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Ishikawa and Mann Iterative Process with Errors for Nonlinear Strongly Accretive Mappings in Banach Spaces

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1. INTRODUCTION

Let X be a Banach space with norm $\|\cdot\|$ and dual X^*. Let $\langle\cdot,\cdot\rangle$ denote the generalized duality pairing. For $1 < p < \infty$, the mapping $J_p: X \to 2^{X^*}$ defined by

$$J_p(x) = \{f^* \in X^* : \mathrm{Re}\,\langle x, f^*\rangle = \|f^*\|\,\|x\|, \|f^*\| = \|x\|^{p-1}\}$$

is called the duality mapping with gauge function $\phi(t) = t^{p-1}$. In particular, the duality mapping with gauge function $\phi(t) = t$, denoted by J, is referred to as the normalized duality mapping. It is a well-known fact that $J_p(x) = \|x\|^{p-2}J(x)$ for $x \in X$, $x \neq 0$, and $1 < p < \infty$ (cf. [31, 33]). A mapping T with domain $D(T)$ and range $R(T)$ in X is said to be accretive if for all x, $y \in D(T)$ and $r > 0$ there holds the inequality

$$\|x - y\| \leqq \|x - y - r(Tx - Ty)\|. \tag{1}$$

T is accretive if and only if for any x, $y \in D(T)$, there is $j \in J(x - y)$ such that

$$\mathrm{Re}\,\langle Tx - Ty, j\rangle \geq 0. \tag{2}$$

The accretive operators were introduced independently by Browder [1] and Kato [13] in 1967. A fundamental result, due to Browder, in the theory

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of accretive operators states that the initial value problem

$$\frac{du}{dt} + Tu = 0, \qquad u(0) = u_0, \tag{3}$$

is solvable if T is a locally Lipschitzian and accretive operator on X. The reader is referred to Browder [2] for more details of the theory of accretive operators.

Let D be a nonempty subset of a Banach space X. Recall that a mapping $T: D \to X$ is said to be strongly accretive if there exists a real number $k > 0$ such that for every $x, y \in D$,

$$\operatorname{Re}\langle Tx - Ty, j\rangle \geq k\|x - y\|^2 \tag{4}$$

holds for some $j \in J(x - y)$, or equivalently, there exists a real number $k > 0$ such that for every $x, y \in D$,

$$\operatorname{Re}\langle Tx - Ty, j_p\rangle \geq k\|x - y\|^p \tag{5}$$

holds for some $j_p \in J_p(x - y)$. Without loss of generality, we assume that $k \in (0, 1)$. Strongly accretive mappings are sometimes also called strictly accretive. These mappings have been studied by many authors (e.g., [2, 7, 9, 11, 18, 21, 23, 24, 28]). In particular, Deimling [8, Theorem 13.1] proved that if X is a uniformly smooth Banach space and $T: X \to X$ is strongly accretive and demicontinuous (i.e., $x_n \xrightarrow{s} x$ implies that $Tx_n \xrightarrow{w} Tx$), then T maps X onto X; that is, for each f in X, the equation $Tx = f$ has a solution in X.

Let D be a nonempty subset of a Banach space X. Recall that a mapping $T: D \to X$ is said to be strictly pseudo-contractive if there exists a constant $t > 1$ such that the inequality

$$\|x - y\| \leqq \|(1 + r)(x - y) - rt(Tx - Ty)\| \tag{6}$$

holds for all $x, y \in D$ and $r > 0$. It is known (see, e.g., [6]) that T is a strictly pseudo-contraction if and only if $(I - T)$ is a strongly accretive with $k = (t - 1)/t$. Strictly pseudo-contractive mappings have been studied by various authors (e.g., [6, 7, 9, 28, 30]).

The objective of this paper is to study the iterative solutions to the equation $Tx = f$ in the case when T is Lipschitzian and strongly accretive and X is uniformly smooth. To this purpose, let us first recall the following two iteration processes due to Mann [19] and Ishikawa [12], respectively.

(I) The Mann iteration process [19] is defined as follows: For a convex subset C of a Banach space X and a mapping T from C into itself, the sequence $\{x_n\}$ in C is defined by

$$x_0 \in C,$$
$$x_{n+1} = (1 - c_n)x_n + c_n T x_n, \qquad n \geq 0,$$

where $\{c_n\}$ is a real sequence satisfying $c_0 = 1$, $0 < c_n \leq 1$, for all $n \geq 1$, and $\sum_{n=0}^{\infty} c_n = \infty$ (The condition $\sum_{n=0}^{\infty} c_n = \infty$ is sometimes replaced by $\sum_{n=0}^{\infty} c_n(1 - c_n) = \infty$).

(II) The Ishikawa iteration process [12, 27] (see [32] for an extension) is defined as follows: With X and C as above, the sequence $\{x_n\}$ in C is defined by

$$x_0 \in C,$$
$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n,$$
$$y_n = (1 - \beta_n)x_n + \beta_n T x_n, \qquad n \geq 0,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0, 1]$ satisfying the conditions $0 \leq \alpha_n \leq \beta_n \leq 1$ for all n, $\lim_{n\to\infty} \beta_n = 0$, and $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$.

The two iteration processes described above have been studied extensively by various authors for approximating either fixed points of nonlinear mappings or solutions of nonlinear operators equations in Banach spases (see, e.g., [3, 4, 6, 7, 9–12, 14–19, 22–32, 34]) and for comparison of the two iteration processes in the one-dimensional case, we refer the reader to [26]. Inspired by [11, 34], we introduce the following concept of the Ishikawa iteration process with errors.

(III) The Ishikawa iteration process with errors is defined as follows: For a nonempty subset K of a Banach space X and a mapping $T: K \to X$, the sequence $\{x_n\}$ in K is defined by

$$x_0 \in K,$$
$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n + u_n,$$
$$y_n = (1 - \beta_n)x_n + \beta_n T x_n + v_n, \qquad n \geq 0,$$

where $\{u_n\}$ and $\{v_n\}$ are two summable sequences in X, i.e., $\sum_{n=0}^{\infty} \|u_n\| < \infty$, $\sum_{n=0}^{\infty} \|v_n\| < \infty$, and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0, 1]$ satisfying certain restrictions.

Note that the Mann and Ishikawa iteration processes are all a special case of the Ishikawa iteration process with errors.

Recently, Chidume [7] proved (Theorem 1) that if $X = L_p$ (or l_p) for $p \geq 2$ then the Mann iteration process converges strongly to a solution of equation $Tx = f$ when T is Lipschitzian and strongly accretive.

Also, he proved a related result (Theorem 2) that deals with the iterative approximation of the fixed point of the class of Lipschitz strictly pseudo-contractive mappings and put forth the following questions.

PROBLEM 1. Is Theorem 1 or 2 extended to L_p (or l_p) spaces for $1 < p < 2$?

PROBLEM 2. Can the Ishikawa iteration process be extended to Theorem 1 and 2?

Afterward, Deng [9] showed that the Ishikawa iteration process can be extended to Theorem 1 and 2 of [7] in L_p (or l_p) spaces for $p \geq 2$. Tan and Xu [28] studied Chidume's above open problem in p-uniformly smooth Banach space. Osilike [23] attempts to show that the Mann and Ishikawa iteration process converge strongly to a solution of the equation $Tx = f$ in case T is a Lipschitzian and strongly accretive operator from a bounded closed convex subset C of a uniformly smooth Banach space into itself. Unfortunately the operator S defined by $Sx = f + x - Tx$, in general, does not map C into C. Therefore, Morales pointed out that the problem of approximating solutions for equations of the form $Tx = f$ with T strongly accretive still remains open (see "Mathematical Reviews" 93*i*: 47091).

In this paper we shall continue to study Chidume's open problem [7] and extend the results of [6, 7] in uniformly smooth Banach space. Our results generalize and unify the corresponding ones of Chidume [6, 7], Tan and Xu [28], Osilike [23], Deng [9], and the author [18] and answer positively the open problem mentioned by Chidume [7] in the more general setting.

2. PRELIMINARIES AND LEMMAS

Let X be an arbitrary Banach space. Recall that the modulus of smoothness $\rho_X(\cdot)$ of X is defined by

$$\rho_X(\tau) = \tfrac{1}{2}\sup\{\|x + y\| + \|x - y\| - 2 : x, y \in X, \|x\| = 1, \|y\| \leqq \tau\}, \tau > 0,$$

and that X is said to be uniformly smooth if $\lim_{\tau \to 0} \rho_X(\tau)/\tau = 0$. Recall also that for a real number $p > 1$, a Banach space X is said to be p-uniformly smooth if $\rho_X(\tau) \leq d\tau^p$ for $\tau > 0$, where $d > 0$ is a constant. It is known [33] that for a Hilbert space H, $\rho_H(\tau) = (1 + \tau^2)^{1/2} - 1$ and, hence, H is 2-uniformly smooth, while if $2 \leq p < \infty$, L_p (or l_p) is 2-uniformly smooth. It is known [31, 33] that X is uniformly smooth if and only if J_p is single-valued and uniformly continuous on any bounded subset of X; X is uniformly convex (smooth) if and only if X^* is uniformly smooth (convex).

We define for positive t,

$$b(t) = \sup\{(\|x + ty\|^2 - \|x\|^2)/t - 2\mathrm{Re}\langle y, J(x)\rangle : \|x\| \leq 1, \|y\| \leq 1\}.$$

Clearly $b:(0, \infty) \to [0, \infty)$ is nondecreasing, continuous and $b(ct) \leq cb(t)$ for all $c \geq 1$ and $t > 0$. Also we have the following.

LEMMA 1 [24]. *Suppose that X is a uniformly smooth Banach space and $b(t)$ is defined as above. Then $\lim_{t\to 0+} b(t) = 0$ and*

$$\|x + y\|^2 \leq \|x\|^2 + 2\mathrm{Re}\langle y, J(x)\rangle + \max\{\|x\|, 1\}\|y\| b(\|y\|)$$

for all $x, y \in X$.

Proof. The proof is the same as Reich [24] proved for a real uniformly smooth Banach space (see also [30]).

We also need the following lemma which is fundamental for our results.

LEMMA 2. *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnegative real sequences satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n$$

with $\{t_n\} \subset [0, 1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $b_n = o(t_n)$, and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Proof. Since $b_n = o(t_n)$, let $b_n = d_n t_n$ and $d_n \to 0$. By a straightforward induction, one obtains

$$0 \leq a_{n+1} \leq a_k \prod_{j=k}^{n} (1 - t_j) + \sum_{j=k}^{n} \left[t_j \prod_{i=j+1}^{n} (1 - t_i) \right] d_j + \sum_{j=k}^{n} c_j \prod_{i=j+1}^{n} (1 - t_i). \tag{7}$$

We have

$$\prod_{j=k}^{n} (1 - t_j) \leq \exp\left(- \sum_{j=k}^{n} t_j \right) \to 0 \qquad (\text{as } n \to \infty)$$

and

$$\sum_{j=k}^{n} t_j \prod_{i=j+1}^{n} (1 - t_i) = 1 - (1 - t_n)(1 - t_{n-1}) \cdots (1 - t_k) \leq 1 \qquad \forall n, k.$$

Since $\lim_{n\to\infty} d_n = 0$ and $\sum_{n=0}^{\infty} c_n < \infty$ for arbitrary $\varepsilon > 0$, there exists a natural number k such that $d_j < \varepsilon$ for all $j \geq k$, and $\sum_{j=k}^{\infty} c_j < \varepsilon$, we have from (7)

$$0 \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n \leq 2\varepsilon.$$

Letting $\varepsilon \to 0$, we obtain $\lim_{n\to\infty} a_n = 0$.

3. The Ishikawa Iteration Process with Errors

In this section we study the Ishikawa iteration process with errors and prove that if X is a uniformly smooth Banach space and $T: X \to X$ is a Lipschitzian strongly accretive mapping, then the Ishikawa iteration process with errors converges strongly to the unique solution of the equation $Tx = f$.

THEOREM 1. *Let X be a uniformly smooth Banach space. Let $T: X \to X$ be a Lipschitzian strongly accretive operator with a constant $k \in (0, 1)$ and a Lipschitz constant $L \geq 1$. Define $S: X \to X$ by $Sx = f + x - Tx$. Let $\{u_n\}$, $\{v_n\}$ be two summable sequences in X and let $\{a_n\}$, $\{\beta_n\}$ be two real sequences in $[0, 1]$ satisfying:*

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n\to\infty} \beta_n < k/(L^2 - k)$.

For arbitrary $x_0 \in X$, the iteration sequence $\{x_n\}$ is defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n S y_n + u_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n S x_n + v_n, \qquad n \geq 0. \end{aligned} \tag{8}$$

Moreover, suppose that the sequence $\{Sy_n\}$ is bounded, then $\{x_n\}$ converges strongly to the unique solution q of the equation $Tx = f$.

Proof. The existence of a solution to $Tx = f$ follows from Morales [21] and the uniqueness from the strong accretiveness of T. Let q denote the solution of $Tx = f$. Now set

$$\begin{aligned} d &= \sup\{\|Sy_n - q\| : n \geq 0\} + \|x_0 - q\|, \\ M &= d + \sum_{n=0}^{\infty} \|u_n\| + 1. \end{aligned} \tag{9}$$

For any $n \geq 0$, using induction, we obtain

$$\|x_n - q\| \leq d + \sum_{i=0}^{n-1} \|u_i\|, \qquad n \geq 0;$$

hence

$$\|x_n - q\| \leq M, \qquad n \geq 0. \tag{10}$$

It follows from (4), (8), and (10) that

$$\begin{aligned}
&\operatorname{Re}\langle y_n - q, J(x_n - q)\rangle \\
&\quad = \operatorname{Re}\langle x_n + \beta_n f - \beta_n T x_n + v_n - q, J(x_n - q)\rangle \\
&\quad = -\beta_n \operatorname{Re}\langle Tx_n - Tq, J(x_n - q)\rangle + \operatorname{Re}\langle x_n - q, J(x_n - q)\rangle \\
&\qquad + \operatorname{Re}\langle v_n, J(x_n - q)\rangle \\
&\quad \leq -k\beta_n \|x_n - q\|^2 + \|x_n - q\|^2 + \|v_n\| \|x_n - q\| \\
&\quad \leq (1 - k\beta_n)\|x_n - q\|^2 + M\|v_n\|.
\end{aligned} \tag{11}$$

Using (4), (8), and (11), we have

$$\begin{aligned}
&\operatorname{Re}\langle Sy_n - q, J(x_n - q)\rangle \\
&\quad = \operatorname{Re}\langle Tq + y_n - Ty_n - q, J(x_n - q)\rangle \\
&\quad = \operatorname{Re}\langle Tx_n - Ty_n, J(x_n - q)\rangle - \operatorname{Re}\langle Tx_n - Tq, J(x_n - q)\rangle \\
&\qquad + \operatorname{Re}\langle y_n - q, J(x_n - q)\rangle \\
&\quad \leq L\|y_n - x_n\| \|x_n - q\| - k\|x_n - q\|^2 + (1 - k\beta_n)\|x_n - q\|^2 + M\|v_n\| \\
&\quad = L\|\beta_n(Tq - Tx_n) + v_n\| \|x_n - q\| + (1 - k - k\beta_n)\|x_n - q\|^2 + M\|v_n\| \\
&\quad \leq L^2\beta_n \|x_n - q\|^2 + L\|v_n\| \|x_n - q\| + (1 - k - \beta_n)\|x_n - q\|^2 + M\|v_n\| \\
&\quad \leq (1 - k - k\beta_n + L^2\beta_n)\|x_n - q\|^2 + M(L + 1)\|v_n\|.
\end{aligned} \tag{12}$$

It then follows from (8), (9), (12), and Lemma 1 that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|(1 - \alpha_n)(x_n - q) + \alpha_n(Sy_n - q) + u_n\|^2 \\
&\leq \|(1 - \alpha_n)(x_n - q) + \alpha_n(Sy_n - q)\|^2 \\
&\quad + 2\operatorname{Re}\langle u_n, J((1 - \alpha_n)(x_n - q) + \alpha_n(Sy_n - q))\rangle \\
&\quad + \max\{\|(1 - \alpha_n)(x_n - q) + \alpha_n(Sy_n - q)\|, 1\}\|u_n\| b(\|u_n\|) \\
&\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n(1 - \alpha_n)\operatorname{Re}\langle Sy_n - q, J(x_n - q)\rangle \\
&\quad + \max\{(1 - \alpha_n)\|x_n - q\|, 1\}\alpha_n \|Sy_n - q\| b(\alpha_n \|Sy_n - q\|) \\
&\quad + 2\|u_n\| \|(1 - \alpha_n)(x_n - q) + \alpha_n(Sy_n - q)\| + Mb(M)\|u_n\| \\
&\leq [(1 - \alpha_n)^2 + 2\alpha_n(1 - \alpha_n)(1 - k - k\beta_n + L^2\beta_n)]\|x_n - q\|^2 \\
&\quad + 2\alpha_n(1 - \alpha_n)(L + 1)M\|v_n\| + M^3\alpha_n b(\alpha_n) \\
&\quad + [2M + Mb(M)]\|u_n\| \\
&\leq [(1 - \alpha_n)^2 + 2\alpha_n(1 - \alpha_n)(1 - k - k\beta_n + L^2\beta_n)]\|x_n - q\|^2 \\
&\quad + M^3\alpha_n b(\alpha_n) + [LM + 2M + Mb(M)](\|u_n\| + \|v_n\|).
\end{aligned}$$

By assumption (ii) on the sequence $\{\beta_n\}$, there exist $\delta \in (0, 2k)$ and a natural number $N \geq 1$ such that

$$L(L^2 - k)\beta_n < k - \delta/2 \qquad \text{for } n \geq N.$$

Consequently,

$$\begin{aligned}\|x_{n+1} - q\|^2 &\leq [(1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n)(1-\delta/2)]\|x_n - q\|^2 \\ &\quad + M^3\alpha_n b(\alpha_n) + [LM + 2M + Mb(M)](\|u_n\| + \|v_n\|) \\ &= (1 - \delta\alpha_n - \alpha_n^2 + \delta\alpha_n^2)\|x_n - q\|^2 + M^3\alpha_n b(\alpha_n) \\ &\quad + [LM + 2M + Mb(M)](\|u_n\| + \|v_n\|) \\ &\leq (1 - \delta\alpha_n)\|x_n - q\|^2 + \alpha_n[M^2\delta\alpha_n + M^3 b(\alpha_n)] \\ &\quad + [LM + 2M + Mb(M)](\|u_n\| + \|v_n\|)\end{aligned}$$

for $n \geq N$. We set $a_n = \|x_n - q\|^2$, $t_n = \delta\alpha_n$, $b_n = \alpha_n[M^2\delta\alpha_n + M^3 b(\alpha_n)]$, and $c_n = [LM + 2M + Mb(M)](\|u_n\| + \|v_n\|)$. Then the above inequality reduces to

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n, \qquad n \geq N.$$

Observe that $\lim_{t\to 0+} b(t) = 0$ and $\lim_{n\to\infty}\alpha_n = 0$, so that $\lim_{n\to\infty} b(\alpha_n) = 0$. It follows from Lemma 2 that $\lim_{n\to\infty} a_n = 0$, so that $\{x_n\}$ converges strongly to the unique solution q of the equation $Tx = f$.

COROLLARY 1. *Let X be a p-uniformly smooth Banach space with $1 < p < \infty$ and let $T: X \to X$ be a Lipschitzian strongly accretive operator with a constant $k \in (0, 1)$ and a Lipschitz constant $L \geq 1$. Define $S: X \to X$ by $Sx = f + x - Tx$. Let $\{u_n\}$, $\{v_n\}$ be two summable sequences in X, and let $\{\alpha_n\}$, $\{\beta_n\}$ be two real sequences in* [0 1] *satisfying* (i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$; *and* (ii) $\limsup_{n\to\infty}\beta_n < k/(L^2 - k)$. *Then for each $x_0 \in X$, the iteration sequence $\{x_n\}$ is defined by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n S y_n + u_n, \\ y_n &= (1-\beta_n)x_n + \beta_n S x_n + v_n, \qquad n \geq 0, \end{aligned} \tag{13}$$

converges strongly to the unique solution q of the equation $Tx = f$.

Proof. From Theorem 1, we only need to prove that the sequence $\{Sy_n\}$ is bounded. By the proof of Theorem 4.1 in [28], we can prove that the sequence $\{x_n\}$ is bounded. Hence $\{Sx_n\}$ is bounded since S is a Lipschitzian mapping. It is easily seen from (13) that the sequence $\{y_n\}$, and thus $\{Sy_n\}$ is bounded.

We now turn to consider approximating fixed point of pseudo-contractive mappings via the Ishikawa iteration process with errors.

THEOREM 2. *Let K be a nonempty closed subset of a uniformly smooth Banach space X. Let $T: K \to X$ be a Lipschitzian and strictly pseudo-contractive mapping with a constant $t > 1$ and a Lipschitz constant $L \geq 1$. Let $\{u_n\}$, $\{v_n\}$ be two summable sequences in X, and let $\{\alpha_n\}$, $\{\beta_n\}$ be two real sequences in* $[0, 1]$ *satisfying:*

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty;$

(ii) $\limsup_{n\to\infty} \beta_n < k/L(1 + L),$

where $k = (t - 1)/t$. If the range $T(K)$ of T is bounded, then $\{x_n\} \subset K$ generated by $x_0 \in K$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n + u_n,$$
$$y_n = (1 - \beta_n)x_n + \beta_n Tx_n + v_n, \qquad n \geq 0,$$

converges strongly to the unique fixed point of T.

Proof. The existence and uniqueness of a fixed point of T are a direct consequence of Proposition 3 of Martin [20]. Let q denote the fixed point of T. It is easily seen from (6) that

$$\mathrm{Re}\langle (I - T)x - (I - T)q, J(x - q)\rangle \geq k\|x - q\|^2,$$

for all $x, y \in K$, where $k = (t - 1)/t$. Hence

$$\mathrm{Re}\langle Tx - Tq, J(x - q)\rangle \leq (1 - k)\|x - q\|^2$$

for all x, y in K.

Proceeding in arguments similar to those in the proof of Theorem 1, we get

$$\mathrm{Re}\langle Ty_n - Tq, J(x_n - q)\rangle \leq (1 - k + L\beta_n + L^2\beta_n)\|x_n - q\|^2 + LM\|v_n\|$$

and

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \|(1 - \alpha_n)(x_n - q) + \alpha_n(Ty_n - q) + u_n\|^2 \\ &\leq [(1 - \alpha_n)^2 + 2\alpha_n(1 - \alpha_n)(1 - k + L\beta_n + L^2\beta_n)]\|x_n - q\|^2 \\ &\quad + M^3\alpha_n b(\alpha_n) + [LM + 2M + Mb(M)](\|u_n\| + \|v_n\|),\end{aligned}$$

where $M = \sup\{\|Ty_n - q\| : n \geq 0\} + \|x_0 - q\| + \sum_{n=0}^{\infty}\|u_n\| + 1$.

By assumption (ii) on the sequence $\{\beta_n\}$, there exist $\delta \in (0, 2k)$ and a natural number $N \geq 1$ such that

$$L(L+1)\beta_n < k_p - \delta/2 \qquad \text{for } n \geq N.$$

Consequently,

$$\begin{aligned}\|x_{n+1} - q\|^2 \leq (1 - \delta\alpha_n)\|x_n - q\|^2 + \alpha_n[M^2\delta\alpha_n + M^3 b(\alpha_n)] \\ + [LM + 2M + Mb(M)](\|u_n\| + \|v_n\|).\end{aligned}$$

Now applying Lemma 2 to the above inequality we get that $\{x_n\}$ strongly converges to the unique fixed point of T.

Remark 1. Theorem 1 improves and generalizes Theorem 4.1 of Tan and Xu [28], Theorem 1 of Chidume [7], and Theorem 1 of Deng [9]. Theorem 2 improves and generalizes Theorem 4.2 of Tan and Xu [28], Theorem 2 of Deng [9], Theorem 2 of Chidume [7], and Theorem 1 of the author [18], and answers positively the open problems mentioned by Chidume [7] in the more general setting.

4. The Mann Iteration Process with Errors

In this section we study the Mann iteration process with errors and prove that if X is a uniformly smooth Banach space and $T :\to X$ is strongly accretive and demicontinuous, then the Mann iteration process with errors converges strongly to the unique solution of the equation $Tx = f$. Note that, by the results of Section 3, if $\beta_n = 0$ for all $n \geq 0$, then $x_n = y_n$ for all $n \geq 0$. Hence, from (12) it is easily seen that T need not be a Lipschitzian operator for the Mann iteration process with errors. We only state theorems without the proof.

Theorem 3. *Let X be a uniformly smooth Banach space. Let $T : X \to X$ be a demicontinuous and strongly accretive operator with a constant $k \in (0, 1)$. Define $S : X \to X$ by $Sx = f + x - Tx$. Let $\{u_n\}$ be a summable sequence in X, and $\{\alpha_n\}$ be a real sequence in $[0, 1]$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. For arbitrary $x_0 \in X$, the iteration sequence $\{x_n\}$ is defined by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Sx_n + u_n, \qquad n \geq 0.$$

Moreover, suppose that the sequence $\{Sy_n\}$ is bounded. Then $\{x_n\}$ converges strongly to the unique solution q of the equation $Tx = f$.

THEOREM 4. *Let K be a nonempty closed subset of a uniformly smooth Banach space X. Let $T: K \to X$ be a strictly pseudo-contractive mapping with a constant $t > 1$. Let $\{u_n\}$ be a summable sequence in X, and $\{\alpha_n\}$ be a real sequence in* $[0, 1]$ *satisfying* $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$. *If the range $T(K)$ of T is bounded, then $\{x_n\} \subset K$ generated by $x_0 \in K$,*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n + u_n, \qquad n \geq 0,$$

converges strongly to the unique fixed point of T.

Remark 2. Theorem 3 improves and generalizes Theorem 1 of author [18], Theorem 4 improves and generalizes the main results of Chidume [6] and answers positively the open problem 1 mentioned by Chidume [7] in the more general setting.

Remark 3. Because an L_p (or l_p) space with $1 < p < \infty$ is min(2, p)-uniformly smooth, our results are all true for L_p (or l_p).

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