Flexible error-reduction method for shape measurement by temporal phase unwrapping: phase averaging method

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Temporal phase unwrapping is an important method for shape measurement in structured light projection. Its measurement errors mainly come from both the camera noise and nonlinearity. Analysis found that least-squares fitting cannot completely eliminate nonlinear errors, though it can significantly reduce the random errors. To further reduce the measurement errors of current temporal phase unwrapping algorithms, in this paper, we proposed a phase averaging method (PAM) in which an additional fringe sequence at the highest fringe density is employed in the process of data processing and the phase offset of each set of the four frames is carefully chosen according to the period of the phase nonlinear errors, based on fast classical temporal phase unwrapping algorithms. This method can decrease both the random errors and the systematic errors with statistical averaging. In addition, the length of the additional fringe sequence can be changed flexibly according to the precision of the measurement. Theoretical analysis and simulation experiment results showed the validity of the proposed method. © 2012 Optical Society of America

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1. Introduction

Spatial phase unwrapping (SPU) and temporal phase unwrapping (TPU) are two important methods in the field of shape measurement. They are widely applied in interferometry and structure light illuminated measurement. SPU algorithms are based on the comparison of neighboring pixels in two-dimensional (2D) space so that it is difficult in practice to tackle object surfaces with complicated geometry and topology because of phase ambiguity and error propagation. TPU algorithms consider interference phase at a given pixel \( \Phi \) as a function of time \( t \), so the one-dimensional (1D) unwrapping along the time axis can avoid the path-dependence problems. Compared with SPU, TPU algorithms have significant improvements in reliability, accuracy, and computation time [1,2].

A lot of TPU algorithms have been presented in the past two decades. According to the type of fringe sequence used, the current TPU algorithms proposed by Huntley can be divided into three categories: linear sequence (LTPU) [1], exponential sequence (ETPU) [3], and reversed exponential sequence (RETPU) [4]. Another classical TPU algorithm (TFHTPU) has been proposed by Reich et al. [5], in which a three-frequency heterodyne is used in phase unwrapping. In addition, some modified TPU algorithms have been known by others, such as a simple version of TPU by Zhao et al. [6], a generalized reversed exponential sequence TPU algorithm by Kinell and Sjodahl [7], a modified TPU algorithm by Peng et al. [8], a generalized TPU algorithm by Tian et al. [9], and a simplified exponential sequence TPU algorithm (SETPU) by Xu and Su [10].
For the purpose of decreasing random error by incorporating data from the intermediate phase values, two approaches were investigated by Huntley and Saldner [4]: one is a simple linear least-squares fitting to the measured phase values, and the other is based on Fourier transform of the phasors calculated from the phase-stepped images. Four TPU algorithms with least-squares fitting or Fourier transform ranging were proposed in the investigation: fitted to linear sequence, fitted to exponential sequence, fitted to reversed exponential sequence, and linear sequence with Fourier transform ranging. The results showed that least-squares fitting can significantly reduce random errors on the condition that the data acquisition time and the reliability of original TPU algorithm remain unchanged.

Related research [11–13] showed that there are two main error sources, electronic noise and nonlinearity, in the shape measurement system using the projected fringe technique, although sometimes other error sources also need to be considered, such as phase-shifting error [14], vibration [15], speckle noise [16], etc. A large quantity of error-reduction methods are presented to deal with nonlinear errors, such as increasing the N value of N-step phase shifting [11,17], building a look-up table (LUT) in advance by analyzing the projector grating directly and using it to compensate the phase errors of wrapped phase maps [18,19], averaging phase values by doubling the fringe sequence of the original TPU algorithm and using different phase offsets according to the period of the phase nonlinear errors [13], etc.

It is obvious that the nonlinear errors from camera or projector have been neglected in Huntley’s investigation [4]. The effects of nonlinear errors cannot be totally ignored, though four-step phase shifting can provide better immunity from nonlinearities than three-step phase shifting and the intensity of nonlinear errors is relatively smaller than that of random errors. Furthermore, limited by the length, highest fringe intensity, and even the intensity distribution of the fringe sequences used in original TPU algorithms, not all the fringe sequences can obtain optimal effects with least-squares fitting.

Take the exponential sequence, for example: the experimental result of the fitted algorithm (Fig. 6(c) in [4]) has no obvious improvement compared with that of the unfitted algorithm (Fig. 6(a) in [4]). For the purpose of reducing both the random errors and nonlinear errors existing in TPU, a more flexible and precise method than those classical error-reduction methods is presented in this study.

The rest of paper is organized as follows: Section 2 analyzes nonlinear errors existing in the four-step phase-shifting algorithm and its effects on error-reduction methods with least-squares fitting. Section 3 presents the phase averaging method (PAM) and two typical applications. Section 4 analyzes and compares the errors of classical TPU algorithms with least-squares fitting and typical algorithms with the PAM. Section 5 shows the experiment results. Section 6 summarizes the main results of this work.

2. Nonlinear Error Analysis of Current Error-Reduction Methods with Least-Squares Fitting

Nonlinear errors existing in phase-shifting interferometry (PSI) have been well examined through experimentation and by computer simulation for a long time [17]. As we know now, overexposure [11] of the image sensor array and the gamma [12] of the projector are two main reasons for nonlinear errors. As proved by Huang et al. [13], there exists mainly third-order phase error caused by nonlinearities of both the camera and the projector in three-step phase shifting. The similar result can be proved that there exists mainly fourth-order phase error in four-step phase shifting.

A. Fourth-Order Nonlinear Errors of Four-Step Phase Shifting

Assume that the input fringe pattern is generated by

\[ x = \frac{M}{2} (1 + \cos(\phi + \delta)), \] (1)

where \( M \) is the maximum gray level, \( \phi \) is the phase, and \( \delta \) is the constant angular phase shift.

If there exist a second-order residual \( \epsilon \) and a third-order residual \( \epsilon' \), the combined response function of the camera and the projector can be expressed by

\[ f(x) = a + bx + \epsilon x^2 + \epsilon' x^3. \] (2)

The intensity of the captured image is

\[
\begin{align*}
  f(x) = a + b \left[ \frac{M}{2} (1 + \cos(\phi + \delta)) \right] & + \epsilon \left[ \frac{M}{2} (1 + \cos(\phi + \delta)) \right]^2 \\
  & + \epsilon' \left[ \frac{M}{2} (1 + \cos(\phi + \delta)) \right]^3.
\end{align*}
\] (3)

The captured four images in four-step phase shifting can be given as follows:

\[
I_i = A + v \cos(\phi + \delta_i) + w \cos 2(\phi + \delta_i) + p \cos 3(\phi + \delta_i) \quad i = 1, 2, 3, 4.
\] (4)

where

\[ \delta_i = \pi/2 \times (i - 1) \quad i = 1, 2, 3, 4. \] (5)

\( A, v, w, \) and \( p \) are constants. Using the four-step phase-shifting formula, we can calculate the phase map as

\[
\tan(\phi') = \frac{I_4 - I_2}{I_1 - I_3} = \frac{v \sin(\phi) - p \sin(3\phi)}{v \cos(\phi) + p \cos(3\phi)}. \] (6)
where \( \phi' \) denotes the measured phase. The phase difference \( \Delta \phi \) introduced by this residual can be obtained as follows:

\[
\tan(\Delta \phi) = \frac{\tan(\phi') - \tan(\phi)}{1 + \tan(\phi') \tan(\phi)} = -\frac{p \sin(4\phi)}{p \cos(4\phi) + v} = -\frac{\sin(4\phi)}{\cos(4\phi) + \frac{v}{p}} = \frac{\sin(4\phi)}{\cos(4\phi) + \frac{k'}{p}}.
\]

(7)

\[
\Delta \phi = \phi' - \phi = -\arctan\left[\frac{\sin(4\phi)}{k'}\right] \quad \text{(for } k' \gg 1). \tag{8}
\]

Equations (3)–(6) show that the third-order residual \( \epsilon \) existing in each captured map is the main factor of nonlinear errors in the four-step phase-shifting algorithm, for the second-order residual \( \epsilon \) can be counteracted by a subtraction operation in Eq. (6). Generally, \( v \) is far greater than \( w \) in Eq. (4), so \( k' \) in Eq. (8) is far smaller than \( k \) in Eq. (19) [15]. That’s why the four-step phase shifting can provide better immunity from nonlinearities than three-step phase shifting.

B. Effects of Nonlinear Errors on Current Error-Reduction Methods with Least-Squares Fitting

Four methods with least-squares fitting are addressed by Huntly: exponential sequence without fitting (method A), fitted to linear sequence (method B), fitted to exponential sequence (method C), and fitted to reversed exponential sequence (method D). The four-step phase-shifting algorithm is adopted in the above methods. Although increasing the value of \( N \) in the \( N \)-step phase-shifting algorithm can provide increasing immunity from nonlinearities [11], greater \( N \) than 4 will lead to the data acquisition time of TPU greatly increasing.

Nonlinear errors of four-step phase shifting should not be completely ignored, though four-step phase shifting can provide better immunity from nonlinearities than three-step phase shifting, which will be verified by experiments in Section 5. To investigate the effects of nonlinear errors on error-reduction methods with least-squares fitting, nonlinear errors corresponding to every fringe density of method D are shown in Fig. 1. It is clear that nonlinear errors at a certain pixel are separately corresponding to a different phase place; some are positive and others are negative. With least-squares fitting, they are added up on separate weighting factors. The distributions of nonlinear errors in methods B and D can be simulated and shown in Fig. 2. It is obvious that the nonlinear phase error of final phase value is greatly reduced, but some peaks obviously remain at some special phase places.

3. Phase Averaging Method

A. Motivation

In most real measurements, some important performance indices should be taken into account simultaneously, including accuracy, reliability, data acquisition time, and computation time [4]. Apart from reliability, which is decided by the original TPU algorithms, the other performance indices can all be reflected from the selected error-reduction method. From the performance comparisons (Table 1 in [4]) of five methods, we can find that all the indices are closely related to the fringe sequences used in original TPU algorithms, including their lengths, highest fringe densities, and even intensity distributions.

As mentioned in [4], least-squares fitting does not significantly improve the estimate of \( \omega \) when an exponentially growing sequence of fringe frequencies is used. The reason for this is that the intermediate phase values are all clustered around the low-\( t \) end of the \( \Phi-t \) graph. Low \( t \) values do not provide reliable estimates of the gradient and so do not contribute significantly to the least-squares estimate for \( \omega \). On the contrary, the reversed exponentially growing sequence of fringe frequencies ensures that the intermediate phase values are clustered at the high-\( t \) end of the \( \Phi-t \) graph, so the precision of the fitted to reversed exponential sequence algorithm
is improved in shape compared to that of the unfitted one.

This comparison above provides us a clue: if all the fringe densities of fringe sequence are centralized on the highest fringe density \((i = s)\), the precision might be further improved, as we know that simple statistical averaging can be used to reduce the errors by \(\sqrt{n}\). This idea seems to be not unrealistic, because such a fringe sequence cannot be unwrapped by any TPU algorithm. But if we append such an additional fringe sequence, generally expressed by \(t = s, s, s, \ldots, s\), over a certain fast TPU algorithm (for example TFHTPU \([5]\) or SETPU \([10]\), this method with statistical averaging will improve the precision of the measured phase better than those methods with least-squares fitting.

But the same problem as least-squares fitting remains because simple statistical averaging can only be used in reducing random errors but cannot help in eliminating nonlinear errors. As shown in Fig. 3, those nonlinear errors are phase locked to the projected fringes, which remain in the same place for each of the averaged phase maps. Periodic nonlinear errors will remain in the averaged phase values.

In this case, the double three-step phase-shifting algorithm provides us a simple and feasible approach to decrease nonlinear errors like this pattern. If another set of four frame fringe patterns with \(\delta_i\) as in Eq. (9) instead of Eq. (5) is used in Eq. (4), its phase error can be computed in Eq. (10):

\[
\delta_i = \pi/2 \times (i - 1) + \pi/4 \quad i = 1, 2, 3, 4. \tag{9}
\]

\[
\Delta \phi' = - \arctan \left[ \frac{\sin(4\phi + \pi)}{k'} \right] = \arctan \left[ \frac{\sin(4\phi)}{k} \right]. \tag{10}
\]

As can be seen from Eqs. (5) and (9), there exists \(\pi/4\) phase offset between them, which will lead to \(\pi/4\) phase offset of wrapped phase maps \(\phi_w(s)\), for

\[
\phi_w(s) = \arctan \left[ \frac{I_4 - I_2}{I_1 - I_3} \right]. \tag{11}
\]

To unwrap the \(\phi_w(s)\) using the original TPU algorithm, a little change should be applied in Eq. (11) as

\[
\phi_w(s) = \arctan \left[ \frac{I_4 - I_2}{I_1 - I_3} \right] - \pi/4. \tag{12}
\]

It is clear that this method can eliminate nonlinear errors between two phase maps at the same fringe density by simply averaging their unwrapped phase values, as the two phases can be separately unwrapped using the same intermediate fringe sequences of the original TPU algorithm, and nonlinear phase errors [Eqs. (8) and (10)] will disappear by addition operation. If the length \((m)\) of averaging fringe sequence \((t = s, s, s, \ldots, s)\) at the highest fringe densities is chosen as an even number, keeping the constant angular phase shift \(\delta_i\) in Eq. (4) of the first half \((m/2)\) sets of four frames unchanged as Eq. (5) and changing that of the last half sets of four frames as Eq. (9), nonlinear errors in the above fringe sequence can be shown as in Fig. 4; then the phase errors caused by nonlinearity can cancel each other out.

Following this idea, for the purpose of improving the accuracy of the estimated \(\Phi-t\) gradient and eliminating the nonlinear errors, another flexible approach that is different from least-squares fitting and presented in this paper; we named it PAM.

B. Description of PAM

As described above, two different sequences should be carefully chosen in PAM, and they are prepared for two processes: the process of phase unwrapping and the process of data processing, respectively. The usage of PAM can be described in the following steps.

Step 1: Select a TPU algorithm, and generate three sets of fringe patterns according to the four-step phase-shifting algorithm. The first set contains

![Fig. 3. Fourth-order nonlinear errors of averaged phase maps in simple averaging sequence \((s = 64)\).](image1)

![Fig. 4. Fourth-order nonlinear error of averaged phase values in phase averaging sequence.](image2)
4(t − 1) frames that correspond to the fringe sequence of the selected original TPU algorithm, except those at the highest fringe density (t = s). The second and the third sets include four pictures corresponding to the highest fringe density (t = s), and keep 0° or 45° phase offset as Eq. (5) or Eq. (9), respectively.

In fact, PAM can be applied with any available TPU algorithm, because all the phase values corresponding to the first set of fringe patterns will not be directly applied in the process of data processing in most cases. But considering the total data acquisition time, computation time, and accuracy, TPU algorithms with shorter fringe sequence are recommended here, such as SETPU [10] and TFHTPU [5].

Step 2: Choose an even number m [Eq. (26)] according to the highest precision requirement of true measurement. Project all the three sets of fringe patterns on the measured object in sequence, and capture them on sampling frequencies as follows: one for every frame of the first set fringe patterns, m/2 for the second set, and m/2 for the third set. The first fringe sequence is mainly used to help phase unwrapping, and the rest of the fringe sequences are prepared for the process of data processing, so we name them the fringe sequence of phase unwrapping (FSPU) and the fringe sequence of data processing (FSDP), respectively.

If we use t, to denote FSPU and t, to denote FSDP, a total fringe sequence used in a certain TPU algorithm with PAM can be generally expressed by

\[
\begin{align*}
t_u &= t \text{ of original TPU except } s \\
t_v &= s, s, s, \ldots, s
\end{align*}
\]

where \(n1\) is the length of FSPU, and \(m\) is the length of FSDP. Take SETPU [10] for example; \(n1 = 2\) and its FSPU can be expressed by \(t_v = 1, \sqrt{s}\).

Step 3: Compute all the wrapped phase maps from \(t_1\) to \(t_{n1+m/2}\) using four-step phase-shifting algorithms. The last half wrapped phase values of FSDP (from \(t_{n1+m/2+1}\) to \(t_{n1+m}\)) should be dealt with in Eq. (13) after four-step phase shifting.

Step 4: Unwrap all the wrapped phase maps of FSDP using the selected TPU algorithm in step 1 with the help of those of FSPU.

Step 5: Take the arithmetic mean of all the unwrapped phase values as the measured phase value of PAM.

Because those unwrapped phase values in step 4 corresponding to the fringe sequence from \(t_{n1+1}\) to \(t_{n1+m}\) are all from the independent sampling at the highest fringe density (t = s), arithmetic average can be applied in them for the purpose of reducing phase error.

C. Two Typical Applications of PAM

From the description of PAM above, we can see that the intermediate phase values corresponding to fringe densities from \(t_1\) to \(t_n\) contribute nothing to the measured results, because they are not appropriate for arithmetic averaging together with those at the highest fringe density (t = s). Accordingly, the longer FSPU will lead to more data acquisition time and computation time. Considering the comprehensive performance of the total algorithm, there are two fast TPU algorithms suitable for application with PAM, TFHTPU [5] and SETPU [10], though their reliabilities are a little lower than some classical TPUs.

When SETPU is used with PAM, the fringe sequence of SETPU with PAM (as shown in Fig. 5) can be written by

\[
\begin{align*}
t_u &= 1, \sqrt{s} \\
t_v &= s, s, s, \ldots, s \\
u &= 1, 2, \ldots, n1 \\
v &= n1 + 1, n1 + 2, \ldots, n1 + m
\end{align*}
\]

where \(s = 64\) and \(n\) is the total length of \(t_u\) and \(t_v\).

TFHTPU is a special TPU algorithm because its fringe densities are very close to each other. All the unwrapped phase values responding to the fringe sequence of TFHTPU can be used in phase averaging to improve the measured precision, which can be explained as follows.

From Eq. (9) in [4], we know that \(\Phi(t)\) is a linear function of time \(t\). In theory, we can compute the phase value \(\Phi(s)\) at the highest fringe density (t = s) from any unwrapped intermediate phase value \(\Phi(t)\) by a transform as

\[
\Phi(s) = \Phi(t) \times s/t = (\omega t + \epsilon_p) \times s/t = \omega s + \epsilon_p \times s/t.
\]

Comparing \(\Phi(s)\) in Eq. (15) with the unwrapped phase value \(\Phi(s)\) in Eq. (9) of [4], the phase error of \(\Phi(s)\) is enlarged \(s/t\) times larger than that of \(\Phi(s)\). The error ratio of \(s/t\) in Eq. (15) is so close to \(1\) that we can safely assume that the errors of \(\Phi(s)\) [Eq. (15)] and \(\Phi(s)\) are taken from the same Gaussian distribution.

Under the transform of Eq. (15), to applied PAM in TFHTPU, the total fringe sequence can be written by

\[
\begin{align*}
t_u &= 59, 64, 59, 64 \\
t_v &= s, s, s, \ldots, s \\
u &= 1, 2, 3, 4 \\
v &= 5, 6, \ldots, n
\end{align*}
\]
where \( s \) is the highest fringe density \( 70 \) according to TFHTPU, and \( n \) keeps the same definitions as Eq. (14).

For the purpose of utilizing the two sets of phase values to improve measurement precision while keeping the total length of the fringe sequence unchanged, some similar methods as adopted in PAM for \( t_u \) should be carried out in this case, including the following: append two sets of fringe patterns with \( 45^\circ \) phase offset in \( t_u \) at \( t_3 \) and \( t_4 \) of Eq. (16), compute their wrapped phase values with Eq. (12), transform \( \Phi(t) \) to \( \Phi(s) \) with Eq. (15), and compute the measured phase integrating all the unwrapped values (including those at \( t_u \) and \( t_v \)) using statistical averaging. The fringe sequence of TFHTPU with PAM is shown in Fig. 6.

4. Precise Analysis and Comparison

To compare the accuracy between PAM and current TPU algorithms, six TPU algorithms will be addressed below: exponential sequence without fitting (method A), fitted to linear sequence (method B), fitted to exponential sequence (method C), fitted to reversed exponential sequence (method D), TFHTPU with PAM (method E), and SETPU with PAM (method F).

A. Random Error Analysis of Classical TPU Algorithms

Random error analysis of methods A–D has been given by Huntley and Saldner [4]. If there are additive and Gaussian errors in the measured phase values, the measured sequence of unwrapped phase values can be written as

\[
\Phi(t) = \omega t + \varepsilon_\Phi. \tag{17}
\]

where \( \varepsilon_\Phi \) is a Gaussian random variable, with a mean of zero and a standard deviation of \( \sigma_\Phi \). \( \omega \) represents the rate of change of phase with time \( t \).

The estimator \( \hat{\omega} \) of \( \omega \) in method A is given by

\[
\hat{\omega}_A = \Phi(s)/s, \tag{18}
\]

with the intermediate phase values playing no part in the calculation. The standard deviation of the random variable \( \hat{\omega}_A \) follows directly from Eq. (18) as

\[
\sigma_A = \sigma_\Phi/s. \tag{19}
\]

In method B, all phase values are unwrapped and used for the fitting (a fitting in the least-squares sense of the line); the estimator \( \hat{\omega} \) of \( \omega \) in method B is given by

\[
\hat{\omega}_B = \frac{\sum t \Phi(t)}{\sum t^2}. \tag{20}
\]

The standard deviation of the random variable \( \hat{\omega} \) follows directly from Eq. (20) as

\[
\sigma_B = \frac{\sqrt{6\sigma_\Phi}}{[s(s+1)(2s+1)]^{1/2}} \approx \sqrt{3\sigma_\Phi/s^{3/2}}. \tag{21}
\]

In methods C and D, the standard deviation of the random variable \( \hat{\omega} \) can be given by

\[
\sigma_C = \frac{\sqrt{3\sigma_\Phi}}{(4s^2-1)^{1/2}} \approx \sqrt{3\sigma_\Phi/(2s)}. \tag{22}
\]

\[
\sigma_D = \frac{\sigma_\Phi}{[s^2(\log_2 s - 2/3) + 2s - (1/3)]^{1/2}} \approx \sigma_\Phi/(s\sqrt{\log_2 s}) \quad \text{for } s \gg 1. \tag{23}
\]

B. Error Analysis of Typical PAM Algorithms

The similar analysis above can be carried out in method E and method F. To represent the general usage of PAM, let us consider SETPU with PAM first.

Method F is a representative application of PAM. Its fringe sequence consists of \( t_u \) and \( t_v \), which can be seen from Eq. (14). As described in step 5 of PAM, the unwrapped phase value at the highest fringe density of \( t_v \) can be used to compute the measured phase value. Because all the unwrapped phase values come from the same \( t \) (\( t = s \)), the measured sequence of unwrapped phase values can be written as
which is different from Eq. (17). Accordingly, the estimator \( \hat{\omega}_F \) of \( \omega \) in method F is given by

\[
\hat{\omega}_F = \frac{\sum_{t=3}^{n} \Phi(t)}{(n)s}.
\]

The standard deviation of the random variable \( \hat{\omega} \) follows directly from Eq. (25) as

\[
\sigma_F = \frac{\sigma_{\phi}}{s\sqrt{m}} = \frac{\sigma_{\phi}}{s\sqrt{(n-2)}}.
\]

where \( m \) is the length of \( t \).

Secondly, let us discuss method E. As explain in Subsection 3.C, under the transform of Eq. (15), the measured sequence of unwrapped phase values can be written in general as

\[
\Phi(t) = \omega_0 + \varepsilon_{\phi} \quad t = 1, 2, \ldots, n.
\]

Then the estimator \( \hat{\omega} \) of \( \omega \) in method E can be given by

\[
\hat{\omega}_E = \frac{\sum_{t=1}^{n} \Phi(t)}{(n)s}.
\]

The standard deviation of the random variable \( \hat{\omega} \) follows directly from Eq. (28) as

\[
\sigma_E = \frac{\sigma_{\phi}}{s\sqrt{n}}.
\]

It can be clearly seen that method E has lower random error than method D with the help of Eq. (15). With these changes, all the intermediate phase values are utilized in phase averaging, and both the random error and the nonlinear error are eliminated to the most extent, while the data acquisition time can be saved. That is why we considered method E as a special application of PAM.

C. Error Comparison of TPU Algorithms

The nonlinear errors existing in PAM are close to zero, while those in classical error-reduction methods are totally ignored in their theoretical analysis. Except for that, from the theoretical analyses of random errors above, we can draw some conclusions as follows.

1. PAM showed more flexibility than those TPU algorithms with least-squares fitting. There are two independent parameters \( (n \text{ and } s) \) in Eqs. (26) or (29), so the phase errors of PAM (methods E and F) are decided by both the length and the highest fringe density of fringe sequence, while those of methods A–D are only decided by the highest fringe density of the fringe sequence, which can be seen in Eqs. (19) and (21)–(23). This feature of PAM is very useful in practice, because we can meet variety of require-
ments to measured accuracy by adjusting the parameter \( n \).

2. On condition of keeping the same length of fringe sequence, PAM showed higher precision than those TPU algorithms with least-squares fitting. Take methods B and D for example; when \( s = 64 \) and \( n = 64 \) are used in Eqs. (21) and (29), the standard deviation of method E is about \( \sqrt{3/7} \) times that of method B; when \( s = 64 \) and \( n = 7 \) is used in Eqs. (29) and (23), the standard deviation of method E is nearly \( \sqrt{6/7} \) times that of method D.

3. To keep similar precision with classical algorithms, a shorter fringe sequence in method E is required than in other methods. For example, on the same condition of \( s = 64 \), to get the similar precision, \( n = 22 \) is needed in method E, rather than \( n = 64 \) required in method B, and \( n = 6 \) is needed in method E, while \( n = 7 \) is required in method D.

4. Compared with method F, method E has more accuracy because all the intermediate phase values are used in data processing.

5. Experiments

The precisions of methods A–E were compared experimentally as an independent verification of the results from the previous sections. For the purpose of directly comparing accuracy with classical algorithms, three numbers (6, 22, and 64) were used as the total length of fringe sequence in method E, respectively.

A. Experimental Setup

The fringes and phase shifts were generated by a computer, and then were projected by a digital projector (Sony VSP-EL7) onto a flat plate, which is employed as the measurement object. A video camera (Nikon MVC1000SAM-GE30ST) with a resolution of 1280 \( \times \) 1024 pixels was used to acquire encoded fringe patterns. Phase maps were calculated using the standard four-frame phase-stepping algorithm. The fringes were horizontal; \( s = 64 \) was selected as the averaged fringe frequency in TPU algorithms. Phase values at row 512 were used in numerical analysis of phase errors.

B. Experimental Results and Verification

It is difficult to evaluate phase errors existing in every method, as the real phase is hard to obtain. In our experiments, measured phase values of method E \( (n = 128) \) were used as real phases to evaluate the errors of all other methods.

The measured errors of methods A–E were shown in Fig. 7, and RMS errors were given in Table 1. To verify the precisions of error-reduction methods (B–E), error ratios (error \( \sigma_E \) in method A to error \( \sigma \) in other methods) and theoretical error ratios were calculated and given in Table 1, too. Fitted curves of phase errors with ninth-order polynomial fitting in methods A–D and E \( (n = 6, 22, \text{ and } 64) \) were shown in Fig. 8.

It can be clearly seen from Fig. 7 that compared with the error distributions of method D [Fig. 7(c)]
and method B [Fig. 7(e)], those of method E with \( n = 6 \) [Fig. 7(d)] and \( n = 22 \) [Fig. 7(f)] are more even along the field of view, and some peaks from the nonlinear errors are significantly weakened. The effectiveness of method E can be further verified from Table 1: comparing RMS errors and error ratios \( \sigma_{\phi} / \sigma \) of method E \( (n = 6 \text{ and } n = 22) \) with those of methods B and D, respectively, method E keeps more accuracies than other methods when keeping the same length of fringe sequence, though the theoretical precisions of them are almost the same. In addition, method E shows more stability than other methods, which can be seen from Fig. 8. Above results verified the conclusions in Subsection 4.C.

But there exist big deviations between error ratios and their theoretical values (Table 1), and the errors show an irregular distribution (Fig. 7) in method E \( (n = 64) \). The reason is that the measured phase values of method E \( (n = 128) \) were used as a real phase to evaluate the errors of all other methods, and there still remained some small errors in it. When the real error of method E \( (n = 128) \) is close to that of a certain method, its effect to the computed phase error cannot be ignored. In spite of that, the trends of error decreasing as \( n \) grows can be clearly seen from Table 1.

Error ratios \( \sigma_{\phi} / \sigma \) of method E at different lengths of fringe sequence were computed and shown in Fig. 9. In addition, those of classical TPU algorithms (methods A–D) at their separate lengths of the fringe sequence were given in the same figure, too. The

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<thead>
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<th>Method</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E (n = 6)</th>
<th>E (n = 22)</th>
<th>E (n = 64)</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMS error ( \times 10^{-4} )</td>
<td>8.745</td>
<td>1.8616</td>
<td>7.712</td>
<td>3.5616</td>
<td>3.2926</td>
<td>1.6472</td>
<td>0.7172</td>
</tr>
<tr>
<td>Error ratio ( \sigma_{\phi} / \sigma )</td>
<td>1</td>
<td>4.698</td>
<td>1.134</td>
<td>2.455</td>
<td>2.656</td>
<td>5.309</td>
<td>12.193</td>
</tr>
<tr>
<td>Theoretical error ratio</td>
<td>1</td>
<td>4.619</td>
<td>1.155</td>
<td>2.828</td>
<td>2.449</td>
<td>4.690</td>
<td>8</td>
</tr>
</tbody>
</table>

Fig. 7. Phase errors of methods A–E: (a) method A, (b) method C, (c) method D, (d) method E \( (n = 6) \), (e) method B, (f) method E \( (n = 22) \), (g) method E \( (n = 64) \).

Fig. 8. Fitted curves of phase error in methods A–E.

Fig. 9. Fitted curves of phase errors with 9th order polynomial fitting.
curve of error ratios in method E reflects that the accuracy of PAM keeps steadily increasing with the length of fringe sequence growing.

6. Conclusions
A new error-reduction method for shape measurement by TPU is investigated in this paper, in which a new fringe sequence with different phase offsets is adopted and arithmetic average is used in the process of data processing. Theoretical analysis and experimental results show that (1) PAM can effectively eliminate the nonlinear errors existing in available error-reduction methods, (2) the precision of the measured phase by this method is higher than other error-reduction methods, and (3) the length of the fringe sequence in this method can be chosen flexibly to meet the needs of various measurements.

References