Positive solutions for singular semipositone boundary value problems on infinite intervals

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A B S T R A C T

By using the fixed point theory on a cone with a special norm and space, we discuss the existence of positive solutions for a class of semipositone boundary value problems on infinite intervals. The work improves many known results including singular and non-singular cases.

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1. Introduction

The main purpose of this paper is to study the existence of positive solutions for the following nonlinear singular and semipositone boundary value problem (BVP) on infinite intervals:

\[
\begin{aligned}
(p(t)x'(t))' + \phi(t)f(t,x(t)) = 0, \quad t \in (0, +\infty),

\alpha_1 x(0) - \beta_1 \lim_{t \to a^+} p(t)x(t) = \sum_{i=1}^{m-2} \gamma_i x(\eta_i),

\alpha_2 \lim_{t \to +\infty} x(t) + \beta_2 \lim_{t \to +\infty} p(t)x(t) = \sum_{i=1}^{m-2} \delta_i x(\eta_i),
\end{aligned}
\]  

(1.1)

where \( \lambda > 0 \) is a parameter, \( \alpha_1, \alpha_2, \gamma_i, \delta_i \geq 0, \beta_1, \beta_2 > 0, \eta_i \in (0, +\infty), (1, 2, \ldots, m-2) \) are given constants, \( f : (0, +\infty) \times (0, +\infty) \to (-\infty, +\infty) \) is a continuous function and \( f(t, u) \) may be singular at \( t = 0 \) and \( u = 0 \), \( p \in C(0, +\infty) \cap C^1(0, +\infty) \) with \( p \geq 0 \) on \( (0, +\infty) \), \( \int_{0}^{+\infty} p(s) ds < +\infty \), \( \rho = \alpha_2 \beta_1 + \beta_1 \beta_2 + \alpha_1 \alpha_2 \beta(0, +\infty) > 0 \) in which \( \beta(t, s) = \int_{0}^{s} \frac{1}{p(t)} dt \).

Boundary value problems on infinite intervals arises from the study of many real world problems such as solution of nonlinear elliptic equations and modeling of gas pressure in a semi-infinite porous medium. A great deal of work has been done in these areas such as those in [1–22] and the references therein. Over the last couple of decades, a great deal of results have been developed for differential and integral boundary value problems. O’Regan et al. studied in [23] the BVP

\[
\begin{aligned}
x''(t) + \phi(t)f(t,x(t)) = 0, \quad t \in (a, +\infty),

x(a) = 0, \quad \lim_{t \to +\infty} x'(t) = 0,
\end{aligned}
\]

where \( f : (a, +\infty) \times (0, +\infty) \to [0, +\infty) \) is a continuous function and \( \phi : (a, +\infty) \to (0, +\infty) \) is continuous. The existence of multiple unbounded positive solutions is discussed using the theory of fixed point index. By using the fixed point theorem and the monotone iterative technique, Zhang [24] studied the problem

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\[ \begin{align*}
\begin{cases}
X'(t) + q(t)f(t, X(t)) = 0, & t \in (0, +\infty), \\
x(0) = \sum_{i=1}^{m-2} \alpha_i x(\eta_i), & \lim_{t \to +\infty} x(t) = x_\infty \geq 0,
\end{cases}
\end{align*} \]

where \( \alpha_i \geq 0 \), \( 0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < +\infty \), \( \sum_{i=1}^{m-2} \alpha_i < 1 \). Let \( q : [0, +\infty) \times [0, +\infty) \to [0, +\infty) \) be a continuous function and \( f : [0, +\infty) \to [0, +\infty) \) be a Lebesgue integrable function. Liu et al. in [25] established the existence of positive solutions for the following equation on infinite intervals by applying the fixed point theorem of cone map

\[ \begin{align*}
\begin{cases}
(p(t)x'(t))' + m(t)f(t, x(t)) = 0, & t \in (0, +\infty), \\
x(0) = \beta_1 x(0) + \beta_2 x(1), & \lim_{t \to +\infty} x(t) = x_\infty \geq 0,
\end{cases}
\end{align*} \]

in which \( f : [0, +\infty) \times [0, +\infty) \to [0, +\infty) \) is a continuous function, \( m : (0, +\infty) \to (0, +\infty) \) is a Lebesgue integrable function and may be singular at \( t = 0 \). Also, by the use of the Krasnosel’skii fixed point theorem, upper and lower solutions, Schauder point theorem and inequality technique, Xing et al. in [26], Lian et al. in [27,28], Li and Zhao in [29] studied many equations with infinite intervals. However, all of the above studies are limited to the cases in which the nonlinear term is positive.

Inspired by the work of the above papers and many known results, in this paper, we study the existence of positive solutions to BVP (1.1), where \( x \in C([0, +\infty)) \) is said to be a positive solution of BVP (1.1) if and only if \( x \) satisfies (1.1) and \( x(t) > 0 \) for any \( t \in [0, +\infty) \). By using the fixed point theorem on a cone, some new existence results are obtained for the case where the nonlinearity is allowed to be sign changing and has singularity. We should address here that our work presented in this paper has various new features. Firstly, our study is on singular nonlinear differential boundary value problems, that is, \( f(t, u) \) is allowed to be singular at \( t = 0 \) and \( u = 0 \), which leads to many difficulties in analysis. Secondly, the techniques used in this paper are the approximation method and a special cone in a special space is established to overcome the difficulties caused by singularity and infinite interval. Also we find another substitute function to solve the problem associated with the semi-positive property of the nonlinear function. Thirdly, we discuss the boundary value problem with multi-point boundary conditions, that is, BVP (1.1) includes two-point and three-point boundary value problems as special cases. To our knowledge, the theory of Sturm–Liouville multi-point boundary value problems on infinite interval is yet to be developed. Our model improves and generalizes the previous results to some degree.

The plan of the paper is as follows. In Section 2, we present the preliminaries and necessary lemmas that are to be used to prove our main results. The main results are given in Section 3, including results for a completely continuous operator, and the conditions for the existence of positive solutions for the BVP (1.1). In Section 4, an example is given to demonstrate the application of our theoretical results.

2. Preliminaries and lemmas

For convenience, we let

\[ \begin{align*}
a(t) &= \beta_1 + \alpha_1 B(0, t), & b(t) &= \beta_2 + \alpha_2 B(t, \infty), \\
a(\infty) &= \lim_{t \to +\infty} a(t) = \beta_1 + \alpha_1 B(0, \infty) < +\infty, & a(0) &= \lim_{t \to 0} a(t) = \beta_1, \\
b(\infty) &= \lim_{t \to +\infty} b(t) = \beta_2, & b(0) &= \lim_{t \to 0} b(t) = \beta_2 + \alpha_2 B(0, \infty) < +\infty, \\
\Delta &= \rho - \sum_{i=1}^{m-2} \gamma_i b(\eta_i) - \sum_{i=1}^{m-2} \gamma_i a(\eta_i), \\
\Delta &= \rho - \sum_{i=1}^{m-2} \delta_i b(\eta_i) - \sum_{i=1}^{m-2} \delta_i a(\eta_i).
\end{align*} \]

It is obvious that \( a(t) \) is increasing and \( b(t) \) is decreasing on \([0, +\infty)\). Define

\[ G(t, s) = \begin{cases} a(s) b(t), & 0 \leq s < t < +\infty, \\
a(t) b(s), & 0 \leq t < s < +\infty.
\end{cases} \] (2.1)

Denote \( \tau(t) = a(t) b(t) \), then for any \( 0 \leq t, s < +\infty \), we get

\[ \begin{align*}
0 &\leq G(t, s) \leq G(s, s) \leq \frac{b(0) a(s)}{\rho}, & 0 &\leq G(t, s) \leq \frac{\tau(t)}{\rho}, \\
G(s) &\leq \lim_{t \to +\infty} G(t, s) = \frac{\rho_2 a(s)}{\rho} \leq G(s, s) < +\infty.
\end{align*} \] (2.2)

**Lemma 2.1.** Suppose \( \theta = \frac{1}{a(\infty) b(0)} \), then \( G(t, s) \geq \theta \tau(t) G(s, s), \ 0 \leq t, s < +\infty. \)
Proof. From (2.2) and the property of $a(t), b(t)$, for $0 \leq s \leq t < +\infty$, we have
\[
\frac{G(t,s)}{G(s,s)} = \begin{cases} \frac{\theta(t)}{\theta(s)}, & s \leq t, \\ \frac{\theta(s)}{\theta(t)}, & t \leq s, \\ \theta(t), & t \leq s \leq t \end{cases}
\]
Therefore, $G(t,s) \geq \theta(t)G(s,s)$, $0 \leq t, s < +\infty$. □

In this paper, we always assume that the following conditions hold.

(H1): $A > 0$, $\rho - \sum_{i=1}^{m-2} \gamma_i b(\eta_i) > 0$, $\rho - \sum_{i=1}^{m-2} \delta_i a(\eta_i) > 0$.

(H2): $f : (0, +\infty) \times (0, +\infty) \rightarrow (-\infty, +\infty)$ is a continuous function and
\[-\psi(t) < f(t, u) < \phi(t) (g(u) + h(u)), \quad (t, u) \in (0, +\infty) \times (0, +\infty),
\]
where $\psi, \phi : (0, +\infty) \rightarrow [0, +\infty)$ is continuous and singular at $t = 0$, $\phi(t) \neq 0$ on $(0, +\infty)$, $g : (0, +\infty) \rightarrow [0, +\infty)$ is continuous and nonincreasing, $h : (0, +\infty) \rightarrow (0, +\infty)$ is continuous, and $g$ and $h$ are bounded in any bounded set of $(0, +\infty)$.

Let
\[
A = 1 \int \frac{\theta(t)}{\theta(s)} ds < +\infty, \quad B = 1 \int \frac{\theta(s)}{\theta(t)} ds < +\infty.
\]
Choose a constant $\overline{\theta}$, such that $\overline{\theta} > a(\infty) b(0) + A a(\infty) + B b(0)$, and denote
\[
\zeta(t) = \frac{\theta(t)}{\theta(s)} + A a(t) + B b(t),
\]
then $\overline{\theta} \theta(t) > 0$, $0 < \zeta(t) \leq 1$.

Lemma 2.2. Suppose (H1) holds, $\int_0^\infty \frac{1}{p(s)} ds < +\infty$ and $\rho > 0$, then the BVP
\[
\begin{cases}
(p(t)x'(t))' + \psi(t) = 0, \quad t \in (0, +\infty), \\
x(t) - \beta_1 \lim_{t \to 0^+} p(t)x'(t) = \sum_{i=1}^{m-2} \gamma_i x(\eta_i), \\
x(t) + \beta_2 \lim_{t \to +\infty} p(t)x'(t) = \sum_{i=1}^{m-2} \delta_i x(\eta_i),
\end{cases}
\]
has a unique solution for any $\psi \in L(0, +\infty)$. Moreover, this unique solution can be expressed in the form
\[
\omega(t) = \int_0^\infty G(t,s)\psi(s)ds + A(\psi)a(t) + B(\psi)b(t),
\]
where $G(t,s)$ is defined as (2.1) and
\[
A(\psi) = 1 \int \frac{\gamma_i}{\delta_i} \int_0^\infty G(\eta_i, s)\psi(s)ds ds - \sum_{i=1}^{m-2} \gamma_i b(\eta_i),
\]
\[
B(\psi) = 1 \int \frac{\delta_i}{\gamma_i} \int_0^\infty G(\eta_i, s)\psi(s)ds ds - \sum_{i=1}^{m-2} \delta_i a(\eta_i).
\]
The proof is similar to [30], so we omit it.

Lemma 2.3. The solution defined by (2.4) satisfies $\omega(t) \leq \frac{\phi(t)}{\rho} \int_0^\infty \psi(s)ds$.

Proof. Since $\omega(t)$ is the unique solution of (2.4), By (2.2)–(2.4), we have
\[
\omega(t) \leq \int_0^\infty \frac{\theta(t)}{\theta(s)} \frac{\psi(s)}{\rho} ds + A(\psi) \int_0^\infty \frac{\psi(s)}{\rho} ds + B(\psi) \int_0^\infty \frac{\psi(s)}{\rho} ds \leq (\theta(t) + A(\psi) + B(\psi)) \int_0^\infty \frac{\psi(s)}{\rho} ds = \frac{\overline{\theta}}{\rho} \int_0^\infty \psi(s)ds.
\]
In this paper, the following space $X$ will be used in the study of BVP (1.1).

$$X = \left\{ x \in C[0, +\infty) : \lim_{t \to +\infty} x(t) \text{ exists} \right\}.$$  

Clearly $(X, \| \|)$ is a Banach space with the supremum norm $\| x \| = \sup_{t \in [0, +\infty)} |x(t)|$, see [31]. Let

$$K = \left\{ x \in X : x(t) \geq \frac{1}{2} \| x \|, \ t \in [0, +\infty) \right\}.$$

It is easy to see that $K$ is a cone in $X$.

Next we consider the following singular nonlinear boundary value problem:

\begin{equation}
\begin{cases}
(p(t)x'(t))' + \lambda(f(t, x(t) - \lambda \omega(t))' + \psi(t)) = 0, \ t \in (0, +\infty), \\
x(0) - \beta_1 \lim_{t \to 0^+} p(t)x'(t) = \sum_{i=1}^{m-2} \gamma_i x(\eta_i), \\
\lim_{t \to +\infty} x(t) + \beta_2 \lim_{t \to +\infty} p(t)x'(t) = \sum_{i=1}^{m-2} \delta_i x(\eta_i),
\end{cases}
\end{equation}

where $\lambda > 0$, $\omega(t)$ is defined in Lemma 2.2, $[z(t)]' = \max\{z(t), 0\}$. □

**Lemma 2.4.** If $x$ is a solution of BVP (2.5) with $x(t) > \lambda \omega(t)$ for any $t \in [0, +\infty)$, then $x(t) - \lambda \omega(t)$ is a positive solution of BVP (1.1).

**Proof.** In fact, if $x$ is a positive solution of BVP (2.5) such that $x(t) > \lambda \omega(t)$ for any $t \in [0, +\infty)$, then from BVP (2.5) and the definition of $[z(t)]'$, we have

\begin{equation}
\begin{cases}
(p(t)u'(t))' + \lambda(f(t, x(t) - \lambda \omega(t)) + \psi(t)) = 0, \ t \in (0, +\infty), \\
x(0) - \beta_1 \lim_{t \to 0^+} p(t)u'(t) = \sum_{i=1}^{m-2} \gamma_i u(\eta_i), \\
\lim_{t \to +\infty} x(t) + \beta_2 \lim_{t \to +\infty} p(t)u'(t) = \sum_{i=1}^{m-2} \delta_i u(\eta_i),
\end{cases}
\end{equation}

Let $u(t) = x(t) - \lambda \omega(t), \ t \in [0, +\infty)$, then $(p(t)u'(t))' = (p(t)u'(t))' + \lambda(p(t)\omega'(t))'$. Thus, (2.6) becomes

\begin{equation}
\begin{cases}
(p(t)u'(t))' + \lambda(f(t, u(t)) = 0, \ t \in (0, +\infty), \\
x(0) - \beta_1 \lim_{t \to 0^+} p(t)u'(t) = \sum_{i=1}^{m-2} \gamma_i u(\eta_i), \\
\lim_{t \to +\infty} u(t) + \beta_2 \lim_{t \to +\infty} p(t)u'(t) = \sum_{i=1}^{m-2} \delta_i u(\eta_i).
\end{cases}
\end{equation}

Then $u(t) = x(t) - \lambda \omega(t)$ is a positive solution of BVP (1.1). □

To overcome singularity, we consider the following approximate problem:

\begin{equation}
\begin{cases}
(p(t)x'(t))' + \lambda(f(t, x(t) - \lambda \omega(t))' + \frac{1}{n}) + \psi(t)) = 0, \ t \in (0, +\infty), \\
x(0) - \beta_1 \lim_{t \to 0^+} p(t)x'(t) = \sum_{i=1}^{m-2} \gamma_i x(\eta_i), \\
\lim_{t \to +\infty} x(t) + \beta_2 \lim_{t \to +\infty} p(t)x'(t) = \sum_{i=1}^{m-2} \delta_i x(\eta_i),
\end{cases}
\end{equation}

in which $n$ is a positive integer. Under the assumptions $(H_1) - (H_3)$, for any $n \in \mathbb{N}$ where $\mathbb{N}$ is a natural number set, we define a nonlinear integral operator $T_n : K \to X$ by

\begin{equation}
(T_n x)(t) = \lambda \int_0^t \left( s, |x(s) - \lambda \omega(s)|' + \frac{1}{n} + \psi(s) \right) ds + A(\lambda(f_n + \psi))a(t) + B(\lambda(f_n + \psi))b(t), \ t \in [0, +\infty),
\end{equation}

where

$$A(\lambda(f_n + \psi)) = \frac{1}{\Delta} \left| \sum_{i=1}^{m-2} \gamma_i \int_0^{\eta_i} G(n, s) (f(s, |x(s) - \lambda \omega(s)|' + \frac{1}{n} + \psi(s) ds) \rho - \sum_{i=1}^{m-2} \delta_i b(\eta_i) \right|.$$
\[ B(f_n + \psi) = \frac{1}{\Delta} \left( \sum_{i=1}^{m-1} \delta_i J_0 G(\eta_i, s) \left( f(s, [x(s) - \lambda \omega(s)]^+ + \frac{1}{n}) + \psi(s) \right) ds - \sum_{i=1}^{m-2} \delta_i a(\eta_i) \right). \]

Obviously, the existence of solutions to BVP (2.7) is equivalent to the existence of solutions in K for operator equation \( T_n x = x \) defined by (2.8).

In this paper, the proof of the main theorem is based on the fixed point theory in cone. We list the following lemmas which are needed in our study.

**Lemma 2.5** [31]. Let \( X \) be defined by (2.1) and \( M \subset X \). Then \( M \) is relatively compact in \( X \) if the following conditions hold:

1. \( M \) is uniformly bounded in \( X \);
2. the functions from \( M \) are equicontinuous on any compact interval of \([0, +\infty)\);
3. the functions from \( M \) are equiconvergent, that is, for any given \( \varepsilon > 0 \), there exists a \( T = T(\varepsilon) > 0 \) such that \( |x(t) - x(\pm \infty)| < \varepsilon \), for any \( t > T \), \( x \in M \).

**Lemma 2.6** [32]. Let \( P \) be a positive cone in a real Banach space \( E \). Denote \( P_r = \{ x \in P : \|x\| < r \} \), \( \mathcal{P}_{r,R} = \{ x \in P : r \leq \|x\| \leq R \} \), \( 0 < r < R < +\infty \). Let \( A : P \rightarrow P \) be a completely continuous operator. If the following conditions are satisfied:

1. \( \|Ax\| \leq \|x\| \), \( \forall x \in \partial P_r \);
2. there exists a \( x_0 \in \partial P_1 \), such that \( x \neq Ax + mx_0 \), \( \forall x \in \partial P_1 \), \( m > 0 \).

Then \( A \) has fixed points in \( \mathcal{P}_{r,R} \).

**Remark 2.1.** If (1) and (2) are satisfied for \( x \in \partial P_r \) and \( x \in \partial P_R \) respectively. Then Lemma 2.6 is still true.

**Lemma 2.7** [33]. Let \( P \) be a positive cone in a Banach space \( E \). \( \Omega_1 \) and \( \Omega_2 \) are bounded open sets in \( E, \theta \in \Omega_1, \Omega_1 \subset \Omega_2 \). \( A : P \cap \Omega_1 \rightarrow P \) is a completely continuous operator. If the following conditions are satisfied:

\[ \|Ax\| \leq \|x\|, \forall x \in P \cap \partial \Omega_1, \|Ax\| = \|x\|, \forall x \in P \cap \partial \Omega_2, \text{ or} \]
\[ \|Ax\| \geq \|x\|, \forall x \in P \cap \partial \Omega_1, \|Ax\| \leq \|x\|, \forall x \in P \cap \partial \Omega_2, \text{ then } A \text{ has at least one fixed point in } P \cap (\Omega_2 \setminus \Omega_1). \]

### 3. Main results

**Lemma 3.1.** Assume that \((H_1)- (H_3)\) hold. Then \( T_n : K \rightarrow K \) is a completely continuous operator for any fixed \( n \in \mathbb{N} \).

**Proof.** First we show that \( T_n : K \rightarrow X \) is well defined and \( T_n(K) \subset K \). For \( x \in K \), there exists \( r > 0 \) such that \( |x(t)| \leq r \), for \( t \in [0, +\infty) \), also \( |x(t) - \lambda \omega(t)| \leq |x(t)| \leq r \), \( t \in [0, +\infty) \). From \((H_2)\) and the definition of \( g \) and \( h \), we have

\[ S_{r,n} := \sup \left\{ g(u) + h(u) : \frac{1}{n} \leq u \leq r + 1 \right\} < +\infty. \]

Thus, by \((H_2)\) and \((H_3)\), for any \( t \in [0, +\infty) \), we know

\[
\lambda \int_0^{\infty} G(t,s) \left( f \left( s, [x(s) - \lambda \omega(s)]^+ + \frac{1}{n} \right) + \psi(s) \right) ds \\
\leq \lambda \int_0^{\infty} G(t,s) \left( \phi(s) \left( g \left( [x(s) - \lambda \omega(s)]^+ + \frac{1}{n} \right) \\
+ h \left( [x(s) - \lambda \omega(s)]^+ + \frac{1}{n} \right) \right) + \psi(s) \right) ds \\
\leq \lambda \int_0^{\infty} G(s,s) (\phi(s)S_{r,n} + \psi(s)) ds \\
\leq \lambda (S_{r,n} + 1) \int_0^{\infty} G(s,s) (\phi(s) + \psi(s)) ds \\
< +\infty.
\]

By (2.2), for \( i = 1, 2, \ldots, m - 2 \), we also have

\[
\lambda \int_0^{\infty} G(\eta_i,s) \left( f \left( s, [x(s) - \lambda \omega(s)]^+ + \frac{1}{n} \right) + \psi(s) \right) ds \\
\leq \lambda (S_{r,n} + 1) \int_0^{\infty} G(s,s) (\phi(s) + \psi(s)) ds < +\infty.
\]
Therefore,

\[
A(\lambda(f_n + \psi))a(t) = \frac{d}{dt} \left[ \sum_{i=1}^{m-2} \gamma_i \int_0^t G(\eta_i, s) \left( f \left( s, \left[ x(s) - \lambda \omega(s) \right]^+ + \frac{1}{n} \right) + \psi(s) \right) ds - \sum_{i=1}^{m-2} \delta_i b(\eta_i) \right] \\
\leq \frac{d}{dt} \left[ \sum_{i=1}^{m-2} \gamma_i \int_0^t G(\eta_i, s) \left( f \left( s, \left[ x(s) - \lambda \omega(s) \right]^+ + \frac{1}{n} \right) + \psi(s) \right) ds - \sum_{i=1}^{m-2} \delta_i b(\eta_i) \right] \\
\quad - \sum_{i=1}^{m-2} \delta_i \lambda(S_n + 1) \int_0^t G(s, s) (\phi(s) + \psi(s)) ds - \sum_{i=1}^{m-2} \delta_i \lambda(\eta_i, s) \\
\leq -\lambda \int_0^t G(s, s) (\phi(s) + \psi(s)) ds < +\infty,
\]

where

\[
\lambda = \frac{1}{\Delta} \left[ \sum_{i=1}^{m-2} \gamma_i \rho - \sum_{i=1}^{m-2} \delta_i \eta_i \right] - \sum_{i=1}^{m-2} \delta_i \lambda(\eta_i, s) \\
\Delta = \sum_{i=1}^{m-2} \gamma_i - \sum_{i=1}^{m-2} \delta_i.
\]

In the same way, we get

\[
B(\lambda(f_n + \psi))b(t) \leq \mathcal{B}b(0)\lambda(S_n + 1) \int_0^t G(s, s) (\phi(s) + \psi(s)) ds < +\infty,
\]

where

\[
\mathcal{B} = \frac{1}{\Delta} \left[ \sum_{i=1}^{m-2} \delta_i \rho - \sum_{i=1}^{m-2} \gamma_i \eta_i \right] - \sum_{i=1}^{m-2} \delta_i \lambda(\eta_i, s) \\
\Delta = \sum_{i=1}^{m-2} \delta_i - \sum_{i=1}^{m-2} \gamma_i.
\]

Hence, by (3.1)–(3.3), we can see that

\[
\lambda = \int_0^\infty G(t, s) \left( f \left( s, \left[ x(s) - \lambda \omega(s) \right]^+ + \frac{1}{n} \right) + \psi(s) \right) ds + A(\lambda(f_n + \psi))a(t) + B(\lambda(f_n + \psi))b(t) \\
\leq (1 + \mathcal{B}a(0) + \mathcal{B}b(0))\lambda(S_n + 1) \int_0^\infty G(s, s) (\phi(s) + \psi(s)) ds < +\infty, \quad t \in [0, +\infty).
\]

Then, \( T_n \chi \) is well defined for any \( \chi \in \mathcal{K} \). On the other hand, for any \( t, t_j \in [0, +\infty), t_j \to t \), by the continuity of \( G(t, s) \), we get

\[
G(t, s) \left( f \left( s, \left[ x(s) - \lambda \omega(s) \right]^+ + \frac{1}{n} \right) + \psi(s) \right) \to G(t, s) \left( f \left( s, \left[ x(s) - \lambda \omega(s) \right]^+ + \frac{1}{n} \right) + \psi(s) \right), \quad s \in [0, +\infty), \quad j \to +\infty.
\]

By (2.2), we know

\[
\int_0^\infty G(t, s) \left( f \left( s, \left[ x(s) - \lambda \omega(s) \right]^+ + \frac{1}{n} \right) + \psi(s) \right) ds \leq (S_n + 1) \int_0^\infty G(s, s) (\phi(s) + \psi(s)) ds < +\infty,
\]

\[
\int_0^\infty G(t, s) \left( f \left( s, \left[ x(s) - \lambda \omega(s) \right]^+ + \frac{1}{n} \right) + \psi(s) \right) ds \leq (S_n + 1) \int_0^\infty G(s, s) (\phi(s) + \psi(s)) ds < +\infty.
\]

So, by (H1), (3.4) and (3.5) and the Lebesgue dominated convergence theorem, we have

\[
\lim_{t_j \to t} \int_0^\infty G(t, s) \left( f \left( s, \left[ x(s) - \lambda \omega(s) \right]^+ + \frac{1}{n} \right) + \psi(s) \right) ds \\
= \int_0^\infty \lim_{t_j \to t} G(t, s) \left( f \left( s, \left[ x(s) - \lambda \omega(s) \right]^+ + \frac{1}{n} \right) + \psi(s) \right) ds \\
= \int_0^\infty G(t, s) \left( f \left( s, \left[ x(s) - \lambda \omega(s) \right]^+ + \frac{1}{n} \right) + \psi(s) \right) ds.
\]

Consequently, together with the continuity of \( a(t) \) and \( b(t) \), we have

\[
|T_n \chi(t) - T_n \chi(t)| \\
= \left| \int_0^t G(t, s) \left( f \left( s, \left[ x(s) - \lambda \omega(s) \right]^+ + \frac{1}{n} \right) + \psi(s) \right) ds \\
+ A(\lambda(f_n + \psi))a(t) + B(\lambda(f_n + \psi))b(t) \\
- \int_0^t G(t, s) \left( f \left( s, \left[ x(s) - \lambda \omega(s) \right]^+ + \frac{1}{n} \right) + \psi(s) \right) ds \\
- A(\lambda(f_n + \psi))a(t) - B(\lambda(f_n + \psi))b(t) \\
\right| \\
\leq \left| \int_0^t (G(t, s) - G(t, s)) \left( f \left( s, \left[ x(s) - \lambda \omega(s) \right]^+ + \frac{1}{n} \right) + \psi(s) \right) ds \right| \\
+ \left| A(\lambda(f_n + \psi))a(t) - A(t) + B(\lambda(f_n + \psi))b(t) - b(t) \right| \\
\to 0, \quad j \to +\infty.
\]
Therefore, $T_n x \in C(0, +\infty)$. In what follows, for any $T_j \in [0, +\infty)$, $T_j \to +\infty$, by (2.2), we have

$$G(T_j, s) \left( f \left( s, \lfloor x(s) - \lambda(\omega s) \rfloor^+ + \frac{1}{n} \right) + \psi(s) \right) \to G(s) \left( f \left( s, \lfloor x(s) - \lambda(\omega s) \rfloor^+ + \frac{1}{n} \right) + \psi(s) \right), \quad s \in [0, +\infty), \quad j \to +\infty.$$ 

Then by the Lebesgue dominated convergence theorem and the property of $a(t)$, $b(t)$, we also have

$$\lim_{j \to +\infty} (T_n x)(T_j) = \lambda \int_0^\infty G(s, s) \left( f \left( s, \lfloor x(s) - \lambda(\omega s) \rfloor^+ + \frac{1}{n} \right) + \psi(s) \right) ds + A(\lambda(f_a + \psi))a(t) + B(\lambda(f_a + \psi))b(t).$$

By Lemma 2.1 and (2.3), we get

$$\begin{align*}
(T_n x)(t) & \geq \lambda \partial(t) \int_0^\infty G(s, s) \left( f \left( s, \lfloor x(s) - \lambda(\omega s) \rfloor^+ + \frac{1}{n} \right) + \psi(s) \right) ds + A(\lambda(f_a + \psi))a(t) + B(\lambda(f_a + \psi))b(t) \\
& \geq \frac{\lambda \partial(t)}{2} \int_0^\infty G(s, s) \left( f \left( s, \lfloor x(s) - \lambda(\omega s) \rfloor^+ + \frac{1}{n} \right) + \psi(s) \right) ds + A(\lambda(f_a + \psi))a(t) + B(\lambda(f_a + \psi))b(t) \\
& \geq \frac{\lambda \partial(t)}{2} \int_0^\infty G(s, s) \left( f \left( s, \lfloor x(s) - \lambda(\omega s) \rfloor^+ + \frac{1}{n} \right) + \psi(s) \right) ds + A(\lambda(f_a + \psi))a(t) + B(\lambda(f_a + \psi))b(t) \\
& \geq \frac{\lambda \partial(t)}{2} \int_0^\infty G(s, s) \left( f \left( s, \lfloor x(s) - \lambda(\omega s) \rfloor^+ + \frac{1}{n} \right) + \psi(s) \right) ds + A(\lambda(f_a + \psi))a(t) + B(\lambda(f_a + \psi))b(t) \\
& \geq \frac{\lambda \partial(t)}{2} \int_0^\infty G(s, s) \left( f \left( s, \lfloor x(s) - \lambda(\omega s) \rfloor^+ + \frac{1}{n} \right) + \psi(s) \right) ds + A(\lambda(f_a + \psi))a(t) + B(\lambda(f_a + \psi))b(t) \\
& \geq \frac{\lambda \partial(t)}{2} \int_0^\infty G(s, s) \left( f \left( s, \lfloor x(s) - \lambda(\omega s) \rfloor^+ + \frac{1}{n} \right) + \psi(s) \right) ds + A(\lambda(f_a + \psi))a(t) + B(\lambda(f_a + \psi))b(t).
\end{align*}$$

Combining (3.7) with (3.8), we have $(T_n x)(t) \geq \frac{\lambda \partial(t)}{2} \| T_n x \|$, for any $t \in [0, +\infty)$. Therefore, $T_n(K) \subseteq K$.

Next, for any positive integers $n$, $k \in \mathbb{N}$, we define an operator $T_{nk} : K \to X$ by

$$\begin{align*}
(T_{nk} x)(t) & = \lambda \int_0^\infty G(t, s) \left( f \left( s, \lfloor x(s) - \lambda(\omega s) \rfloor^+ + \frac{1}{n} \right) + \psi(s) \right) ds + A_k(\lambda(f_a + \psi))a(t) + B_k(\lambda(f_a + \psi))b(t), \quad t \in [0, +\infty),
\end{align*}$$

where

$$A_k(\lambda(f_a + \psi)) = \frac{1}{\Lambda} \left( \sum_{i=1}^{m-2} \gamma_i \int_0^1 G(\eta_i, s) \left( f \left( s, \lfloor x(s) - \lambda(\omega s) \rfloor^+ + \frac{1}{n} \right) + \psi(s) \right) ds \right) - \frac{\rho}{\Lambda} \sum_{i=1}^{m-2} \delta_i \int_0^1 G(\eta_i, s) \left( f \left( s, \lfloor x(s) - \lambda(\omega s) \rfloor^+ + \frac{1}{n} \right) + \psi(s) \right) ds.$$
Though calculation, we obtain

\[
B_k(\lambda(f_u + \psi)) = \frac{1}{\Delta} \left| \sum_{i=1}^{m-2} \delta \int_0^\infty G(\eta, s) f(s, [x(s) - \lambda \omega(s)]) + \frac{1}{n} + \psi(s)) ds - \frac{\sum_{i=1}^{m-2} \delta a(\eta)}{\sum_{i=1}^{m-2} \delta a(\eta)} \right| \]

As in the above discussion, we can prove that \( T_{n,k} : K \to X \) is well defined and \( T_{n,k}(K) \subseteq K \). In what follows, we will prove that \( T_{n,k} : K \to K \) is completely continuous, for each \( k \geq 1 \). Firstly, we show that \( T_{n,k} : K \to K \) is continuous. Let \( x_n, x \in K \) are such that \( \|x_n - x\| \to 0 \) as \( \nu \to +\infty \). By (3.9) and (H3), we know

\[
\left| \int_0^\infty G(s, s)(\phi(s) + \psi(s)) ds \right| < +\infty.
\]

where \( \mathcal{S}_{n,\nu} := \sup \{g(u) + h(u) : 1/2 \leq u \leq \nu + 1 \} < +\infty \) (by (H2)), \( \nu' \) is a real number such that \( \nu' \geq \max \{\|x\|, \|x_n\|\} \). Denote

\[
A_{k,n}(\lambda(f_u + \psi)) = \frac{1}{\Delta} \left| \sum_{i=1}^{m-2} \delta \int_0^\infty G(\eta, s) f(s, [x(s) - \lambda \omega(s)]) + \frac{1}{n} + \psi(s)) ds - \frac{\sum_{i=1}^{m-2} \delta b(\eta)}{\sum_{i=1}^{m-2} \delta b(\eta)} \right| \]

\[
B_{k,n}(\lambda(f_u + \psi)) = \frac{1}{\Delta} \left| \sum_{i=1}^{m-2} \delta \int_0^\infty G(\eta, s) f(s, [x(s) - \lambda \omega(s)]) + \frac{1}{n} + \psi(s)) ds - \frac{\sum_{i=1}^{m-2} \delta a(\eta)}{\sum_{i=1}^{m-2} \delta a(\eta)} \right| \]

Though calculation, we obtain

\[
|A_{k,n}(\lambda(f_u + \psi)) - A_k(\lambda(f_u + \psi))|a(t) \leq \frac{a(\infty)}{\Delta} \left| \sum_{i=1}^{m-2} \delta \int_0^\infty G(s, s) f(s, [x(s) - \lambda \omega(s)]) + \frac{1}{n} + \psi(s)) ds - \frac{\sum_{i=1}^{m-2} \delta b(\eta)}{\sum_{i=1}^{m-2} \delta b(\eta)} \right| \]

\[
\leq \frac{A\lambda}{\Delta} \int_0^\infty G(s, s) \left( \phi(s) \left( f(s, [x(s) - \lambda \omega(s)]) + \frac{1}{n} + \psi(s)) ds - 2\lambda \int_0^\infty G(s, s) \left( \phi(s) + \psi(s)) \right) ds \right)
\]

\[
\leq 2A(\infty)\lambda(\mathcal{S}_{n,\nu} + 1) \int_0^\infty G(s, s)(\phi(s) + \psi(s)) ds < +\infty.
\]

In the same way, we get

\[
|B_{k,n}(\lambda(f_u + \psi)) - B_k(\lambda(f_u + \psi))|b(t) \leq 2Bb(0)\lambda(\mathcal{S}_{n,\nu} + 1) \int_0^\infty G(s, s)(\phi(s) + \psi(s)) ds < +\infty.
\]

From (3.10), (3.12), (3.13), for any \( \varepsilon > 0 \), by (H3), there exists a sufficiently large \( A_0 (A_0 > 1/k) \), such that

\[
\max \left\{ 1, \frac{a(\infty)A}{\lambda(\mathcal{S}_{n,\nu} + 1)} \int_0^\infty G(s, s)(\phi(s) + \psi(s)) ds \right\} < \frac{\varepsilon}{12}
\]

On the other hand, by the continuity of \( f(s, u + \frac{1}{n}) \) on \( [1/k, A_0] \times [0, r'] \), for the above \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that for any \( s \in [1/k, A_0] \) and \( u, v \in [0, r'] \), when \( |u - v| = |(u + \frac{1}{n}) - (v + \frac{1}{n})| < \delta \), we have

\[
\left| f(s, u + \frac{1}{n}) - f(s, v + \frac{1}{n}) \right| < \varepsilon \left( \max \left\{ 1, \frac{a(\infty)A}{\lambda(\mathcal{S}_{n,\nu} + 1)} \right\} \int_0^\infty G(s, s) ds \right)^{-1}.
\]

\[ \text{(3.14)} \]
From \(|x_n - x| \to 0\ (n \to +\infty)\) and the definition of the norm \(\| \cdot \|\) in the space \(X\), for the above \(\delta > 0\), there exists a sufficiently large nature number \(V_0\) such that when \(v > V_0\), for all \(s \in [1/k, A_0]\), we have

\[
\left| \left( x_n(s) - \lambda \omega(s) \right)^* + \frac{1}{n} - \left( x(s) - \lambda \omega(s) \right)^* + \frac{1}{n} \right| \\
\leq \frac{1}{2} \left| x_n(s) - \lambda \omega(s) + x(s) - \lambda \omega(s) \right| - 2 \left| x(s) - \lambda \omega(s) \right| + \frac{1}{2} \left| x(s) - x_n(s) \right| \\
\leq \frac{1}{2} \left| x_n(s) - x(s) \right| \\
= \frac{1}{2} \left| x_n(s) - x(s) \right| \\
= \frac{1}{2} \left| x_n(s) - x(s) \right| \\n(3.15)
\]

Hence, by (3.14)-(3.16), when \(v > V_0\), we have the following inequality

\[
|A_{k_0}(\lambda(f_n + \psi)) - A_k(\lambda(f_n + \psi))| \leq |A_{k_0}(\lambda(f_n + \psi)) - A_k(\lambda(f_n + \psi))| + |A_k(\lambda(f_n + \psi))| \leq \frac{\varepsilon}{3}.
\]

Using the same method as (3.17), when \(v > V_0\), we get

\[
|B_{k_0}(\lambda(f_n + \psi)) - B_k(\lambda(f_n + \psi))| \leq \frac{\varepsilon}{3}.
\]

Then, by (3.17) and (3.18) and the above discussion, when \(v > V_0\), we obtain

\[
|\langle T_{n+k}x_n(x) \rangle(t) - \langle T_{n+k}x \rangle(t) | \\
\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\]

This implies that the operator \(T_{n+k} : K \to K\) is continuous for any natural numbers \(n, k\).
It what follows, we need to prove that $T_{n,k} : K \to K$ is a compact operator for natural numbers $n, k$. Let $M$ be any bounded subset of $K$. Then there exists a constant $R > 0$ such that $\|x\| \leq R$ for any $x \in M$. By (3.9), (H2) and (H1), for any $x \in M$, we have

\[
\left| \lambda \int_0^\infty G(t,s) \left( f \left( s, |x(s) - \lambda \omega(s)|^n + \frac{1}{n} \right) + \psi(s) \right) ds \right| \\
\leq \lambda \int_0^\infty G(s,s) \left( \phi(s) \left( f \left( s, |x(s) - \lambda \omega(s)|^n + \frac{1}{n} \right) + \psi(s) \right) ds \\
+ h \left( |x(s) - \lambda \omega(s)|^n + \frac{1}{n} \right) + \psi(s) \right) ds \\
\leq \lambda \int_0^\infty G(s,s)(\phi(s)S_{n,k} + \psi(s))ds \\
\leq \lambda(S_{n,k} + 1) \int_0^\infty G(s,s)(\phi(s) + \psi(s))ds < +\infty,
\]

where $S_{n,k} := \sup \{g(u) + h(u) : \frac{1}{n} < u \leq R + 1\}$. By the similar proof as (3.2) and (3.3), for any $t \in [0, +\infty)$, we have

\[
A_k(\lambda(f_n + \psi))a(t) \leq \overline{A}(\infty)\lambda(S_{n,k} + 1) \int_0^\infty G(s,s)(\phi(s) + \psi(s))ds < +\infty,
\]

\[
B_k(\lambda(f_n + \psi))b(t) \leq \overline{B}(0)\lambda(S_{n,k} + 1) \int_0^\infty G(s,s)(\phi(s) + \psi(s))ds < +\infty.
\]

Therefore, from (3.18) and (3.19), $T_{n,k}M$ is bounded in $K$.

Given $\frac{1}{2} > 0$, for any $x \in M$ and $t, t' \in [0, \frac{1}{2}]$, by (3.9), we get

\[
\left| \lambda \int_0^\infty G(t,s) \left( f \left( s, |x(s) - \lambda \omega(s)|^n + \frac{1}{n} \right) + \psi(s) \right) ds \\
- \lambda \int_0^\infty G(t',s) \left( f \left( s, |x(s) - \lambda \omega(s)|^n + \frac{1}{n} \right) + \psi(s) \right) ds \right| \\
\leq \lambda \int_0^\infty \left| G(t,s) + G(t',s) \right| \left( f \left( s, |x(s) - \lambda \omega(s)|^n + \frac{1}{n} \right) + \psi(s) \right) ds \\
\leq 2\lambda(S_{n,k} + 1) \int_0^\infty G(s,s)(\phi(s) + \psi(s))ds < +\infty
\]

and so, for any $\epsilon' > 0$, we can find a sufficiently large $H_0$ ($H_0 > \frac{1}{2}$) such that

\[
\lambda(S_{n,k} + 1) \int_0^{H_0} G(s,s)(\phi(s) + \psi(s))ds < \frac{\epsilon'}{12}.
\]

By the uniformly continuity of $G(t,s)$ on $[0, 1] \times [\frac{1}{2}, H_0]$, for the above $\epsilon' > 0$, there exists $\delta' > 0$ such that for any $t, t' \in [0, 1], s \in [\frac{1}{2}, H_0]$ and $|t - t'| < \delta'$, we have

\[
|G(t,s) - G(t',s)| < \frac{\epsilon'}{6} \left( \lambda(S_{n,k} + 1) \int_0^{H_0} (\phi(s) + \psi(s))ds \right)^{-1}.
\]

Therefore, we get

\[
\left| \lambda \int_0^\infty G(t,s) \left( f \left( s, |x(s) - \lambda \omega(s)|^n + \frac{1}{n} \right) + \psi(s) \right) ds \\
- \lambda \int_0^\infty G(t',s) \left( f \left( s, |x(s) - \lambda \omega(s)|^n + \frac{1}{n} \right) + \psi(s) \right) ds \right| \\
\leq \lambda \int_0^{H_0} |G(t,s) - G(t',s)| \left( f \left( s, |x(s) - \lambda \omega(s)|^n + \frac{1}{n} \right) + \psi(s) \right)ds \\
+ \lambda \int_0^\infty |G(t,s) - G(t',s)| \left( f \left( s, |x(s) - \lambda \omega(s)|^n + \frac{1}{n} \right) + \psi(s) \right)ds \\
\leq \frac{\epsilon'}{6} + 2\lambda(S_{n,k} + 1) \int_0^\infty G(s,s)(\phi(s) + \psi(s))ds < \frac{\epsilon'}{3}.
\]
Also by the uniformly continuity of \( a(t), b(t) \) on \([0, \bar{\tau}]\), for the above \( \epsilon' > 0 \), there exists \( \delta'' > 0 \) such that for any \( t, t' \in [0, \bar{\tau}] \) and \( |t - t'| < \delta'' \), we have
\[
|a(t) - a(t')| \leq \frac{\epsilon'}{3} \left( A_{\lambda}(S_{k,n} + 1) \int_0^\infty G(s,s)(\phi(s) + \psi(s))\,ds \right)^{-1},
\]
\[
|b(t) - b(t')| \leq \frac{\epsilon'}{3} \left( B_{\lambda}(S_{k,n} + 1) \int_0^\infty G(s,s)(\phi(s) + \psi(s))\,ds \right)^{-1}.
\]
(3.21)

By (3.20) and (3.21), for the above \( \epsilon' > 0 \), let \( \delta_0 = \min\{\delta', \delta''\} \), then for any \( t, t' \in [0, \bar{\tau}] \) with \( |t - t'| < \delta_0 \), and for any \( x \in M \), we have
\[
|T_{n,k}x(t) - T_{n,k}x(t')| = \lambda \int_0^\infty G(t,s) \left( f(s, [x(s) - \lambda \omega(s)] + \frac{1}{n}) + \psi(s) \right) \, ds + A_k(\lambda(f'_n + \psi))(a(t) - a(t'))
\]
\[
= \lambda \int_0^\infty G(t,s) \left( f(s, [x(s) - \lambda \omega(s)] + \frac{1}{n}) + \psi(s) \right) \, ds - A_k(\lambda(f'_n + \psi))(a(t) - a(t'))
\]
\[
\leq \lambda \int_0^\infty G(t,s) - \bar{G}(s) \left( f(s, [x(s) - \lambda \omega(s)] + \frac{1}{n}) + \psi(s) \right) \, ds + A_k(\lambda(f'_n + \psi))(a(t) - a(t'))
\]
\[
\leq \lambda \int_0^\infty G(t,s) - \bar{G}(s) \left( f(s, [x(s) - \lambda \omega(s)] + \frac{1}{n}) + \psi(s) \right) \, ds + A_k(\lambda(f'_n + \psi))(a(t) - a(t'))
\]
< \epsilon'.

So, \( \{T_{n,k} : x \in M\} \) are equicontinuous on \([0, \bar{\tau}]\). Since \( \bar{\tau} > 0 \) is arbitrary, \( \{T_{n,k} : x \in M\} \) are locally equicontinuous on \([0, +\infty)\). Let \( T_{n,k}(x) = \lim_{t \to +\infty} T_{n,k}x(t) \), by a simple calculation, we can see that \( \lim_{t \to +\infty} T_{n,k}(x) = +\infty \), so we get
\[
|T_{n,k}(x)(t) - T_{n,k}(x)(t')| \leq \lambda \int_0^\infty G(t,s) \left( f(s, [x(s) - \lambda \omega(s)] + \frac{1}{n}) + \psi(s) \right) \, ds + A_k(\lambda(f'_n + \psi))(a(t) - a(t'))
\]
\[
\leq \lambda \int_0^\infty G(t,s) - \bar{G}(s) \left( f(s, [x(s) - \lambda \omega(s)] + \frac{1}{n}) + \psi(s) \right) \, ds + A_k(\lambda(f'_n + \psi))(a(t) - a(t'))
\]
< \epsilon'.

By the similar method as for (3.20), we get that, for any \( \bar{\tau} > 0 \), there exists \( N' \) such that, when \( t > N' \), it is true that
\[
\lambda \int_0^\infty G(t,s) \left( f(s, [x(s) - \lambda \omega(s)] + \frac{1}{n}) + \psi(s) \right) \, ds < \frac{\rho}{\delta}.
\]

Together with the continuity of \( a(t), b(t) \) on \([0, +\infty)\), we get that for the above \( \bar{\tau} > 0 \), there exists \( N' \) such that, when \( t > N' \), we have \( |T_{n,k}x(t) - T_{n,k}x(+\infty)| < \bar{\tau} \). Hence, the functions \( \{T_{n,k} : x \in M\} \) are equiuniform at \(+\infty\), which implies that \( \{T_{n,k} : x \in M\} \) is relatively compact (by Lemma 2.5). Therefore, we get that the operator \( T_{n,k} : K \to K \) is completely continuous for natural numbers \( n \).

Finally, we show that \( T_n : K \to K \) is a completely continuous operator. For any \( t \in [0, +\infty) \) and \( x \in S = \{x \in K : \|x\| \leq 1\} \), by (2.8) and (3.9), we have
\[
\lambda \int_0^\infty G(s,s) \left( f(s, [x(s) - \lambda \omega(s)] + \frac{1}{n}) + \psi(s) \right) \, ds \leq \lambda \int_0^\infty G(s,s) \left( \phi(s) \left( [x(s) - \lambda \omega(s)] + \frac{1}{n} \right) + \psi(s) \right) \, ds \leq \lambda \int_0^\infty G(s,s)(\phi(s)S_{1,n} + \psi(s))\,ds \to 0, \quad k \to +\infty.
\]
(3.22)

where \( S_{1,n} : = \sup\{g(u) + h(u) : \frac{1}{n} \leq u \leq 2\} < +\infty \). And then we get
\[
|A_k(\lambda(f'_n + \psi)) - A_k(\lambda(f'_n + \psi))(a(t))| \leq \frac{a(\infty)}{\lambda} \left[ \sum_{i=1}^{m-1} \frac{\beta_i}{\sum_{i=1}^{m-1} \alpha_i} \right] \left( \sum_{i=1}^{m-1} \frac{\beta_i}{\sum_{i=1}^{m-1} \alpha_i} \right) |(f(s, [x(s) - \lambda \omega(s)] + \frac{1}{n}) + \psi(s))\,ds \leq 2A_k(\lambda(S_{1,n} + 1)a(\infty)) \int_0^\infty G(s,s)(\phi(s) + \psi(s))\,ds \to 0, \quad k \to +\infty.
\]
(3.23)
Using the similar method, we also get
\begin{equation}
|B(\hat{\lambda}(f_n + \psi)) - B_k(\hat{\lambda}(f_n + \psi))|b(t) \leq 2B\hat{\lambda}(S_{t, n} + 1)b(0) \int_0^t G(s, s)(\psi(s) + \psi(s))ds \to 0, \quad k \to +\infty.
\end{equation}

(3.24)

Together (3.22)–(3.24) imply that
\begin{equation}
\|T_n - T_{n,k}\| = \sup_{x \in \mathbb{R}}||x_n x - T_{n,k} x|| \to 0, \quad k \to +\infty.
\end{equation}

Therefore, by $T_{n,k} : K \to K$ is a completely continuous operator, we get that $T_n : K \to K$ is a completely continuous operator. □

**Theorem 3.1.** Assume that $(H_4)_1$ and $(H_5)$ hold and $f$ satisfies the following condition:

$(H_4)_2$ There exists $[a, b] \subset (0, +\infty)$ such that
\begin{equation}
\lim_{u \to +\infty} \frac{f(t, u)}{u} = +\infty.
\end{equation}

Then there exists $\bar{t} > 0$ such that BVP (1.1) has at least one positive solution for any $\hat{\lambda} \in (0, \bar{t})$.

**Proof.** Choose $r_1 > \max \left\{ 4, \frac{r}{p} \int_0^\infty \psi(s)ds \right\}$, where $\tilde{a}$ is defined as (2.3), let
\begin{equation}
\bar{\lambda} = \min \left\{ 1, \left( 1 + a(\infty) A + b(0) B \right) \int_0^\infty G(s, s) \left( \phi(s) \left( g(\zeta(s)) + \tilde{h}(r_1) \right) + \psi(s) \right) ds \right\},
\end{equation}

where $\tilde{h}(r) = \sup_{u \in \mathbb{R}} \tilde{h}(u), A$ and $B$ are defined by (3.2) and (3.3). Let $K_{r_1} = \{ x \in K : \|x\| \leq r_1 \}$. For any $x \in \partial K_1, t \in [0, +\infty)$, by the definition of $\| \cdot \|$ and Lemma 2.3, we have
\begin{equation}
|\lambda(t) - \hat{\lambda} \omega(t)|^+ \leq |x(t)| \leq \|x\| \leq r_1,
\end{equation}

\begin{equation}
x(t) - \hat{\lambda} \omega(t) \geq x(t) - \frac{\tilde{a}(t)}{\rho} \int_0^\infty \psi(s)ds \geq x(t) - \frac{\tilde{a}(t)}{\rho r_1} \int_0^\infty \psi(s)ds \geq x(t) - \frac{2\tilde{a}(t)}{\rho r_1} \int_0^\infty \psi(s)ds \geq x(t) - \frac{x(t)}{2} \geq \frac{\omega(t)}{4} \geq \frac{\omega(t)}{4}.
\end{equation}

For any $\hat{\lambda} \in (0, \bar{t})$, we have
\begin{equation}
\|T_n x(t)\| \leq \bar{\lambda} \int_0^\infty G(t, s) \left( \left| f \left( s, |x(s) - \hat{\lambda} \omega(s)|^+ + \frac{1}{n} \right) \right| + \psi(s) \right) ds
\end{equation}

\begin{equation}
+ A(\hat{\lambda}(f_n + \psi))a(t) + B(\hat{\lambda}(f_n + \psi))b(t)
\leq \bar{\lambda} \int_0^\infty G(s, s) \left( \phi(s) \left( \left| x(s) - \hat{\lambda} \omega(s) \right|^+ + \frac{1}{n} \right) + h \left( \left| x(s) - \hat{\lambda} \omega(s) \right|^+ + \frac{1}{n} \right) \right) \psi(s)ds
\end{equation}

\begin{equation}
+ A(\hat{\lambda}(f_n + \psi))a(\infty) + B(\hat{\lambda}(f_n + \psi))b(0)
\leq \bar{\lambda} \left( 1 + \tilde{A}(\infty) + \tilde{B}(0) \right) \int_0^\infty G(s, s) \left( \phi(s) \left( g(\zeta(s)) + \tilde{h}(r_1) \right) + \psi(s) \right) ds
\end{equation}

\begin{equation}
\leq r_1.
\end{equation}

Thus,
\begin{equation}
\|T_n x\| \leq \|x\|, \quad \text{for any } x \in \partial K_{r_1}.
\end{equation}

On the other hand, by the inequality in $(H_4)_1$, choose $l$ such that $\lambda \hat{\sigma} t \int_0^\infty G(s, s)ds > 4, \tau = \min_{t \in [0, 1]} \tau(t)$, then there exists $N^* > 0$ such that
\begin{equation}
f(t, u) \geq tu, \quad u \geq N^*, \quad t \in [a, b].
\end{equation}

Let
\begin{equation}
r_2 = \max \left\{ r_1, \frac{\tilde{a}(t)}{\rho} \right\}, \quad \zeta = \min_{t \in [0, 1]} \zeta(t), \quad K_{r_2} = \{ x \in K : \|x\| < r_2 \}. \quad \text{Take } q_1 \equiv 1 \in \partial K_1, K_1 = \{ x \in K : \|x\| < 1 \}. \quad \text{For any } x \in \partial K_{r_2}, \mu > 0, n \in \mathbb{N}, \text{ we will show}
\end{equation}

\begin{equation}
x \neq T_n x + \mu q_1.
\end{equation}

Otherwise, there exist $x_0 \in \partial K_{r_2}$ and $\mu_0 > 0$, such that $x_0 = T_n x_0 + \mu_0 q_1$. From $x_0 \in \partial K_{r_2}$, we know that $\|x_0\| = r_2$, then, for $t \in [a, b]$, we have
\begin{equation}
x_0(t) - \hat{\lambda} \omega(t) \geq x_0(t) - \frac{\tilde{a}(t)}{\rho} \int_0^\infty \psi(s)ds \geq x_0(t) - \frac{\tilde{a}(t)}{\rho r_2} \int_0^\infty \psi(s)ds \geq x_0(t) - \frac{2\tilde{a}(t)}{\rho r_2} \int_0^\infty \psi(s)ds \geq x_0(t) - \frac{x_0(t)}{2} \geq \frac{\omega(t)}{4} \geq \frac{\omega(t)}{4}.
\end{equation}

\begin{equation}
N^*.
\end{equation}
Hence, we conclude that
\[
x_0(t) = \lambda \int_0^\infty G(t, s) \left( f(s, [x_0(s) - \lambda \omega(s)]^n + \frac{1}{n}) + \psi(s) \right) ds \\
+ A(\lambda f_n + \psi))a(t) + B(\lambda f_n + \psi)b(t) + \mu_0
\]
\[
\geq \lambda \int_0^\infty \theta(t)G(t, s) \left( f \left( s, [x_0(s) - \lambda \omega(s)]^n + \frac{1}{n} \right) + \psi(s) \right) ds + \mu_0
\]
\[
\geq \lambda \int_0^b \theta(t)G(s, x_0) \frac{kr_2}{4} ds + \mu_0
\]
\[
\geq \frac{\lambda \theta(t) r_2}{4} \int_a^b G(s, x_0) ds + \mu_0
\]
\[
\geq r_2 + \mu_0 > r_2.
\]
This implies that \( r_2 > r_2 \), which is a contradiction. This yields that (3.27) holds.

It follows from the above discussion, (3.26) and (3.27), Lemmas 2.6 and 3.1, that for any \( n \in \mathbb{N} \), \( \lambda \in (0, 7) \), \( T_\lambda \) has a fixed point \( x_n \in K_{r_1} \setminus K_{r_1} \).

Let \( \{x_n\}_{n=1}^\infty \) be the sequence of solutions of BVP (2.7). It is easy to see that they are uniformly bounded. From \( x_n \in K_{r_1} \setminus K_{r_1} \), by the similar discussion as (3.25), we know that
\[
[x_n(t) - \lambda \omega(t)]^n \leq x_0(t) \leq ||x_n|| \leq r_2, \quad t \in [0, +\infty).
\]
\[
x_n(t) - \lambda \omega(t) \geq x_0(t) - \frac{\lambda \alpha(t)}{\rho} \int_0^\infty \psi(s) ds \geq x_0(t) - \frac{\lambda \alpha(t)}{\rho} \int_0^\infty \psi(s) ds \geq x_0(t) - \frac{2\lambda x_0(t)}{\rho \Gamma_1} \int_0^\infty \psi(s) ds \geq \frac{x_0(t)}{2}
\]
\[
\geq \frac{\zeta(t)}{2}, \quad t \in [0, +\infty).
\]
(3.28)

Next, given \( \varpi > 0 \), we will prove that \( \{x_n\}_{n=1}^\infty \) are equicontinuous on \([0, \varpi]\). For any \( \varpi > 0 \), by \( \int_0^\infty G(s, s) (\phi(s)g(z(s)) + \bar{h}(r_2)) + \psi(s) ds < +\infty \), where \( \bar{h}(r_2) = \sup(h(u) : 0 \leq u \leq r_2 + 1) \), we can find a sufficiently large \( \bar{H}_0 > 0 \) such that
\[
\lambda \int_0^{\bar{H}_0} G(s, s) (\phi(s)g(z(s)) + \bar{h}(r_2)) + \psi(s) ds < \frac{\varpi}{2}.
\]
By the uniformly continuity of \( G(t, s) \) on \([0, \varpi] \times [0, \bar{H}_0]\), for the above \( \varpi > 0 \), there exists \( \bar{\varpi} > 0 \) such that for any \( t, t' \in [0, \bar{\varpi}], s \in [0, \bar{H}_0] \) and \( |t - t'| < \bar{\varpi} \), we have
\[
|G(t, s) - G(t', s)| < \frac{\varpi}{6} \left( \lambda \int_0^{\bar{H}_0} (\phi(s)g(z(s)) + \bar{h}(r_2)) + \psi(s) ds \right)^{-1}.
\]
Therefore, for any \( n \in \mathbb{N} \), \( t, t' \in [0, \bar{\varpi}], s \in [0, \bar{H}_0] \) and \( |t - t'| < \bar{\varpi} \), we get
\[
\left| \lambda \int_0^\infty G(t, s) \left( f \left( s, [x_n(s) - \lambda \omega(s)]^n + \frac{1}{n} \right) + \psi(s) \right) ds \\
- \lambda \int_0^\infty G(t', s) \left( f \left( s, [x_n(s) - \lambda \omega(s)]^n + \frac{1}{n} \right) + \psi(s) \right) ds \right|
\]
\[
\leq \lambda \int_0^{\bar{H}_0} |G(t, s) - G(t', s)| \left( f \left( s, [x_n(s) - \lambda \omega(s)]^n + \frac{1}{n} \right) + \psi(s) \right) ds \\
+ \lambda \int_0^\infty |G(t, s) - G(t', s)| \left( f \left( s, [x_n(s) - \lambda \omega(s)]^n + \frac{1}{n} \right) + \psi(s) \right) ds
\]
\[
\leq \lambda \int_0^{\bar{H}_0} |G(t, s) - G(t', s)| (\phi(s)g(z(s)) + \bar{h}(r_2)) + \psi(s)) ds \\
+ \lambda \int_0^\infty |G(t, s) - G(t', s)| (\phi(s)g(z(s)) + \bar{h}(r_2)) + \psi(s)) ds
\]
\[
\leq \frac{\varpi}{6} + 2\lambda \int_0^\infty G(s, s) (\phi(s)g(z(s)) + \bar{h}(r_2)) + \psi(s)) ds < \frac{\varpi}{3}.
\]
(3.29)

For
\[
A_n(\lambda f_n + \psi) = \frac{1}{\Delta} \left| \sum_{i=1}^{m-2} \frac{\lambda}{\rho} \int_0^\infty G(\eta_i, s) (f(s, [x_n(s) - \lambda \omega(s)]^n + \frac{1}{n} \psi(s) ds - \sum_{i=1}^{m-2} \frac{\lambda}{\rho} \int_0^\infty G(\eta_i, s) (f(s, [x_n(s) - \lambda \omega(s)]^n + \frac{1}{n} \psi(s) ds
\]
\[
B_n(\lambda f_n + \psi) = \frac{1}{\Delta} \left| \sum_{i=1}^{m-2} \frac{\lambda}{\rho} \int_0^\infty G(\eta_i, s) (f(s, [x_n(s) - \lambda \omega(s)]^n + \frac{1}{n} \psi(s) ds - \sum_{i=1}^{m-2} \frac{\lambda}{\rho} \int_0^\infty G(\eta_i, s) (f(s, [x_n(s) - \lambda \omega(s)]^n + \frac{1}{n} \psi(s) ds
\]
\[
\rho - \sum_{i=1}^{m-2} \frac{\lambda}{\rho} \frac{\partial b(\eta_i)}{\eta_i} \right|.
\]
\[
\rho - \sum_{i=1}^{m-2} \frac{\lambda}{\rho} \frac{\partial a(\eta_i)}{\eta_i} \right|.
\]
through a simple calculation, we get
\[
A_n(\lambda(f_n + \psi)) \leq \overline{A} \int_0^\infty G(s,s)(\phi(s)(g(\zeta(s)) + \bar{h}(r_2))) + \psi(s))ds < +\infty,
\]
\[
B_n(\lambda(f_n + \psi)) \leq \overline{B} \int_0^\infty G(s,s)(\phi(s)(g(\zeta(s)) + \bar{h}(r_2))) + \psi(s))ds < +\infty,
\]
where \(\overline{A}\) and \(\overline{B}\) are defined as (3.2) and (3.3). So by the uniformly continuity of \(a(t), b(t)\) on \([0, \overline{r}]\), for the above \(\overline{r} > 0\), there exists \(\overline{r}^* > 0\) such that, for any \(t, t' \in [0, \overline{r}]\) and \(|t - t'| < \overline{r}^*\), we have
\[
|a(t) - a(t')| < \frac{\overline{A}}{3} \left( \int_0^\infty G(s,s)(\phi(s)(g(\zeta(s)) + \bar{h}(r_2))) + \psi(s))ds \right)^{-1},
\]
\[
|b(t) - b(t')| < \frac{\overline{B}}{3} \left( \int_0^\infty G(s,s)(\phi(s)(g(\zeta(s)) + \bar{h}(r_2))) + \psi(s))ds \right)^{-1}.
\]
(3.30)

Then, by (3.29) and (3.30), for the above \(\overline{r}^* > 0\), let \(\delta_0 = \min\{\overline{r}, \overline{r}^*\}\), then for any \(n \in \mathbb{N}, t, t' \in [0, \overline{r}]\) and \(|t - t'| < \delta_0\), we have
\[
|\lambda_n(t) - \lambda_n(t')| = \left| \int_0^\infty G(t,s) \left( f(s, \lambda_n(s) - \lambda \phi(s)) + \frac{1}{n} \right) + \psi(s) ds \right|
\]
\[
+ A_n(\lambda(f_n + \psi))a(t) + B_n(\lambda(f_n + \psi))b(t)
\]
\[
- \lambda \int_0^\infty G(t', s) \left( f(s, \lambda_n(s) - \lambda \phi(s)) + \frac{1}{n} \right) + \psi(s) ds
\]
\[
- A_n(\lambda(f_n + \psi))a(t') - B_n(\lambda(f_n + \psi))b(t')
\]
\[
\leq \frac{\lambda}{\lambda_0} \left( \int_0^\infty G(t,s) - G(t', s) \right) \left( f(s, \lambda_n(s) - \lambda \phi(s)) + \frac{1}{n} \right) + \psi(s) ds
\]
\[
+ A_n(\lambda(f_n + \psi))a(t) - a(\infty) + B_n(\lambda(f_n + \psi))b(t) - b(\infty). \]

By the similar method as for (3.29), we get that, for any \(\overline{r}_0 > 0\), there exists \(\overline{N}\) such that, when \(t > \overline{N}\), it follows
\[
\lambda \left| \int_0^\infty (G(t,s) - G(t', s)) \left( f(s, \lambda_n(s) - \lambda \phi(s)) + \frac{1}{n} \right) + \psi(s) ds \right| < \frac{\overline{r}_0}{3}.
\]
(3.31)

Together with the continuity of \(a(t), b(t)\) on \([0, +\infty)\), we get that for the above \(\overline{r}_0 > 0\), there exists \(\overline{N}\) such that, when \(t > \overline{N}\), we have \(\lambda_n(t) - \lambda_n(\infty) < \overline{r}_0\). Hence, the functions from \(\{\lambda_n\}_{n=1}^\infty\) are equiconvergent at \(+\infty\), which implies that \(\lambda_n\) is relatively compact (by Lemma 2.5). Therefore, the sequence \(\{\lambda_n\}_{n=1}^\infty\) has a subsequence being uniformly convergent on \([0, +\infty)\). Without loss of generality, we still assume that \(\{\lambda_n\}_{n=1}^\infty\) itself uniformly converges to \(x\) on \([0, +\infty)\). Since \(\{\lambda_n\}_{n=1}^\infty \subseteq K \setminus K_n \subseteq K\), we have \(\lambda_n \geq 0\). By (2.7), we have
\[
\lambda_n(t) = \lambda_n \left( t \right) + x_n \left( t \right) - \int_t^\infty ds \int_0^s \frac{p(\xi)\phi(x_n(s) - 2\lambda(\xi))}{p(\xi)} d\xi
\]
\[
- \int_0^t ds \int_0^r \frac{p(\xi)\phi(x_n(s) - 2\lambda(\xi))}{p(\xi)} d\xi, \quad t \in (0, +\infty).
\]
(3.31)

As \(\{\lambda_n(t)\}_{n=1}^\infty\) is bounded, without loss of generality, we may assume \(\lambda_n(\frac{1}{2}) \to c_0\) as \(n \to +\infty\). Then, by (3.31) and the Lebesgue dominated convergence theorem, we have
\[
x(t) = x \left( \frac{1}{2} \right) + c_0 \left( t \right) - \int_t^\infty ds \int_0^s \frac{p(\xi)\phi(x_n(s) - 2\lambda(\xi))}{p(\xi)} d\xi - \int_0^t ds \int_0^r \frac{p(\xi)\phi(x_n(s) - 2\lambda(\xi)) + \psi(s))}{p(\xi)} d\xi, \quad t \in (0, +\infty).
\]
(3.32)
By (3.32), the direct computation shows that
\[
\left(p(t)x'(t)\right)' + \lambda(f(t, x(t)) - \lambda \alpha(t)) + \psi(t) = 0, \quad t \in (0, +\infty).
\]

On the other hand, let \( n \rightarrow +\infty \) in the following boundary conditions:
\[
\begin{align*}
\alpha_1 x_n(0) - \beta_1 & \lim p(t)x'_n(t) = \sum_{i=1}^{m-2} a_i x_n(\eta_i), \\
\alpha_2 & \lim x_n(t) + \beta_2 \lim p(t)x'_n(t) = \sum_{i=1}^{m-2} \delta x_n(\eta_i).
\end{align*}
\]

Therefore, we deduce that \( x \) is a solution of BVP (2.7).

Let \( \tau \equiv \min_{t \in [0, a]} (\tau_n) \). For any \( \varepsilon > 0 \), there exists \( N > 0 \) such that for any \( t \in [a, b] \), we have
\[
\int_0^\varepsilon \psi(s) sds > N.
\]

By (3.32), the direct computation shows that
\[
\int_0^\varepsilon \psi(s) sds \geq \frac{\lambda \nu}{\rho \tau} \int_0^\varepsilon \psi(s) sds > N.
\]

In the following proof, we suppose \( \lambda \geq \lambda \). Let
\[
R_1 = \frac{4\lambda \nu}{\rho} \int_0^\varepsilon \psi(s) sds.
\]

Assume \( K_{R_1} = \{x \in K : ||x|| < R_1 \} \). For any \( x \in \partial K_{R_1} \), \( t \in [c, d] \),
\[
\begin{align*}
x(t) - \lambda \alpha(t) & \geq \int_0^t \psi(s) ds \\
& \geq \frac{\lambda \nu}{2\rho} \int_0^\varepsilon \psi(s) sds \\
& \geq \frac{\lambda \nu}{\rho} \int_0^\varepsilon \psi(s) sds > N.
\end{align*}
\]

Then, by (3.33), we have
\[
|T_n(x)(t)| = \int_0^t G(t, s) \left(f(s, |x(s)| - \lambda \alpha(s)) + \psi(s)\right) ds \\
& > \frac{\lambda \nu}{\rho} \int_0^\varepsilon \psi(s) sds > N.
\]

Remark 3.1. From the proof of Theorem 3.1, we get the main results under the condition that the function \( f(t, u) \) not only has singularity on \( t \) but also has singularity on \( u \). In addition, we obtain the positive solution of BVP (1.1) under the condition that the function \( f(t, u) \) is semipositive, which are the great improvement of [23–25]. Also our method is different from those in [26–29].

Theorem 3.2. Assume that (H_1)–(H_3) hold, \( f \) and \( h \) satisfy the following condition:
\[ \begin{align*}
\text{(H_5)} \quad & \exists \text{ exists } [c, d] \subset (0, +\infty), \text{ such that} \\
& \lim \inf_{t \to +\infty} \min_{u \in [c, d]} f(t, u) > \frac{4\lambda \nu}{\rho \tau} \int_0^\varepsilon \psi(s) sds, \quad \lim \sup_{t \to +\infty} h(u) = 0,
\end{align*} \]

where \( \tau = \min_{t \in [a, b]} (\tau_n) \). \( m = \int_0^\varepsilon G(s, s) ds. \) Then there exists \( \lambda > 0 \) such that BVP (1.1) has at least one positive solution for any \( \lambda \in (\lambda, +\infty) \).

Proof. By the first inequality of (H_5), we have that there exists \( N > 0 \) such that for any \( t \in [a, b], u > N, \) we have
\[
f(t, u) > \frac{4\lambda \nu}{\rho \tau} \int_0^\varepsilon \psi(s) sds.
\]

Select
\[
\lambda = \frac{N_1, \rho}{\int_0^\varepsilon \psi(s) sds}, \quad \text{where } \lambda = \min_{t \in [c, d]} \lambda(t).
\]

In the following proof, we suppose \( \lambda \geq \lambda \). Let
\[
R_1 = \frac{4\lambda \nu}{\rho} \int_0^\varepsilon \psi(s) sds.
\]

Assume \( K_{R_1} = \{x \in K : ||x|| < R_1 \} \). For any \( x \in \partial K_{R_1}, t \in [c, d], \)
\[
\begin{align*}
x(t) - \lambda \alpha(t) & \geq \int_0^t \psi(s) ds \\
& \geq \frac{\lambda \nu}{2\rho} \int_0^\varepsilon \psi(s) sds \\
& \geq \frac{\lambda \nu}{\rho} \int_0^\varepsilon \psi(s) sds > N.
\end{align*}
\]

Then, by (3.33), we have
\[
|T_n(x)(t)| = \lambda \int_0^t G(t, s) \left(f(s, |x(s)| - \lambda \alpha(s)) + \psi(s)\right) ds \\
& > \int_0^t \psi(s) sds > N.
\]
Noticing that \( \overline{\sigma} = \int_0^1 G(s, s) ds \), we have
\[
\| T_n x \| \geq \| x \|, \quad \text{for any} \quad x \in \partial K_{R_1}.
\]
(3.34)

On the basis of the second inequality in (H2) and the continuity of \( h(u) \) on \([0, +\infty)\), we have
\[
\lim_{u \to +\infty} \frac{h(u)}{u} = 0.
\]

For
\[
\overline{c} = \max \left\{ 1, \left( 4 \lambda (1 + \overline{\Lambda}a(\infty) + \overline{B}b(0)) \int_0^\infty G(s, s)(\phi(s) + \psi(s)) ds \right)^{-1} \right\},
\]
there exists \( N^* > 0 \), such that when \( x \geq N^* \), for any \( 0 \leq y \leq x \), we have \( h(y) \leq \overline{c} x \). Select
\[
R_2 = \left\{ 2, R_1, N^*, 2 \lambda (1 + \overline{\Lambda}a(\infty) + \overline{B}b(0)) \int_0^\infty G(s, s) \phi(s) g \left( \frac{\zeta(s)}{2} \right) ds \right\}.
\]

Then, for any \( x \in \partial K_{R_0}, t \in [0, +\infty) \), we have
\[
[x(t) - \lambda \omega(t)]^+ \leq x(t) \leq \frac{\lambda \overline{c}(t)}{\rho} \int_0^\infty \psi(s) ds \geq x(t) - \frac{2 \lambda \overline{c}(t)}{\rho R_2} \int_0^\infty \psi(s) ds \geq \frac{x(t)}{2} \geq \frac{\zeta(t) R_2}{4} \geq \zeta(t) > 0.
\]

Hence, we gain
\[
\| T_n x \| \leq \| x \|, \quad \text{for any} \quad x \in \partial K_{R_1}.
\]
(3.35)

It follows from the above discussion, (3.34) and (3.35), Lemmas 2.7 and 3.1, for \( n \in \mathbb{N}, \lambda \in (\overline{\gamma}, +\infty) \), \( T_n \) has a fixed point \( x_n \in K_{R_1} \setminus K_{R_2} \) satisfying \( R_1 \leq \| x_n \| \leq R_2 \). The rest of proof is similar to Theorem 3.1. That's the proof of Theorem 3.2. \( \square \)

**Corollary 3.1.** The conclusion of Theorem 3.2 is valid if \((H_2)\) is replaced by:

(\(H_5\)) There exists \([c, d] \subset (0, +\infty), \) such that
\[
\liminf_{u \to +\infty} f(t, u) = +\infty, \quad \lim_{u \to +\infty} \frac{h(u)}{u} = 0.
\]

**Remark 3.2.** From Theorems 3.1 and 3.2, we get the positive solution of BVP (1.1) when the parameter \( \lambda \) is sufficiently large or small, and the solution \( x \) in BVP (1.1) satisfies \( x(t) > 0 \) for any \( t \in [0, +\infty) \).

4. Example

**Example 4.1.** Consider the following BVP
\[
\begin{aligned}
\left( (1 + t)^2 x(t) \right)' + \lambda \left( \frac{1}{4} + \frac{x^2}{(1 + t^2)^2} - \frac{1}{\tau (1 + t^2)} \right) &= 0, \quad t \in (0, +\infty), \\
x(0) - \lim_{t \to 0^+} (1 + t)^2 x(t) &= \frac{1}{2} x(\frac{1}{2}), \quad \lim_{t \to +\infty} x(t) + \lim_{t \to +\infty} (1 + t)^2 x(t) &= \frac{1}{6} x(\frac{1}{2}).
\end{aligned}
\]
(4.1)

It is easy to see that $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$, $p(t) = (1 + t)^2$, $a(t) = 2 - \frac{1}{1 + t^2}$, $b(t) = 1 + \frac{1}{1 + t^2}$, $\rho = 3$, $\rho - \frac{1}{2} b^{\frac{1}{2}}(\frac{1}{2}) = \frac{67}{27}$, $\rho - \frac{1}{2} a^{\frac{1}{2}}(\frac{1}{2}) = \frac{53}{8}$, and

$$\Delta = \left| \rho - \frac{1}{2} b^{\frac{1}{2}}(\frac{1}{2}) \right| + \left| \frac{1}{2} a^{\frac{1}{2}}(\frac{1}{2}) \right| = \left| \frac{67}{27} \frac{53}{8} \right| > 0.$$

As $f(t, u) = \left( \frac{1}{2} + x^2 \right) \frac{1}{1 + t^2} - \frac{1}{\rho + t^2}$, we can suppose

$$\psi(t) = 1 + t^2, \quad \phi(t) = 1 + t^2, \quad g(u) = u, \quad h(u) = u^2.$$

By direct calculation, we have

$$\int_0^\infty \psi(s)ds < +\infty, \quad \int_0^\infty G(s, s)(\psi(s) + \psi(s))ds < +\infty,$$

$$\lim_{s \to +\infty} \min_{\mu \geq 0} \frac{f(t, u)}{u} = +\infty.$$

So all conditions of Theorem 3.1 are satisfied. By Theorem 3.1, BVP (4.1) has at least one positive solution provided $\lambda$ is small enough.

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