
\]

(3)

Then, according to definition 1, it is self-evident.

**Lemma 2** [29, 41]. For \(E, A \in \mathbb{R}^{m \times m}\), assume that \((E, A)\) is regular. We have:

(i). If there exist \(X = X^T \geq 0\) and \(Y = Y^T > 0\) satisfying that

\[
E^T X A + A^T X E = -E^T Y E,
\]

then \((E, A)\) is admissible.

(ii). If \((E, A)\) is admissible, then there exist \(X = X^T > 0\) and \(Y = Y^T > 0\) satisfying Lyapunov equation (4).

**Lemma 3** [41]. For \(E, A \in \mathbb{R}^{m \times m}, B \in \mathbb{R}^{m \times r}\) and \(C \in \mathbb{R}^{q \times m}\), assume that \((E, A)\) is regular and impulse free, and \((E, A, B)\) is R-controllable. Then, for any given positive definite matrices \(W_b\) and \(R_b\), there exists a positive definite \(V_b\) satisfying the following generalized Riccati equation

\[
E^T V_b A + A^T V_b E - E^T V_b B R_b^{-1} B^T V_b E + E^T W_b E = 0.
\]

(5)

**2.2 Problem formulation**

Consider a descriptor multi-agent system consisting of \(n\) identical agents indexed by \(i = 1, 2, \cdots, n\). The dynamics of the agent \(i\) is modeled by the following descriptor system

\[
\begin{align*}
\dot{x}_i &= A x_i + B u_i, \\
y_i &= C x_i,
\end{align*}
\]

(6)

where \(x_i \in \mathbb{R}^m\) is the agent \(i\’s\) state, \(u_i \in \mathbb{R}^p\) is the agent \(i\’s\) control input and \(y_i \in \mathbb{R}^q\) is the agent \(i\’s\) measured output. \(E, A, B, C\) are constant matrices with appropriate dimensions.

**Assumption 1.** For descriptor system (6), we always assume that \((E, A)\) is regular and impulse free, \((E, A, B)\) is R-controllable and \((E, A, C)\) is R-observable.
Considering the limited capability of the agent, we are interesting in the distributed consensus protocol for each agent, which only depends on the information of the agent itself and its neighbors. The neighbor relations can be modeled by a weighted digraph $\mathcal{G}$, which is denoted by $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, A\}$, where $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges and the weighted adjacency matrix $A = [a_{ij}]$ has nonnegative adjacency elements $a_{ij}$. The neighbor set of node $v_i$ is defined by $\mathcal{N}_i = \{ j | (v_i, v_j) \in \mathcal{E} \}$. The degree matrix $D = \text{diag}\{d_1, d_2, \ldots, d_n\} \in \mathbb{R}^{n \times n}$ of digraph $\mathcal{G}$ is a diagonal matrix with diagonal elements $d_i = \sum_{j \in \mathcal{N}_i} a_{ij}$. Then the Laplacian matrix of $\mathcal{G}$ is defined as $L = D - A \in \mathbb{R}^{n \times n}$, which satisfies $L \mathbf{1}_n = 0$. The Laplacian matrix $L$ associated with weighted digraph $\mathcal{G}$ has at least one zero eigenvalue and all of the non-zero eigenvalues are located on the open right half plane. Furthermore, $L$ has exactly one zero eigenvalue if and only if the directed graph $\mathcal{G}$ has a directed spanning tree. Let $r = (r_1, r_2, \cdots, r_n)^T \in \mathbb{R}^n$ be the right eigenvector associated with the only zero eigenvalue, which satisfies $r^T \mathbf{1} = 1$. It is well-known that $r$ is a nonnegative vector. Conveniently, let $\lambda_i(L)$, $i = 1, 2, \cdots, n$ be $i$-th eigenvalue of $L$ with $\lambda_1(L) = 0$.

Assume that $\{v_1\} \cup \mathcal{N}_i = \{j_1, j_2, \cdots, j_l\}$. Then, a state feedback

$$u_i = k_i(x_{j_1}, \cdots, x_{j_l}) \quad (7)$$

is said to be a protocol with topology $\mathcal{G}$.

The descriptor multi-agent system is said to be achieved consensus, if the states of all agents satisfy

$$\lim_{t \to \infty} (x_i(t) - x_j(t)) = 0, i, j = 1, 2, \cdots, n$$

for any initial state $x_i(0) (i = 1, 2, \cdots, n)$. We say that the protocol (7) can solve the consensus problem, if the closed-loop feedback system achieves consensus.

### 3 Distributed observer-based consensus protocol

In many applications, the agents can not obtain its neighbor’s state variables due to measurement constraints or economic cost, but can access the neighbor’s output variables. Normally, a feasible scheme is to adopt an observer for the agent to estimate those state variables by the measured outputs. Three different architectures were first proposed by [20] to solve the leader-following multi-agent consensus problem with general continuous-time linear dynamics systems. Now, the three system architectures are generalized to solve descriptor multi-agent consensus problem in leaderless case.

Assume that the state of the observer adopted by the following agent $i$ is $\hat{v}_i(t)$. Denote $\hat{y}_i(t) = Cv_i(t)$ as the estimated output. For convenience, some auxiliary variables for agent $i$ are denoted as

$$\xi_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij}(v_j(t) - \hat{v}_j(t)), \quad (8)$$

$$\xi_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij}(y_i(t) - \hat{y}_j(t)), \quad (9)$$

$$\tilde{\xi}_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij}(\hat{y}_i(t) - \hat{y}_j(t)). \quad (10)$$
3.1 Neighbor-based observers and local controllers

To solve the descriptor multi-agent consensus problem, a local controller together with a neighbor-based observer is proposed for agent $i$ with form

$$ E\ddot{v}_i = (A + BK)v_i + c_1 G(\dot{\xi}_i - \xi_i) $$

$$ u_i = Kv_i $$ (11)

where $v_i \in \mathbb{R}^n$ is the protocol state, $c_1$ is the coupling constant, $G \in \mathbb{R}^{m \times q}$ is a given observer gain matrix, and $K$ is a given feedback gain matrix.

Then, after manipulations with combining (6), (9), and (11), the dynamics of the closed-loop system is given as

$$ \left[ \begin{array}{cc} I \otimes E & 0 \\ 0 & I \otimes E \end{array} \right] \frac{d}{dt} \left[ \begin{array}{c} x \\ v \end{array} \right] = \left[ \begin{array}{cc} I \otimes A & I \otimes (BK) \\ -L \otimes (c_1 GC) & I \otimes (A + BK) + L \otimes (c_1 GC) \end{array} \right] \left[ \begin{array}{c} x \\ v \end{array} \right] $$ (12)

**Theorem 1.** For the multi-agent system (6) whose interconnection topology graph $\mathcal{G}$ contains a directed spanning tree, if pair $(E, A + BK)$ and all pairs $(E, A + \lambda_i(L)c_1 GC)$, $i = 2, 3, \cdots, n$ are admissible, then the descriptor multi-agent system can achieve consensus via protocol (11). Moreover, the consensus value $\bar{x}(t)$ is the solution of $E\ddot{x}(t) = A\dot{x}(t)$ with initial value $\bar{x}(0) = \sum_{j=1}^{n} r_j [x_j(0) - v_j(0)]$

**Proof.** Let $r^T = (r_1, r_2, \cdots, r_n)$ is the left zero eigenvector of $L$ with $r^T 1 = 1$. Let $S = [1, S_1]$ and $S^{-1} = \left[ \begin{array}{c} r^T \\ Q_1 \end{array} \right]$ such that

$$ S^{-1}LS = J = \left[ \begin{array}{cc} 0 & 0 \\ 0 & \Delta \end{array} \right] $$ (13)

where $\Delta$ is an upper triangular matrix whose diagonal entries are $\lambda_i(i = 2, 3, \cdots, n)$.

Denote $\dot{x} = (S^{-1} \otimes I_m)x$ and $\dot{v} = (S^{-1} \otimes I_m)v$, where $S$ is defined in (13). The dynamics system (12) is transformed into the following equivalent system

$$ \left[ \begin{array}{cc} I_n \otimes E & 0 \\ 0 & I_n \otimes E \end{array} \right] \frac{d}{dt} \left[ \begin{array}{c} \dot{x} \\ \dot{v} \end{array} \right] = \left[ \begin{array}{cc} I_n \otimes A & I_n \otimes (BK) \\ -J \otimes (c_1 GC) & I_n \otimes (A + BK) + J \otimes (c_1 GC) \end{array} \right] \left[ \begin{array}{c} \dot{x} \\ \dot{v} \end{array} \right] $$ (14)

which can be divided into the following two subsystems

$$ \left[ \begin{array}{cc} E & 0 \\ 0 & E \end{array} \right] \frac{d}{dt} \left[ \begin{array}{c} \dot{x}^0 \\ \dot{\varphi}^0 \end{array} \right] = \left[ \begin{array}{cc} A & BK \\ 0 & A + BK \end{array} \right] \left[ \begin{array}{c} \dot{x}^0 \\ \dot{\varphi}^0 \end{array} \right] $$ (15)

and

$$ \left[ \begin{array}{cc} I_{n-1} \otimes E & 0 \\ 0 & I_{n-1} \otimes E \end{array} \right] \frac{d}{dt} \left[ \begin{array}{c} \dot{x}^1 \\ \dot{\varphi}^1 \end{array} \right] = \left[ \begin{array}{cc} I_{n-1} \otimes A & I_{n-1} \otimes (BK) \\ -\Delta \otimes (c_1 GC) & I_{n-1} \otimes (A + BK) + \Delta \otimes (c_1 GC) \end{array} \right] \left[ \begin{array}{c} \dot{x}^1 \\ \dot{\varphi}^1 \end{array} \right] $$ (16)
with $\dot{x} = [x^0 , \dot{x}^1]^{T}$, $\dot{x}^0 \in C^m$, $\dot{v} = [v^0 , v^1]^{T}$ and $v^0 \in C^m$. If $\lim_{t \to \infty} \begin{bmatrix} \bar{x}^1 \\ \bar{v}^1 \end{bmatrix} = 0$, we have

$$
\bar{x}(t) = (S \otimes I_m) \bar{x}(t) = ([1, S] \otimes I_m) \begin{bmatrix} \bar{v}^0(t) \\ \bar{x}^1(t) \end{bmatrix}
$$

$$
\rightarrow ([1, S] \otimes I_m) \begin{bmatrix} \bar{v}^0 \\ 0 \end{bmatrix} = 1 \otimes \bar{x}^0(t), \text{ as } t \to \infty,
$$

that is, the descriptor multi-agent system achieves consensus. Thus, we know that the descriptor multi-agent system achieves consensus if descriptor system (16) is admissible.

Next, we prove that descriptor system (16) is admissible. Denote $\dot{\bar{v}}^1 = \bar{v}^1 - \dot{x}^1$. With variable substitution

$$
\begin{bmatrix} \dot{x}^1 \\ \dot{v}^1 \end{bmatrix} = \begin{bmatrix} I_{n-1} & 0 \\ -I_{n-1} & I_{n-1} \end{bmatrix} \otimes I_m \begin{bmatrix} \dot{x}^1 \\ \dot{v}^1 \end{bmatrix},
$$

system (16) is equivalent to

$$
\begin{bmatrix} I_{n-1} \otimes E & 0 \\ 0 & I_{n-1} \otimes E \end{bmatrix} \begin{bmatrix} \dot{x}^1 \\ \dot{v}^1 \end{bmatrix} = \begin{bmatrix} I_{n-1} \otimes (A + BK) & I_{n-1} \otimes (BK) \\ 0 & I_{n-1} \otimes (A + \Delta \otimes (c_1GC)) \end{bmatrix} \begin{bmatrix} \dot{x}^1 \\ \dot{v}^1 \end{bmatrix}. \tag{18}
$$

Noticing that descriptor system (18) has form as (2) and by Lemma 1, it is easy to see that descriptor system (18) is admissible if pair $(E, A + BK)$ and all pairs $(E, A + \bar{c}_iGC)$, $i = 2, 3, \cdot \cdot \cdot , n$ are admissible.

From the second equation of descriptor system (15), we have $\lim_{t \to \infty} \bar{v}^0 = 0$. In view of (17), $\lim_{t \to \infty} [x_i(t) - (\bar{x}^0 - \bar{v}^0)] = \lim_{t \to \infty} (x_i(t) - \bar{x}) = 0$. From (15), we can get $E \frac{d}{dt} (\bar{x}^0 - \bar{v}^0) = A(\bar{x}^0 - \bar{v}^0)$ with initial value $\bar{x}^0 - \bar{v}^0 = \sum_{j=1}^{n} r_j [x_j(0) - v_j(0)]$. Thus, the consensus value $\bar{x}(t)$ is the solution of $E \dot{\bar{x}}(t) = A \bar{x}(t)$ with initial value $\bar{x}(0) = \sum_{j=1}^{n} r_j [x_j(0) - v_j(0)]$.

### 3.2 Local observers and neighbor-based controllers

In this subsection, a neighbor-based controller together with a local observer is proposed for agent $i$ as follows

$$
E \dot{v}_i(t) = Av_i(t) + Bu_i(t) + G(\dot{\bar{s}}_i(t) - y_i(t)),
$$

$$
u_i(k) = c_2 K \bar{e}_i(t), \tag{19}
$$

where $v_i \in R^m$ is the protocol state, $c_2$ is the coupling strength, $G \in R^{m \times q}$ and $K$ are given gain matrices.

Then, after manipulations with combining (6), (8) and (19), the dynamics of the closed-loop system is given as

$$
\begin{bmatrix} I \otimes E & 0 \\ 0 & I \otimes E \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} I \otimes A & L \otimes (c_2 BK) \\ -I \otimes (GC) & I \otimes (A + GC) + L \otimes (c_2 BK) \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}. \tag{20}
$$
Using same variable substitutions as above subsection, the dynamics system (20) can be divided into the following two subsystems

\[
\begin{bmatrix}
E & 0 \\
0 & E
\end{bmatrix} \frac{d}{dt} \begin{bmatrix} \dot{x}^0 \\ \dot{\vartheta}^0 \end{bmatrix} = \begin{bmatrix} A & 0 \\
-GC & A+GC \end{bmatrix} \begin{bmatrix} x^0 \\ \vartheta^0 \end{bmatrix} \tag{21}
\]

and

\[
\begin{bmatrix}
I_{n-1} \otimes E & 0 \\
0 & I_{n-1} \otimes E
\end{bmatrix} \frac{d}{dt} \begin{bmatrix} \dot{x}^1 \\ \dot{\vartheta}^1 \end{bmatrix} = \begin{bmatrix} I_{n-1} \otimes A + \Delta \otimes (c_2 BK) \\
-I_{n-1} \otimes (GC) & I_{n-1} \otimes (A+GC) + \Delta \otimes (c_2 BK) \end{bmatrix} \begin{bmatrix} \dot{x}^1 \\ \dot{\vartheta}^1 \end{bmatrix} \tag{22}
\]

Similarly, we know that the descriptor multi-agent system achieves consensus if descriptor system (22) is admissible. With variable substitution \( \hat{\vartheta}^1 = \hat{\vartheta}^1 - \hat{\vartheta}^1 \), system (16) is equivalent to

\[
\begin{bmatrix}
I_{n-1} \otimes E & 0 \\
0 & I_{n-1} \otimes E
\end{bmatrix} \frac{d}{dt} \begin{bmatrix} \dot{x}^1 \\ \dot{\vartheta}^1 \end{bmatrix} = \begin{bmatrix} I_{n-1} \otimes A + \Delta \otimes (c_2 BK) \\
I_{n-1} \otimes (A+GC) \end{bmatrix} \begin{bmatrix} \dot{x}^1 \\ \dot{\vartheta}^1 \end{bmatrix} \tag{23}
\]

Thus, descriptor system (23) is admissible if all pairs \( (E, A + \lambda_i c_2 BK), i = 2, 3, \ldots, n \) and pair \( (E, A + GC) \) are admissible. Obviously, the consensus value is \( \hat{\vartheta}^0 \) which satisfies (21). Then, we obtain the following result.

**Theorem 2.** For the multi-agent system (6) whose interconnection topology graph \( \mathcal{G} \) contains a directed spanning tree, if all pairs \( (E, A + \lambda_i c_2 BK), i = 2, 3, \ldots, n \) and pair \( (E, A + GC) \) are admissible, then the multi-agent system can achieve consensus via protocol (19). Moreover, the consensus value \( \bar{x}(t) \) is the solution of \( E \ddot{x}(t) = A \dddot{x}(t) \) with initial value \( \bar{x}(0) = \sum_{j=1}^{n} r_j x_j(0) \).

### 3.3 Neighbor-based observers and controllers

In this subsection, a neighbor-based controller together with a neighbor-based observer is adopted for agent \( i \) with form

\[
\begin{align*}
E \dot{v}_i(t) &= A v_i(t) + B u_i(t) + c_1 G (\hat{\xi}_i - \xi_i), \\
u_i(k) &= c_2 K \hat{v}_i(t),
\end{align*}
\tag{24}
\]

where \( v_i \in \mathbb{R}^m \) is the protocol state, \( c_1 \) and \( c_2 \) are the coupling strength constants, \( G \in \mathbb{R}^{m \times q} \) and \( K \) are given gain matrices.

Then, after manipulations with combining (6), (8), (9) and (24), the dynamics of the closed-loop system is given as

\[
\begin{bmatrix}
I \otimes E & 0 \\
0 & I \otimes E
\end{bmatrix} \frac{d}{dt} \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} I \otimes A + L \otimes (c_2 BK) \\
-L \otimes (c_1 GC) & I \otimes A + L \otimes (c_1 GC + c_2 BK) \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}. \tag{25}
\]

Similarly, the dynamics system (25) can be divided into

\[
\begin{bmatrix}
E & 0 \\
0 & E
\end{bmatrix} \frac{d}{dt} \begin{bmatrix} \dot{x}^0 \\ \dot{\vartheta}^0 \end{bmatrix} = \begin{bmatrix} A & 0 \\
0 & A \end{bmatrix} \begin{bmatrix} x^0 \\ \vartheta^0 \end{bmatrix} \tag{26}
\]
and
\[
\begin{bmatrix}
I \otimes E & 0 \\
0 & I \otimes E
\end{bmatrix}
\frac{d}{dt}
\begin{bmatrix}
\hat{x}^1 \\
\hat{e}^1
\end{bmatrix}
=\begin{bmatrix}
I \otimes A & \Delta \otimes (c_2 B K) \\
-\Delta \otimes (c_1 G C) & I \otimes A + \Delta \otimes (c_1 G C + c_2 B K)
\end{bmatrix}
\begin{bmatrix}
\hat{x}^1 \\
\hat{e}^1
\end{bmatrix}.
\tag{27}
\]

Certainly, the descriptor multi-agent system achieves consensus if descriptor system (27) is admissible. With variable substitution \( \hat{e}^1 = \hat{v}^1 - \hat{x}^1 \), system (27) is equivalent to
\[
\begin{bmatrix}
I \otimes E & 0 \\
0 & I \otimes E
\end{bmatrix}
\frac{d}{dt}
\begin{bmatrix}
\hat{x}^1 \\
\hat{e}^1
\end{bmatrix}
=\begin{bmatrix}
I \otimes A + \Delta \otimes (c_2 B K) & \Delta \otimes (c_2 B K) \\
0 & I \otimes A + \Delta \otimes (c_1 G C)
\end{bmatrix}
\begin{bmatrix}
\hat{x}^1 \\
\hat{e}^1
\end{bmatrix}.
\tag{28}
\]

Thus, the descriptor system (28) is admissible if all pairs \((E, A + \lambda_i c_2 B K)\), \(i = 2, 3, \ldots, n\) and pair \((E, A + G C)\) are admissible. Similarly, we know that the consensus value \( \bar{x}(t) \) is the solution of \( E \hat{x}(t) = A \bar{x}(t) \) with initial value \( \bar{x}(0) = \sum_{j=1}^{n} r_j x_j(0) \). Then, we can obtain the following result.

**Theorem 3.** For the multi-agent system (6) whose interconnection topology graph \( \mathcal{G} \) contains a directed spanning tree, if all pairs \((E, A + \lambda_i c_2 B K)\) and \((E, A + \lambda_i c_1 G C)\), \(i = 2, 3, \ldots, n\) are admissible, then the multi-agent system can achieve consensus via protocol (24). Moreover, the consensus value \( \bar{x}(t) \) is the solution of \( E \hat{x}(t) = A \bar{x}(t) \) with initial value \( \bar{x}(0) = \sum_{j=1}^{n} r_j x_j(0) \).

### 4 Protocol gain design approach

In this section, our main objective is to determine \( c_1, c_2, G, K \) used in protocols (11), (19) and (24), which can solve the multi-agent consensus problem. The following algorithm is provided to construct \( c_1, c_2, G \) and \( K \).

**Algorithm 1.** For descriptor system (6), suppose that Assumption 1 is satisfied. Then, the coupling strength \( \kappa \), the gain matrix \( G \) and feedback matrix \( K \) are constructed as follows:

1. For any given positive definite matrices \( W_b \) and \( R_b \), solve the generalized Riccati equation (5) to obtain the positive definite solution \( V_b \). Then, the gain matrix \( K \) is chosen by
   \[
   K = \frac{1}{2} R_b^{-1} B^T V_b E.
   \tag{29}
   \]

2. For any given positive definite matrices \( W_c \) and \( R_c \), solve the generalized Riccati equation
   \[
   EV_c A^T + AV_c E^T - EV_c C^T R_c^{-1} CV_c E^T + EW_c E^T = 0
   \tag{30}
   \]
   to obtain the positive definite solution \( V_c \). Then, the gain matrix \( G \) is chosen by
   \[
   G = \frac{1}{2} EV_c C^T R_c^{-1}.
   \tag{31}
   \]

3. Select the coupling strengths \( c_1 \) and \( c_2 \) satisfying
   \[
   c_1 \geq \frac{1}{2 \min_{\lambda_i(L) \neq 0} \text{Re}(\lambda_i(L))}, \quad c_2 \geq \frac{1}{2 \min_{\lambda_i(L) \neq 0} \text{Re}(\lambda_i(L))}.
   \tag{32}
   \]
Remark 1. For the regular descriptor system (6), the $R$-observability of $(E, A, C)$ means that $(E^T, A^T, C^T)$ is $R$-controllable. According to Lemma 3, we know that generalized Riccati equation (30) is solvable, that is, there exists a positive definite $V_c$ satisfying (30) for any given positive definite matrices $W_c$ and $R_c$.

Lemma 4 For descriptor system (6), suppose that Assumption 1 is satisfied. Then, for any $s \in \mathbb{C}$ with $\text{Re}(s) \geq 1$, $(E, A + sBK)$ and $(E, A + sGC)$ are admissible, where $K$ and $G$ are chosen as (29) and (31) respectively.

Proof. Since $V_b$ is positive definite solution of Riccati equation (5), we have

$$
E^T V_b (A + sBK) + (A + sBK)^* V_b E = (1 - \text{Re}(s)) E^T V_b B R_b^{-1} B^T V_b E - E^T W_b E, \quad (33)
$$

where $\tilde{W}_b = W_b + (\text{Re}(s) - 1) V_b B R_b^{-1} B^T V_b > 0$. According to Lemma 2, pair $(E, A + sBK)$ is admissible for any $\text{Re}(s) \geq 1$.

Similarly, since $V_c$ is positive definite solution of Riccati equation (30), we have

$$
EV_c (A + sGC)^* + (A + sGC) V_c E^T = (1 - \text{Re}(s)) EV_c C R_c^{-1} C^T E^T - E W_c E^T = -E \tilde{W}_c E^T, \quad (34)
$$

where $\tilde{W}_c = W + (\text{Re}(s) - 1) V_c C^T R_c^{-1} C V_c > 0$. According to Lemma 2, all pairs $(E^*, (A + sGC)^*)$ with $\text{Re}(s) \geq 1$ are admissible, which implies that $(E, A + sGC)$ is admissible for $\text{Re}(s) \geq 1$.

Now, Lemma 4 is used to analyze effectiveness of the proposed design approaches. Here, $c_1, c_2, G$ and $K$ are constructed by algorithm 1. Obviously, $(E, A + BK)$ and $(E, A + GC)$ are admissible by Lemma 4. While $G$ contains a directed spanning tree, we know that $\text{Re}(\lambda_i(L)) > 0$, for $i = 2, 3, \cdots, n$, which means $\text{Re}(c_1 \lambda_i(L)) > 1$ and $\text{Re}(c_2 \lambda_i(L)) > 1$. Thus, we know that all pairs $(E, A + \lambda_i c_2 BK)$ and $(E, A + \lambda_i c_1 GC)$, $i = 2, 3, \cdots, n$ are admissible. According to Theorems 1, 2 and 3, we can obtain the following result directly.

Theorem 4. For multi-agent system (6) whose interconnection topology graph $G$ contains a directed spanning tree, all three kinds of protocols (11), (19) and (24) designed by by algorithm 1 can solve the descriptor multi-agent consensus problem.

5 More discussion

5.1 Distributed state variable feedback consensus protocol

In this subsection, we consider a special case that the relative states of neighbouring agents are available. The state feedback consensus protocol for agent $i$ is taken as

$$
u_i = c_2 K \xi_i \quad (35)$$
with the neighborhood state disagreement error of agent $i$ denoted by

$$
\xi_i = \sum_{j \in \mathcal{N}_i} a_{ij} (x_i - x_j),
$$

where $c_2$ is the positive coupling strength and $K \in \mathbb{R}^{p \times m}$ is feedback gain matrix. Let $x = (x_1^T, x_2^T, \cdots, x_n^T)^T \in \mathbb{R}^{nm}$. Then, the overall closed-loop system dynamics is

$$(I_n \otimes E)\dot{x} = [I_n \otimes A + L \otimes (c_2 B K)]x$$

(37)

It is easy to obtain the following result via feedback consensus protocol, and the proof is omitted.

**Theorem 5.** For multi-agent system (6) whose interconnection topology graph $\mathcal{G}$ contains a directed spanning tree, the coupling strength $c_2$ and gain matrix $K$ are constructed by Algorithm 1. Then, the proposed protocol (35) can solve the descriptor multi-agent consensus problem. Furthermore, $x_i(t) \rightarrow \bar{x}(t)$, as $t \rightarrow \infty$, where $\bar{x}(t)$ is the solution of $\dot{x} = Ax$ with initial value $\bar{x}(0) = \sum_{j=1}^{n} r_j x_j(0)$.

### 5.2 Decay rate

Normally, the consensus speed is determined by the finite generalized eigenvalues of the error dynamical systems. To get large decay rate, all finite generalized eigenvalues of the error dynamical system are assigned into the left half plane $\text{Re} \lambda < -\alpha$ ($\alpha > 0$). Based on the analysis of Section 3, while all finite generalized eigenvalues of $(E, A + BK)$ $(E, A + GC)$ $(E, A + \lambda c_2 BK), (E, A + \lambda_0c_1 GC)$ are in the left half plane $\text{Re} \lambda < -\alpha$ ($\alpha > 0$), the consensus decay rate is at least larger than $\alpha$.

From Definition 1, it is easy to see that a regular descriptor system $E \dot{x} = Ax$ is impulse free and with all its finite generalized eigenvalues satisfying $\text{Re} \lambda < -\alpha$ if and only if the pair $(E, \alpha E + A)$ is admissible. By Lemma 2, $(E, \alpha E + A)$ is admissible if there exist $X = X^T \geq 0$ and $Y = Y^T > 0$ satisfying that

$$
E^T X A + A^T X E + 2 \alpha E^T X E = -E^T Y E.
$$

(38)

Under Assumption 1, it can be verified directly that $(E, \alpha E + A)$ is also regular and impulse free, $(E, \alpha E + A, B)$ is $R$-controllable, and $(E, \alpha E + A, C)$ is $R$-observable. According to Lemma 3, for any given positive definite matrices $W_b$ and $R_b$, there exists a positive definite $\bar{V}_b$ satisfying the following generalized Riccati equation

$$
E^T \bar{V}_b A + A^T \bar{V}_b E - E^T \bar{V}_b B R_b^{-1} B^T \bar{V}_b E + 2 \alpha E^T \bar{V}_b E + E^T W_b E = 0.
$$

(39)

Similarly, the following generalized Riccati equation has a positive-definition solution $\bar{V}_c$

$$
E \bar{V}_c A^T + A \bar{V}_c E - E \bar{V}_c C R_c^{-1} C^T \bar{V}_c E + 2 \alpha E \bar{V}_c E^T + E W_c E^T = 0
$$

(40)

with $W_c > 0$ and $R_c > 0$.

**Algorithm 2.** For descriptor system (6), suppose that Assumption 1 is satisfied. Then, the coupling strengths $c_1, c_2$ and the gain matrices $G, K$ are constructed as follows:
(1) Solve generalized Riccati equation (39) to a positive-definition solution $\tilde{V}_b$. Then, the gain matrix $K$ is chosen by

$$K = -\frac{1}{2} R_b^{-1} B^T \tilde{V}_b E.$$  \hfill (41)

(2) Solve generalized Riccati equation (40) to obtain a positive-definition solution $\tilde{V}_c$. Then, the gain matrix $G$ is chosen by

$$G = -\frac{1}{2} E \tilde{V}_c C^T R_c^{-1}.$$  \hfill (42)

(3) Select the coupling strengths $c_1$ and $c_2$ as (32)

Similarly, for any $Re(s) > 1$, we can prove

$$E^T \tilde{V}_b (A + sB) + (A + sB) \tilde{V}_b E + 2\alpha E^T \tilde{V}_b E$$

$$= (1 - Re(s)) E^T \tilde{V}_b BR_b^{-1} B^T \tilde{V}_b E - E^T \tilde{W}_b E = -E^T \tilde{W}_b E,$$  \hfill (43)

where $\tilde{W}_b = W_b + (Re(s) - 1) \tilde{V}_b BR_b^{-1} B^T \tilde{V}_b > 0$, and

$$E \tilde{V}_c (A + sGC) + (A + sGC) \tilde{V}_c E^T + 2\alpha E \tilde{V}_c E^T$$

$$= (1 - Re(s)) E \tilde{V}_c C^T R_c^{-1} C \tilde{V}_c E^T - EW_c E^T = -E \tilde{W}_c E^T,$$  \hfill (44)

where $\tilde{W}_c = W + (Re(s) - 1) \tilde{V}_c C^T R_c^{-1} C \tilde{V}_c > 0$.

According to (38), we know that for any $Re(s) \geq 1$, $(E, A + sB)$ and $(E, A + sGC)$ are admissible with all its finite generalized eigenvalues satisfying $Re\lambda < -\alpha$. Then, the following result can be obtained easily.

Theorem 6. For multi-agent system (6) whose interconnection topology graph $\mathcal{G}$ contains a directed spanning tree, all protocols (11), (19) and (24) designed by algorithm 2 can solve the descriptor multi-agent consensus problem with a convergence rate larger than $\alpha$.

5.3 Under balanced switching interaction topologies

Now, we begin to discuss consensus problem under switching interaction topologies. Concisely, we only discuss observer-based consensus problem via protocol (24). The similar result by using protocols (11) and (19) can be established by similar line. To describe the switching topology, we denote $\Gamma = \{\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_M\}$ as a set of all possible topologies with finite index set $\mathcal{P} = \{1, 2, \cdots, M\}$. Let $0 = t_1, t_2, t_3, \cdots$ be an infinite time sequence at which the interaction graph of the considered multi-agent system switches. Assume there is a dwell time $\tau_0 > 0$ such that $t_{j+1} - t_j \geq \tau_0$ are satisfied for all $j = 1, 2, 3, \cdots$.

A directed graph $\mathcal{G}$ is said to be balanced if $\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ji}, i = 1, 2, \cdots, n$, that is, graph $\mathcal{G}$ is balanced if and only if $1^T L = 0$ (see [9]). Obviously, any undirected weighted graph is balanced. If $\mathcal{G}_i$ is balanced and contains a directed spanning, then matrix $H_i^T + H_i$ is positive definite [11]. Assume that all interaction topology graphs $\mathcal{G}_i, i \in \mathcal{P}$ are balanced and contain a directed spanning tree.
Under the switching topology, the dynamics of the closed-loop system is given as
\[
\begin{bmatrix}
I \otimes E & 0 \\
0 & I \otimes E
\end{bmatrix}
\frac{d}{dt}
\begin{bmatrix}
x \\
v
\end{bmatrix} =
\begin{bmatrix}
I \otimes A & L_{\sigma(t)} \otimes (c_2 BK) \\
-L_{\sigma(t)} \otimes (c_1 GC) & I \otimes A + L_{\sigma(t)} \otimes (c_1 GC + c_2 BK)
\end{bmatrix}
\begin{bmatrix}
x \\
v
\end{bmatrix},
\]
(45)
where \( \sigma : [0, \infty) \rightarrow \mathcal{P} \) is the piecewise constant switching signal.

Take an orthogonal matrix with form \( U = \frac{1}{\sqrt{n}}[1, U_1] \). Since \( L_i \) is balanced, we have
\[
U^T L_i U = \begin{pmatrix} 0 & 0 \\ 0 & H_i \end{pmatrix} \triangleq L_i.
\]

Define
\[
\hat{\lambda} = \min_{i \in \mathcal{P}} \{ \lambda_{\min}(H_i + H_i^T) | \mathcal{G}_i \text{ is balanced and has a directed spanning tree} \}.
\]
(46)
It is easy to see that \( \hat{\lambda} \) is well-defined and positive constant.

Similarly, denote \( \tilde{x} = (U^T \otimes I_m)x \) and \( \tilde{v} = (U^T \otimes I_m)v \). The dynamics system (45) can be transformed into the following equivalent system
\[
\begin{bmatrix}
I_n \otimes E & 0 \\
0 & I_n \otimes E
\end{bmatrix}
\frac{d}{dt}
\begin{bmatrix}
\tilde{x} \\
\tilde{v}
\end{bmatrix} =
\begin{bmatrix}
I \otimes A & L_{\sigma(t)} \otimes (c_2 BK) \\
-L_{\sigma(t)} \otimes (c_1 GC) & I \otimes A + L_{\sigma(t)} \otimes (c_1 GC + c_2 BK)
\end{bmatrix}
\begin{bmatrix}
\tilde{x} \\
\tilde{v}
\end{bmatrix},
\]
(47)
which can be divided into
\[
\begin{bmatrix}
E & 0 \\
0 & E
\end{bmatrix}
\frac{d}{dt}
\begin{bmatrix}
\tilde{x}^0 \\
\tilde{x}^1
\end{bmatrix} =
\begin{bmatrix}
A & 0 \\
0 & A
\end{bmatrix}
\begin{bmatrix}
\tilde{x}^0 \\
\tilde{x}^1
\end{bmatrix},
\]
(48)
and
\[
\begin{bmatrix}
I \otimes E & 0 \\
0 & I \otimes E
\end{bmatrix}
\frac{d}{dt}
\begin{bmatrix}
\tilde{x}^1 \\
\tilde{v}^1
\end{bmatrix} =
\begin{bmatrix}
I \otimes A & H_{\sigma(t)} \otimes (c_2 BK) \\
-H_{\sigma(t)} \otimes (c_1 GC) & I \otimes A + H_{\sigma(t)} \otimes (c_1 GC + c_2 BK)
\end{bmatrix}
\begin{bmatrix}
\tilde{x}^1 \\
\tilde{v}^1
\end{bmatrix}.
\]
(49)
If \( \lim_{t \to \infty} \tilde{x}^1(t) = 0 \), we can obtain \( \lim_{t \to \infty} (x(t) - \frac{1}{\sqrt{n}}[1, \tilde{x}^0(t)]) = \lim_{t \to \infty} \left[ \left( \frac{1}{\sqrt{n}}[1, U_1] \otimes I_m \right) \left[ \begin{array}{c} x(t) \\ \tilde{x}(t) \end{array} \right] - \frac{1}{\sqrt{n}}1 \otimes \tilde{x}(t) \right] = 0 \). Thus, the descriptor multi-agent system achieves consensus if the switching descriptor system (49) is stable.

Now, we investigate the consensus problem under switching interconnection topology and present our result as follows.

**Theorem 7.** For the multi-agent system (6), suppose that the interconnection topology graph \( \mathcal{G}_{\sigma(t)} \) associated with any interval \([t_j, t_{j+1})\) is balanced and contains a directed spanning tree. Choose the coupling strengths \( c_1 \) and \( c_2 \) satisfying
\[
c_1 \geq \frac{2}{\hat{\lambda}}, \quad c_2 \geq \frac{2}{\hat{\lambda}}.
\]
(50)
Then, the protocol (24), whose gain matrices \( G \) and \( K \) are constructed by Algorithm 1, can solve descriptor multi-agent consensus problem.
Proof. Now, we prove the switching descriptor system (49) is stable. Denote \( \dot{\xi}^1 = \xi^1 - \tilde{x}^1 \). system (49) is equivalent to

\[
\begin{bmatrix}
I \otimes E & 0 \\
0 & I \otimes E
\end{bmatrix}
\begin{bmatrix}
\dot{\xi}^1 \\
\dot{\tilde{x}}^1
\end{bmatrix}
= \begin{bmatrix}
I \otimes A + H_{\sigma(t)} \otimes (c_2 B K) & H_{\sigma(t)} \otimes (c_2 B K) \\
0 & I \otimes A + H_{\sigma(t)} \otimes (c_1 G C)
\end{bmatrix}
\begin{bmatrix}
\xi^1 \\
\tilde{x}^1
\end{bmatrix}
\tag{51}
\]

Since \((E, A)\) is regular and impulse free, there exist two nonsingular matrices \(M\) and \(N\) such that \((E, A)\) has following Weierstrass Form (see [42])

\[
\tilde{E} = MEN = \begin{bmatrix}
I & 0 \\
0 & 0
\end{bmatrix}, \quad \tilde{A} = MAN = \begin{bmatrix}
A_1 & 0 \\
0 & I
\end{bmatrix}.
\tag{52}
\]

Let \(\tilde{B} = MB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\) and \(\tilde{C} = CN = [C_1, C_2]\). Furthermore, denote

\[
\tilde{V}_b = M^{-T} V_b M^{-1} = \begin{bmatrix}
V_{11} & V_{12} \\
V_{12}^T & V_{22}
\end{bmatrix}, \quad \tilde{W}_b = M^{-T} W_b M^{-1} = \begin{bmatrix}
W_{11} & W_{12} \\
W_{12}^T & W_{22}
\end{bmatrix}.
\tag{53}
\]

Pre-multiplying by \(N^T\) and post-multiplying \(N\) both sides of the generalized Riccati equation (5), we can get

\[
E^T \tilde{V}_b \tilde{A} + \tilde{A}^T \tilde{V}_b E - E^T \tilde{V}_b B R_b^{-1} B^T \tilde{V}_b E + E^T \tilde{W}_b E = 0
\]

which implies that

\[
V_{11} A_1 + A_1^T V_{11} - (V_{11} B_1 + V_{12} B_2) R_b^{-1} (B_1^T V_{11} + B_2^T V_{12}^T) + W_{11} = 0,
\tag{54}
\]

and

\[
V_{12} = 0.
\]

Since \((E, A, B)\) is \(R\)-controllable, \((A_1, B_1)\) is controllable. Therefore, the Riccati equation (54) has a unique solution \(V_{11} > 0\) for any \(W_{11} > 0\). Similarly, denote

\[
\tilde{V}_c = N^{-1} V_c N^{-T} = \begin{bmatrix}
V_{11} & V_{12} \\
V_{12}^T & V_{22}
\end{bmatrix}, \quad \tilde{W}_c = N^{-1} W_c N^{-T} = \begin{bmatrix}
W_{11} & W_{12} \\
W_{12}^T & W_{22}
\end{bmatrix}.
\tag{55}
\]

Pre-multiplying by \(M\) and post-multiplying \(M\) both sides of the generalized Riccati equation (30), we can obtain

\[
V_{11} A_1^T + A_1 V_{11} - (V_{11} C_1^T + V_{12} C_2^T) R_c^{-1} (C_1 V_{11} + C_2 V_{12}^T) + W_{11} = 0,
\tag{56}
\]

and

\[
\tilde{V}_{12} = 0.
\]

From the \(R\)-observability of \((E, A, C)\), we also know that the Riccati equation (56) has a unique solution \(\tilde{V}_{11} > 0\) for any \(\tilde{W}_{11} > 0\).

Let \(\tilde{x} = I \otimes N^{-1} \tilde{x}^1\) and \(\tilde{e} = I \otimes N^{-1} \tilde{e}^1\). The switching descriptor system (51) has equivalence form

\[
\begin{bmatrix}
I \otimes \tilde{E} & 0 \\
0 & I \otimes \tilde{E}
\end{bmatrix}
\begin{bmatrix}
\dot{\tilde{x}}^1 \\
\dot{\tilde{e}}^1
\end{bmatrix}
= \begin{bmatrix}
I \otimes \tilde{A} + H_{\sigma(t)} \otimes (c_2 \tilde{B} K \tilde{N}) & H_{\sigma(t)} \otimes (c_2 \tilde{B} K \tilde{N}) \\
0 & I \otimes \tilde{A} + H_{\sigma(t)} \otimes (c_1 M G \tilde{C})
\end{bmatrix}
\begin{bmatrix}
\tilde{x}^1 \\
\tilde{e}^1
\end{bmatrix}.
\tag{57}
\]
In view of (29) and (31),
\[
c_2BKN = \begin{bmatrix}
-\frac{1}{2}c_2B_1R_b^{-1}B_1^TV_{11} & 0 \\
-\frac{1}{2}c_2B_2R_b^{-1}B_2^TV_{11} & 0
\end{bmatrix},
\]
and
\[
c_1MGC = \begin{bmatrix}
-\frac{1}{2}c_1\bar{V}_{11}C_1^TR_c^{-1}C_1 & -\frac{1}{2}c_1\bar{V}_{11}C_1^TR_c^{-1}C_2 \\
0 & 0
\end{bmatrix}.
\]
The switching descriptor system (57) is transformed into the following equivalent decomposition form
\[
\begin{bmatrix}
\dot{x}^1 \\
\dot{\bar{e}}^1
\end{bmatrix} = \begin{bmatrix}
I \otimes A_1 + H_{\sigma(t)} \otimes (\frac{1}{2}c_2B_1R_b^{-1}B_1^TV_{11}) & H_{\sigma(t)} \otimes (\frac{1}{2}c_2B_1R_b^{-1}B_1^TV_{11}) \\
0 & I \otimes A_1 - H_{\sigma(t)} \otimes (\frac{1}{2}c_1\bar{V}_{11}C_1^TR_c^{-1}C_1)
\end{bmatrix} \begin{bmatrix}
x^1 \\
\bar{e}^1
\end{bmatrix}
\]
\[
\triangleq F_{\sigma(t)} \begin{bmatrix}
x^1 \\
\bar{e}^1
\end{bmatrix}
\] (58)
with
\[
\dot{x}^2 = H_{\sigma(t)} \otimes (\frac{1}{2}c_2B_2R_b^{-1}B_1^TV_{11})\dot{x}^1 + H_{\sigma(t)} \otimes (B_2R_b^{-1}B_1^TV_{11})\dot{\bar{e}}^1
\]
\[
\dot{\bar{e}}^2 = 0
\] (59)
where \(x^1, \dot{x}^2, \bar{e}^1\) and \(\dot{\bar{e}}^2\) are extracted some components from \(x, \bar{e}\), which are compatible with the decomposition form. Obviously, by (59) \(\lim_{t \to \infty} \begin{bmatrix}
\dot{x}^2 \\
\dot{\bar{e}}^2
\end{bmatrix} = 0\) if the switching system (58) is stable. Thus, the stability of (58) means the stability of (57). To prove the stability of the switching system (58), consider the following parameter-dependent common Lyapunov matrix as follows
\[
P = \begin{bmatrix}
I \otimes V_{11} & 0 \\
0 & \omega I \otimes \bar{V}_{11}^{-1}
\end{bmatrix} > 0
\]
where \(\omega\) is a given large enough positive constant. From (46) and (50), \(I - \frac{1}{2}c_1(H_{\sigma(t)} + H_{\sigma(t)}^T) < 0\) and \(I - \frac{1}{2}c_2(H_{\sigma(t)} + H_{\sigma(t)}^T) < 0\) hold. The unique positive solution \(V_{11}\) of Riccati equation (54) satisfies
\[
\begin{align*}
(I \otimes A_1 - H_{\sigma(t)} \otimes (\frac{1}{2}c_2B_1R_b^{-1}B_1^TV_{11})) (I \otimes \bar{V}_{11}) &+ (I \otimes \bar{V}_{11})(I \otimes A_1 - H_{\sigma(t)} \otimes (\frac{1}{2}c_2B_1R_b^{-1}B_1^TV_{11}))^T \\
= -I \otimes \bar{W}_{11} + [I - \frac{1}{2}c_2(H_{\sigma(t)} + H_{\sigma(t)}^T)] \otimes (\frac{1}{2}c_2B_1R_b^{-1}B_1^TV_{11}) \\
&\leq -I \otimes \bar{W}_{11} < 0.
\end{align*}
\] (60)
The unique positive solution \(\bar{V}_{11}\) of Riccati equation (56) satisfies
\[
\begin{align*}
(I \otimes A_1 - H_{\sigma(t)} \otimes (\frac{1}{2}c_1\bar{V}_{11}C_1^TR_c^{-1}C_1)) (I \otimes \bar{V}_{11}) &+ (I \otimes \bar{V}_{11})(I \otimes A_1 - H_{\sigma(t)} \otimes (\frac{1}{2}c_1\bar{V}_{11}C_1^TR_c^{-1}C_1))^T \\
= -I \otimes \bar{W}_{11} + [I - \frac{1}{2}c_1(H_{\sigma(t)} + H_{\sigma(t)}^T)] \otimes (\bar{V}_{11}C_1^TR_c^{-1}C_1) \\
&\leq -I \otimes \bar{W}_{11} < 0,
\end{align*}
\] (61)
from which we have
\[
\begin{align*}
(I \otimes A_1 - H_{\sigma(t)} \otimes (\frac{1}{2}c_1\bar{V}_{11}C_1^TR_c^{-1}C_1))^T (I \otimes \bar{V}_{11}^{-1}) &+ (I \otimes \bar{V}_{11}^{-1})(I \otimes A_1 - H_{\sigma(t)} \otimes (\frac{1}{2}c_1\bar{V}_{11}C_1^TR_c^{-1}C_1)) \\
&\leq -I \otimes (\bar{V}_{11}^{-1}\bar{W}_{11}\bar{V}_{11}^{-1}) \triangleq -I \otimes \bar{W}_{11} < 0.
\end{align*}
\] (62)
Furthermore, by applying Schur Complement Lemma, we can obtain
\[
F_{\sigma(t)}P + PF_{\sigma(t)}^T \leq \begin{bmatrix}
-I \otimes W_{11} & H_{\sigma(t)}^T \otimes \left(\frac{1}{2}c_1R_1^{-1}B_1V_{11}^{-1}\right) \\
H_{\sigma(t)} \otimes \left(\frac{1}{2}c_2B_1^{-1}B_1^TV_{11}^{-1}\right)^T & -\omega I \otimes \hat{W}_{11}
\end{bmatrix} < 0
\]
for large enough $\omega$ satisfying
\[
\omega > \frac{1}{4} \lambda_{\max}(W_{11}^{-1}V_{11}^{-1}V_{11}^{-1}B_1^{-1}B_1^TV_{11}^{-1}B_1^{-1}B_1^TV_{11}^{-1}) \max_{i \in \mathcal{P}} \lambda_{\max}(H_i^TH_i).
\]
Therefore, the switching system (58) is stable. Now, the proof is completed.

\textbf{Remark 2}. Certainly, as a special case, the result of Theorem 2 is also right under connected undirected switching topology. Furthermore, if there exists a common positive definite matrix $P_H$ and a positive constant $\mu$ such that
\[
H_i^TP_H + P_HP_H \geq \mu P_H, \forall i \in \mathcal{P},
\]
then the dynamic protocol (24) constructed by Algorithm 1 with $c_1 \geq \frac{2}{\mu}c_2$ can guarantee the closed-loop descriptor system achieves consensus.

Although it is assumed that chattering does not occur, that is, $\sigma(t)$ switches a finite number of times in every finite time interval, the interaction topology is arbitrarily switching in the possible topology set $\Gamma$. Our established consensus condition based on balanced interaction topology is relatively strict. To weaken the consensus condition, we can probe the problem by using the the concept of average dwell time [43, 44]. If the average dwell time is large enough and all topology graphs are directed and contain a directed spanning tree, we can also prove that the multi-agent system is able to achieve consensus by the proposed protocols.

### 6 Numerical example

In this section, a numerical simulation is provided to illustrate our obtained theoretical results. Consider descriptor multi-agent system (6) consisting of $n = 4$ agents, whose system matrices are given as
\[
E = \begin{bmatrix}
2 & 1 & 0 & 0 \\
0 & 1 & -2 & -1 \\
2 & 2 & -2 & -1 \\
0 & 2 & 0 & 1
\end{bmatrix}, \quad A = \begin{bmatrix}
0 & 0 & 1 & 0 \\
1 & -2 & 0 & 0 \\
1 & -3 & 1 & 0 \\
-3 & 1 & -1 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
0 \\
1 \\
-2
\end{bmatrix}, \quad C = \begin{bmatrix}
-2 & -2 & 2 & 1
\end{bmatrix}.
\]

Certainly, it can be verified that $(E, A)$ is impulse free, $(E, A, B)$ is $R$-controllable, $(E, A, C)$ is $R$-observable. The interaction topologies are arbitrarily switched with period 0.5s among three graphs $\mathcal{G}_i(i = 1, 2, 3)$, whose Laplacian matrices are respectively given by
\[
L_1 = \begin{bmatrix}
2 & -0.5 & 0 & -1.5 \\
0 & 1 & -1 & 0 \\
0 & 0 & 1 & -1 \\
0 & -1.5 & 0 & 1.5
\end{bmatrix}, \quad L_2 = \begin{bmatrix}
2 & -0.5 & -1.5 & 0 \\
0 & 1 & 0 & -1 \\
0 & -1 & 3 & -2 \\
-1 & 0 & 0 & 1
\end{bmatrix}, \quad L_3 = \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 3 & 0 & -3 \\
-2 & 0 & 3 & -1 \\
0 & -0.5 & -1.5 & 2
\end{bmatrix}.
\]
Obviously, all the topologies $G_i$ are directed and contain a directed spanning tree. Each agent adopts the observer-based protocol (11). The distributed protocol is constructed by the Algorithm 1. For convenience, $R_b$, $R_c$, $W_b$ and $W_c$ are taken as the unit matrix with appropriate dimensions. Solve the Riccati equations (5) and (30) to get the positive define solutions

$$V_b = \begin{bmatrix} 5.4276 & 4.2314 & -3.5342 & -0.7022 \\ 4.2314 & 8.6705 & -5.7458 & 0.8285 \\ -3.5342 & -5.7458 & 4.5320 & 0.3326 \\ -0.7022 & 0.8285 & 0.3326 & 1.2101 \end{bmatrix}$$

and

$$V_c = \begin{bmatrix} 2.0849 & -1.5000 & -0.6250 & 1.4198 \\ -1.5000 & 3.0000 & 0.7500 & 1.5000 \\ -0.6250 & 0.7500 & 0.4769 & 0.2039 \\ 1.4198 & 1.5000 & 0.2039 & 5.2475 \end{bmatrix}.$$  

Then, the gain matrix $G$ can be constructed by

$$G = -\frac{1}{2} E V C^T R^{-1} = [1.0000, 0.8155, 1.8155, 0.1845]^T$$

and

$$K = -\frac{1}{2} R^{-1} B^T V_b E = [0.3507, 0.8996, -1.4484, -0.7242].$$

Select the coupling strength $c_1 = c_2 = 0.5$. Denote the consensus error $\delta_i = x_i - \frac{1}{n} \sum_{j=1}^{n} x_j$ for agent $i$. It is easy to see that if all $\lim_{t \to \infty} \|\delta_i\| = 0 (i = 1, 2, \cdots, n)$ are satisfied, then the multi-agent system achieves consensus. The consensus protocols for agent $i$ are adopted by (11), (19) and (24) respectively with randomly compatible initial state. The trajectories of $\|\delta_i\|$ are depicted in Figure 1, Figure 2 and Figure 3 respectively, which show that the proposed three kinds of proposed protocols are able to solve the descriptor multi-agent consensus problem.

While the interaction topology is fixed and directed, the simulation results also show that all proposed protocols (11), (19) and (24) are able to solve the descriptor consensus problem. According to Theorem 7, while interaction topology switches arbitrarily in a finite set of balanced topologies with a directed spanning tree, the proposed protocols can solve the descriptor consensus problem. While interaction topology switches arbitrarily in a finite set of directed topology, the multi-agent system may not reach consensus. Here, although the interaction topology is switching and directed, the simulation results also show that the multi-agent system can achieve consensus via the protocols (11), (19) and (24). The reason is that the interaction topology involved in this example is not arbitrary switching with fixed dwell $\tau_0 = 0.5s$. In Remark 2, we have pointed that the consensus condition can be weakened by using the the concept of average dwell time [43, 44].

7 Conclusions

In this paper, the consensus problem for descriptor multi-agent systems with general form of linear dynamics and undirected topologies has been investigated. Based on the relative outputs of neighboring
agents, a distributed observer-based consensus protocol is proposed to each agent to track their neighboring agents. A multi-step algorithm has been proposed to construct the consensus protocol, and the two control gain matrices used in the consensus protocol can be constructed by solving the generalized Riccati equation. Some sufficient consensus conditions are established for descriptor multi-agent systems. The interaction topology is modeled by directed graph, whose connectivity is assumed as key conditions to achieve consensus. Other topics such as functional observer, robust consensus, discrete-time descriptor multi-agent systems will be probed in our future work.

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