Exact number of solutions of stationary reaction–diffusion equations

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Abstract

We study both existence and the exact number of positive solutions of the problem

\begin{equation}
(P_{1}) \begin{cases}
\hat{\lambda}(|u|^{p-2}u)'+f(u) = 0 \text{ in } (0,1), \\
u(0) = u(1) = 0.
\end{cases}
\end{equation}

where $\hat{\lambda}$ is a positive parameter, $p > 1$, the nonlinearity $f$ is positive in $(0,1)$, and $f(0) = f(1) = 0$. Assuming that $f$ satisfies the condition $\lim_{t \to 1^-} \frac{f(t)}{t} = \omega > 0$ where $\omega \in (0,p - 1)$, we study its behavior near zero, and we obtain existence and exactness results for positive solutions. We prove the results using the shooting method. We show that there always exist solutions with a flat core for $\hat{\lambda}$ sufficiently small. As an application, we prove the existence of a non-negative solution for a class of singular quasilinear elliptic problems in a bounded domain in $\mathbb{R}^N$ having a flat core in a ball.

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1. Introduction

We give exact multiplicity results for the nonlinear eigenvalue problem

\begin{equation}
(P_{1}) \begin{cases}
\hat{\lambda}(|u|^{p-2}u)'+f(u) = 0 \text{ in } (0,1), \\
u(0) = u(1) = 0.
\end{cases}
\end{equation}

We will assume throughout that $f$ satisfies the following four hypotheses:

\begin{itemize}
  \item [(H_1)] The nonlinearity $f : [0, +\infty) \to \mathbb{R}$ is continuous, with $f > 0$ in $(0,1)$ and $f(0) = f(1) = 0$.
  \item [(H_2)] We have $\lim_{s \to 1^-} \frac{\theta f(s)}{(1-s)^2} = \omega > 0$, where $\theta \in (0,p - 1)$.
  \item [(H_3)] We have $\lim_{s \to 0^+} \frac{\theta f(s)}{(s-1)^2} = f_0 \in [0, +\infty]$.
  \item [(H_4)] Let $G(s) = \theta F(s) - \Theta(s)$, where $F(s) = \int_0^s f(t) \, dt$. Then:
    \begin{itemize}
      \item [(i)] When $f_0 \in (0,\infty)$, the function $G(s) = \theta F(s) - \Theta(s)$ is strictly monotone increasing in $(0,1)$.
      \item [(ii)] When $f_0 = 0$, we assume that $f \not\in C^1((0,1))$. In addition, we suppose that there exist numbers $0 < a_0 < a_1 < 1$ such that the function $\gamma(s, a) = G(a) - G(as)$ satisfies the following two conditions:
        \begin{itemize}
          \item [(Y_1)] $\gamma(s, a)$ is strictly monotone increasing in $s \in (0,1)$ for $a \in (0, a_0)$.
          \item [(Y_2)] We have that $\frac{\partial}{\partial a} \gamma(s, a) > 0$ for $a \in (a_0, 1)$ and $s \in (0,1)$, and that $\gamma(s, a) > 0$ for $a \in (a_1, 1)$ and $s \in (0,1)$.
        \end{itemize}
    \end{itemize}
\end{itemize}

Since we know the behavior of $f$ only in $[0,1]$, we mainly study existence of positive solutions $u$ of the Problem $(P_{1})$ satisfying $\sup_{[0,1]} |u(x)| \leq 1$. These solutions are classical in the following sense: $u \in C([0,1]), |u|^{p-2}u \in C^1((0,1)), u(0) = u(1) = 0$, and $-\hat{\lambda}(|u(t)|^{p-2}u(t))' = f(u(t))$ for every $t \in (0,1)$.

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As typical examples,
\begin{itemize}
  \item (a) \( f(u) = |u|^p \ln |u|^p \), with \( 0 < q < p - 1 \) and \( 0 < p < p - 1 \),
  \item (b) \( f(u) = |u|^{p-1} - (1 - |u|) \), with \( q > 1 \), \( r > 0 \) and \( p > 2 \),
  \item (c) \( f(u) = \sin(\pi u)^2 \), with \( p = 2 \)
\end{itemize}

are nonlinearities \( f \) satisfying hypotheses \((H_1)\) through \((H_4)\). As a matter of fact, it is not difficult to see that Examples (a) and (c) satisfy the preceding four hypotheses. Observe that the values \( \theta, \omega \) of \((H_3)\), \( f_0 \) of \((H_3)\), and \( a_0, a_1 \) of \((H_4)\) of Example (b) are the following.
\begin{itemize}
  \item If \( p > 2 \), then \( \theta = 1 \) and \( \omega = r > 0 \).
  \item If \( q < p \), then \( f_0 = +\infty \).
  \item If \( q = p \), then \( f_0 = 1 \).
  \item If \( q > p \), then \( f_0 = 0 \) and \( 0 < a_0 = \left( \frac{a + b}{a + b} \right)^\frac{1}{2} < a_0 \left( \frac{a + b}{a + b} \right)^\frac{1}{2} = a_1 < 1 \).
\end{itemize}

Thus Example (b) satisfies \((H_1)\), \((H_2)\) and \((H_3)\), and \((H_4)\) in the case \( q \leq p \) because it verifies part (i) of the hypothesis. However, checking \((H_4)\) in the case \( q > p \), is a little more complicated. Let
\[
G(s) = \left( \frac{p-q}{q} \right) s^q + \left( \frac{q-p+r}{q+r} \right) s^{q+r},
\]
and
\[
\mathcal{T}(s, a) = \left( \frac{p-q}{q} \right) (1 - s^q) a^q + \left( \frac{q-p+r}{q+r} \right) (1 - s^{q+r}) a^{q+r}.
\]

Differentiating \((1.2)\) with respect to \( s \), we have
\[
\frac{\partial \mathcal{T}}{\partial s}(s, a) = \left[ q - p - (q - p + r) a^{q^{q^q}} \right] a^q s^{q-1}.
\]

For \( s \in (0, 1) \) and \( a \in (0, a_0) \) it follows that
\[
\frac{\partial \mathcal{T}}{\partial s}(s, a) > \left[ q - p - (q - p + r) a^{q^{q^q}} \right] a^q s^{q-1}.
\]

Thus hypothesis \((\mathcal{T}1)\) is verified in this case.

On the other hand, an easy computation shows that
\[
\frac{\partial \mathcal{T}}{\partial a}(s, a) = [q - p] (1 - s^q) + (q - p + r) (1 - s^{q+r}) a^q s^{q-1} := f a^{q-1}.
\]

For \( s \in (0, 1) \) and \( a \in (a_0, 1) \) it follows that
\[
J > (p-q)(1 - s^q) + (q - p + r)(1 - s^{q+r}) a^q s^{q-1}.
\]

Thus the first part of hypothesis \((\mathcal{T}2)\) is verified. For the second, it is convenient to rewrite \((1.2)\) as
\[
\mathcal{T}(s, a) = a^q \left[ \frac{q-p+r}{q+r} a^{q-1} - \frac{q-p}{q} \right] + \left( \frac{q-p+r}{q+r} a^{q-1} \right) s^q.
\]

Then for \( s \in (0, 1) \) and \( a \in (a_1, 1) \), we have
\[
\mathcal{T}(s, a) > a^q \left[ \frac{q-p+r}{q+r} a^{q-1} - \frac{q-p}{q} \right] + \left( \frac{q-p+r}{q+r} a^{q-1} \right) s^q
\]
\[
= a^q \left( \frac{q-p+r}{q+r} a^{q-1} - \frac{q-p}{q} \right) (1 - s^q)
\]
\[
= a^q \left( \frac{q-p}{q+r} a^{q-1} - \frac{q-p}{q} \right) (1 - s^q)
\]
\[
= a^q (q-p) \left( \frac{a^{q-1}}{q} - \frac{1}{q} \right) (1 - s^q) > 0.
\]

Hence the second part of hypothesis \((\mathcal{T}2)\) is verified as well, and therefore Example (b) also satisfies \((H_4)\) in the case \( q > p \).
We point out that Example (b) has been studied by many authors. In [11], Takeuchi and Yamada show that if $p > q$, then there exists a unique solution $u_1$ for every $\lambda > 0$. If $p = q$, there exists a unique positive solution $u_1$ and the set (called the flat core of $u_1$)

$$\mathcal{C}_1 = \mathcal{C}_1(u_1) = \{ x \in (0, 1) : u_1(x) = 1 \}$$

is non-empty for $\lambda$ sufficiently small. Further, $\mathcal{C}_1$ has bifurcations from the trivial solution (the same result is obtained by Guedda and Veron in [6, Theorem 2.2]). If $p < q$, the structure of the set of solutions is essentially different from those in the other cases. In other words, there exists a $\lambda > 0$ so that when $\lambda < \lambda_1$, the Problem (P) has no solutions; when $\lambda = \lambda_1$, the Problem (P) has a unique positive solution; when $\lambda > \lambda_1$, the Problem (P) has exactly two positive solutions, $u_1$ and $u_2$, satisfying $u_1 > u_2$ in $(0, 1)$. For an extension of the results of [11] in the $N$-dimensional case, see [10].

Let $\lambda_1$ be the first eigenvalue of $-\Delta u = -\text{div}(\nabla u \nabla^2 u)$ under zero Dirichlet boundary conditions. Using a suitable lower solution obtained from the eigenfunction associated to $\lambda_1$, in [8] Kamin and Veron show that, for $\lambda$ sufficiently large, the unique solution of the problem

$$\begin{cases}
    \Delta u + \lambda a - 1 (1 - u^2) = 0 & \text{in } \Omega, \\
    u = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $\Omega$ is a bounded smooth domain of $R^N$, with $q = p$, has a flat core. In addition, they extend the results of [6].

In [7], Guo considers the more general problem

$$-\text{div}(\nabla u \nabla^2 u) = \lambda f(u) \quad \text{in } \Omega, u = 0 \quad \text{on } \partial \Omega,$$

where $p > 1$ and $\Omega \subset R^N$ is a bounded smooth domain, and studies the structure of positive solutions of the preceding problem for a class of logistic type nonlinearities $f(u)$. These nonlinearities satisfy (i) $f(0) = f(a) = 0$ for some $a > 0$, (ii) $f(u) > 0$ for $u \in (0, a)$, (iii) $\lim_{u \to 0} \frac{f(u)}{u^p} = 0$, and (iv) $a > 0$ is a zero of $f$ of $o(\omega)$-order. When $\omega \in (0, p - 1)$, the author shows that the problem has a unique positive solution $u_1$, which has the flat core $\{x \in \Omega : u_1(x) = a\} \neq \emptyset$. In addition, he studies the asymptotic behavior of the flat core as $\lambda \to \infty$.

Our results depend essentially on the asymptotic behavior of the function

$$I(a) = \int_0^a (F(a) - F(u))^{-\frac{1}{2}} du, \quad a \in (0, 1).$$

The following is consequence of the monotonicity properties of this function.

**Theorem 1.1.** Assume that the nonlinearity $f$ satisfies hypotheses (H1) through (H3). Then:

(a) Under the conditions of part (i) of hypothesis (H1),

(i) when $f_0 = +\infty$, there exists a unique positive solution of the Problem (P) for each $\lambda > 0$;

(ii) when $0 < f_0 < +\infty$, there exists a $\lambda_0 > 0$ such that the Problem (P) has no positive solutions for $\lambda \geq \lambda_0$, and a unique positive solution for $\lambda < \lambda_0$.

(b) Under the conditions of part (ii) of hypothesis (H1), there exists a positive number $\lambda_0$ such that the Problem (P) has exactly two positive solutions for $\lambda < \lambda_0$, one for $\lambda = \lambda_0$, and no positive solutions for $\lambda > \lambda_0$.

**Remark 2.** In Theorem 1.1, there always exist solutions with a flat core for $\lambda > 0$ sufficiently small.

We combine the preceding result with the upper and lower solutions method to study the quasilinear elliptic problem with singular weights

$$\begin{cases}
    \lambda \text{ div}(|x|^{-\theta p} |\nabla u|^{p-2} \nabla u) + |x|^{-(a+1)p} f(u) = 0 & \text{in } \Omega, \\
    u = 0 & \text{on } \partial \Omega, \tag{1.3}
\end{cases}$$

where $1 < p < N, c > 0, -\infty < a < \min \{ \frac{p N - p - 1}{p}, \frac{p N - 1}{p}, \frac{p N - 1}{p - 1} \}, 0 \in \Omega$. $\Omega \subset R^N$ is a bounded domain with $C^1$-boundary, and $\lambda$ is a positive parameter. More precisely, we obtain the following.

**Theorem 1.3.** Assume that the nonlinearity $f$ satisfies (H1) through (H4) and the following condition:

(H5) There exists a continuous, non-decreasing function $f_0 : [0, +\infty) \to R$ such that $f_0(0) = 0$ and that the map $s \mapsto f(s) + f_0(s)$ is non-decreasing.

Then there exists a $\lambda_0 > 0$ such that the Problem (1.3) has a solution $u_1(x)$ satisfying $0 \leq u_1(x) \leq 1$ in $\Omega$, for any $\lambda < \lambda_0$. Moreover, let $R = \text{dist}(0, \partial \Omega)$. Then there exists a non-negative solution $u_1$ such that

$$\lim_{\lambda \to 0} m \{ x \in \Omega : u_1(x) = 1 \} \geq m(B_R(0)),$$

where $m$ denotes the Lebesgue measure in $R^N$. 

---

Observe that results similar to those of Theorem 1.3 are obtained in [5] and [10].

The paper is organized as follows. In Section 2, we establish notation. We outline basic facts used to study the function \( I(a) \), and we prove Theorem 1.1. Section 3 is devoted to proving Theorem 1.3, which is an important application of our results.

2. Time-mapping analysis

In this section, we briefly introduce the well known time-mapping function. Since we are interested only in positive solutions of the Problem \((P_i)\), we consider the initial value problem

\[
(P_i) \begin{cases}
\lambda \left( |u|^{p-2} u \right)' + f(u) = 0, \\
u(0) = 0, \quad u'(0) = E > 0.
\end{cases}
\]

Let \( p > 1 \), and assume that \( u \) is a positive solution of the Problem \((P_i)\). Then the energy relation associated with this problem is given by

\[
F(u(x)) + \lambda \left( \frac{p-1}{p} |u'(x)|^p \right) = \lambda \left( \frac{p-1}{p} |E|^p \right) \quad \text{for all } x \in (0, 1),
\]

where \( F(u) = \int_0^1 f(t) \, dt \). Define \( E_0 \) and \( u_E \) by

\[
F(1) = \lambda \left( \frac{p-1}{p} |E_0|^p \right) \quad \text{and} \quad F(u_E) = \lambda \left( \frac{p-1}{p} |E|^p \right), \quad \text{with } 0 < u_E < 1.
\]

Since we study the solutions of the Problem \((P_i)\), satisfying \( u(1) = 0 \) and \( \sup_{x \in [0, 1]} |u(x)| \leq 1 \), we must take \( E \leq E_0 \). We associate to the Problem \((P_i)\) its time-mapping function

\[
T_i(E) = \left\{ \lambda \left( \frac{p-1}{p} \right) \right\}^{\frac{1}{p}} \int_0^{u_E} \left( F(u_E) - F(u) \right)^{\frac{1}{p}} \, du
\]

\[
= C_{i,p} \int_0^{u_E} (F(u_E) - F(u))^\frac{1}{p} \, du
\]

\[
= C_{i,p} I(u_E) \quad \text{with} \quad C_{i,p} = \left\{ \lambda \left( \frac{p-1}{p} \right) \right\}^{\frac{1}{p}},
\]

which is a function from \((0, E_0]\) to \((0, +\infty)\). Observe that \( E \rightarrow u_E \) is a strictly increasing function of class \( \mathcal{C}^1 \) in \((0, E_0)\). Therefore, it is convenient to study the function \( I(a) \) rather than \( T_i(E) \).

The next Proposition gives us the behavior of the function \( I \).

**Proposition 2.1.** For all \( p > 1 \), the function \( I(\cdot) \) is continuous in \((0,1)\). In particular, \( I(1) = \lim_{a \rightarrow -1} I(a) \) is finite. Then:

(i) When \( f_0 = +\infty \), the function \( I(\cdot) \) is strictly increasing and

\[
\lim_{a \rightarrow -0^+} I(a) = 0.
\]

(ii) When \( 0 < f_0 < +\infty \), the function \( I(\cdot) \) is strictly increasing and

\[
\lim_{a \rightarrow -0^+} I(a) = I_0.
\]

(iii) When \( f_0 = 0 \), there exists an \( a^* \in (0, 1) \) such that \( I(\cdot) \) is strictly decreasing in \((0, a^*) \) and strictly increasing in \((a^*, 1) \). Moreover, \( I(\cdot) \) satisfies

\[
\lim_{a \rightarrow -0^*} I(a) = +\infty.
\]

**Remark 2.2.** We point out that the limits of (i), (ii), and (iii) are already known. See, for example, [4] (when \( f_0 \in [0, +\infty) \)) and [2,3,9,12] (when \( f_0 = +\infty \)).

**Proof of Proposition 2.1.** The proof follows the same lines as that of Takeuchi and Yamada [11, Lemma 3.1].

It is not difficult to see that \( I(\cdot) \) is continuous in \((0,1)\). In particular, it follows from hypothesis \((H_2)\) that \( (F(1) - F(u))^\frac{1}{p} = O(1 - u)^{\frac{1}{p}} \) near \( u = 1 \). In other words, \( I(1) \) exists and is finite.

We begin the proofs of (i) and (ii).
According to Remark 2.2, it suffices to prove the conclusions about the behavior of the function \( I \). For this, we make the change of variables \( u = as \) to obtain

\[
I(a) = a \int_0^1 (F(a) - F(as))^{\frac{1}{p}} ds.
\]

A direct calculation yields

\[
I'(a) = a \int_0^1 \left( \frac{1}{p} \right) \Psi'(s, a)^{\frac{1}{p} - 1} \Psi'_a(s, a) ds + \int_0^1 \Psi'(s, a)^{\frac{1}{p}} ds
\]

\[
= \frac{1}{p} \int_0^1 \mathcal{Y}(s, a) \Psi'(s, a)^{1 - \frac{1}{p}} ds \quad (*),
\]

where \( \mathcal{Y}(s, a) = p \Psi'(s, a) - a \Psi'_a(s, a) \) and \( \Psi'(s, a) = F(a) - F(as) \). Here \( \Psi'_a \) denotes \( \frac{\partial \Psi}{\partial a} \). It follows from \( (H_1) \) that \( \Psi'(s, a) > 0 \) for all \( s, a \in (0, 1) \). According to part (i) of hypothesis \( (H_4) \), we have

\[
\mathcal{Y}(s, a) = p \Psi'(s, a) - a \Psi'_a(s, a)
\]

\[
= pF(a) - af(a) - (pF(as) - asf(as))
\]

\[
= G(a) - G(as) > 0 \quad \text{for all } s \in (0, 1).
\]

This shows that \( I'() \) is strictly increasing in \((0, 1)\) in the cases (i) and (ii).

Concerning (iii), since \( \mathcal{Y}(1, a) = 0 \), we conclude that \( \mathcal{Y}(1, a) < 0 \) for \( s \in (0, 1) \) and \( a \in (0, a_0) \) by hypothesis \( (\gamma 1) \) of \((H_4)\). By \((*)\), \( I'(a) < 0 \) in \((0, a_0)\). Thus \( I() \) is strictly monotone decreasing in \((0, a_0)\). We next show that \( I'(a) \to -\infty \) when \( a \to 1^- \).

Indeed, we have

\[
I'(a) = \frac{1}{p} \int_0^1 (F(a) - F(as))^{\frac{p-1}{p}} \left( pF(a) - pF(as) - af(a) + asf(as) \right) ds
\]

\[
= \frac{1}{p} \int_0^{a(1-\delta)} (F(a) - F(t))^{\frac{p-1}{p}} \left( pF(a) - pF(t) - af(a) + tf(t) \right) dt
\]

\[
+ \frac{1}{p} \int_{1-\delta}^1 (F(a) - F(as))^{\frac{p-1}{p}} \left( pF(a) - pF(as) - af(a) + asf(as) \right) ds
\]

\[
= I_1(a) + I_2(a),
\]

which implies that \( |I_1(1)| = +\infty \). Now according to condition \((H_2)\), given \( \varepsilon > 0 \) there exists \( 0 < \delta < 1 - a_1 \) such that if \( 0 < 1 - \delta < s < \), then

\[
(\omega - \varepsilon)(1 - s)^\eta < f(s) < (\omega + \varepsilon)(1 - s)^\eta.
\]

Choose \( \varepsilon_0 \) so that \( 0 < \varepsilon_0 < \delta \). Then \((1 - \delta, 1 - \varepsilon_0) \subset (a_1, 1)\). According to hypothesis \( (\gamma 2) \) of condition \((H_4)\), we have

\[
\mathcal{Y}(s, a) = pF(a) - pF(as) - af(a) + asf(as) > 0 \quad \text{in } (0, 1) \times (1 - \delta, 1).
\]

Hence

\[
\liminf_{a \to 1^-} I_2(a) = \frac{1}{p} \liminf_{a \to 1^-} \int_{1-\delta}^1 (F(a) - F(as))^{\frac{p-1}{p}} \mathcal{Y}(s, a) ds
\]

\[
\geq \left( \frac{1}{p} \int_{1-\delta}^{1-\varepsilon_0} (F(1) - F(s))^{\frac{p-1}{p}} [pF(1) - pF(s) + sf(s)] ds \right.
\]

\[
= \int_{1-\delta}^{1-\varepsilon_0} (F(1) - F(s))^{\frac{1}{p}} ds + \frac{1}{p} \int_{1-\delta}^{1-\varepsilon_0} (F(1) - F(s))^{\frac{p-1}{p}} sf(s) ds
\]

\[
\geq L(1 - \varepsilon_0) - L(1 - \delta) + \frac{1}{p} \int_{1-\delta}^{1-\varepsilon_0} (f(\eta(s))(1 - s))^{\frac{p-1}{p}} sf(s) ds
\]

\[
\geq L(1 - \varepsilon_0) - L(1 - \delta) + \frac{(1 - \delta)(\omega + \varepsilon)}{p} \frac{(1 - \eta(s))^{\eta(1 - s)}}{(1 - s)^{\eta(1 - s)}} ds
\]

\[
\geq L(1 - \varepsilon_0) - L(1 - \delta) + C(\varepsilon, \delta, p) \int_{1-\delta}^{1-\varepsilon_0} (1 - s)^{\frac{p-1}{p}} ds,
\]

where \( s < \eta(s) < 1 \) and \( I(r) = \int_0^r (F(1) - F(t))^{\frac{1}{p}} dt \). When \( \varepsilon_0 \to 0 \), we have \( I_2(a) \to +\infty \), whence \( I'(a) \to +\infty \).

Differentiating \((*)\) with respect to \( a \), we have

\[
I'(a) = \frac{1}{p} \int_0^1 (F(a) - F(as))^{\frac{p-1}{p}} \left( pF(a) - pF(as) - af(a) + asf(as) \right) ds
\]

\[
= \frac{1}{p} \int_0^{a(1-\delta)} (F(a) - F(t))^{\frac{p-1}{p}} \left( pF(a) - pF(t) - af(a) + tf(t) \right) dt
\]

\[
+ \frac{1}{p} \int_{1-\delta}^1 (F(a) - F(as))^{\frac{p-1}{p}} \left( pF(a) - pF(as) - af(a) + asf(as) \right) ds
\]

\[
= I_1(a) + I_2(a),
\]

which implies that \( |I_1(1)| = +\infty \). Now according to condition \((H_2)\), given \( \varepsilon > 0 \) there exists \( 0 < \delta < 1 - a_1 \) such that if \( 0 < 1 - \delta < s < \), then

\[
(\omega - \varepsilon)(1 - s)^\eta < f(s) < (\omega + \varepsilon)(1 - s)^\eta.
\]

Choose \( \varepsilon_0 \) so that \( 0 < \varepsilon_0 < \delta \). Then \((1 - \delta, 1 - \varepsilon_0) \subset (a_1, 1)\). According to hypothesis \( (\gamma 2) \) of condition \((H_4)\), we have

\[
\mathcal{Y}(s, a) = pF(a) - pF(as) - af(a) + asf(as) > 0 \quad \text{in } (0, 1) \times (1 - \delta, 1).
\]

Hence

\[
\liminf_{a \to 1^-} I_2(a) = \frac{1}{p} \liminf_{a \to 1^-} \int_{1-\delta}^1 (F(a) - F(as))^{\frac{p-1}{p}} \mathcal{Y}(s, a) ds
\]

\[
\geq \left( \frac{1}{p} \int_{1-\delta}^{1-\varepsilon_0} (F(1) - F(s))^{\frac{p-1}{p}} [pF(1) - pF(s) + sf(s)] ds \right.
\]

\[
= \int_{1-\delta}^{1-\varepsilon_0} (F(1) - F(s))^{\frac{1}{p}} ds + \frac{1}{p} \int_{1-\delta}^{1-\varepsilon_0} (F(1) - F(s))^{\frac{p-1}{p}} sf(s) ds
\]

\[
= L(1 - \varepsilon_0) - L(1 - \delta) + \frac{1}{p} \int_{1-\delta}^{1-\varepsilon_0} (f(\eta(s))(1 - s))^{\frac{p-1}{p}} sf(s) ds
\]

\[
\geq L(1 - \varepsilon_0) - L(1 - \delta) + \frac{(1 - \delta)(\omega + \varepsilon)}{p} \frac{(1 - \eta(s))^{\eta(1 - s)}}{(1 - s)^{\eta(1 - s)}} ds
\]

\[
\geq L(1 - \varepsilon_0) - L(1 - \delta) + C(\varepsilon, \delta, p) \int_{1-\delta}^{1-\varepsilon_0} (1 - s)^{\frac{p-1}{p}} ds,
\]

where \( s < \eta(s) < 1 \) and \( I(r) = \int_0^r (F(1) - F(t))^{\frac{1}{p}} dt \). When \( \varepsilon_0 \to 0 \), we have \( I_2(a) \to +\infty \), whence \( I'(a) \to +\infty \).
\[ I'(a) = \frac{1}{p} \int_0^1 \left( \tau(s, a) \left( -1 + \frac{1}{p} \right) \psi'(s, a)^{2-p} \psi(s, a) + \psi(s, a)^{1-p} \tau_a(s, a) \right) \, ds \]
\[ = \frac{p+1}{p^2} \int_0^1 \tau(s, a) \psi'(s, a)^{2-p} \psi(s, a) \, ds + \frac{1}{p} \int_0^1 \psi(s, a)^{1-p} \tau_a(s, a) \, ds \]
\[ = \frac{p+1}{p^2} \int_0^1 \tau(s, a) \psi'(s, a)^{2-p} \psi(s, a) \, ds + \frac{1}{p} \int_0^1 \psi(s, a)^{1-p} \tau_a(s, a) \, ds \]
\[ = \frac{p^2(p+1)}{ap^2} I'(a) + \frac{p+1}{ap^2} \int_0^1 \tau(s, a) \psi'(s, a)^{2-p} \psi(s, a) \, ds + \frac{1}{p} \int_0^1 \psi(s, a)^{1-p} \tau_a(s, a) \, ds. \]

According to hypothesis (\( T' \)) of condition \((H_4), \) for \( a \in (a_0, 1), \) we have
\[ p^2(p+1)I'(a) + ap^2I'(a) = (p+1) \int_0^1 \tau(s, a) \psi'(s, a)^{2-p} \psi(s, a) \, ds + ap \int_0^1 \psi(s, a)^{1-p} \tau_a(s, a) \, ds > 0. \]

We combine the decreasing property of \( I(\cdot) \) in \((0, a_0)\) with the fact that \( I'(a) \to +\infty \) when \( a \to 1^- \) to conclude that \( I(\cdot) \) must have at least one critical point in \((a_0, 1). \) Let \( a^* \) be a critical point of \( I(\cdot), \) that is, \( I'(a^*) = 0. \) Now the preceding inequality implies that \( I'(a^*) > 0, \) which shows that \( I(\cdot) \) does not admit a local maximum. Therefore, there exists an \( a^* \in (a_0, 1) \) such that \( I(\cdot) \) is strictly monotone decreasing (resp. increasing) in \((0, a^*) \) (resp. \((a^*, 1)). \) This completes the proof of Proposition 2.1. \( \square \)

**Proof of Theorem 1.1.**

(a)-(i): By part (i) of Proposition 2.1, the function \( I \) is strictly increasing and \( I(a) \to 0. \) Then there exists a unique \( E^* \in (0, E_0) \) such that \( 2T_s(E^*) = 1. \) where
\[ T_s(E) = \left\{ \frac{1}{2} \int_0^1 (F(uE) - F(u)) \, du \right\}^{1/2} = C_{s,p} I(u_E). \]

Now since
\[ 2T_s(E_0) = 2C_{s,p} I(1) = \left( \frac{\lambda}{\lambda_0(1)} \right)^{1/2}, \]
where \( \lambda_0(1) = \frac{p}{p+1} (2I(1))^{-p}, \) we have that \( E^* < E_0 \) if \( \lambda_0(1) < \lambda \) and that \( E^* = E_0 \) if \( \lambda \leq \lambda_0(1). \) This means that for \( \lambda < \lambda_0(1), \) our solution \( u_E \) has a flat core in the interval \([T_s(E_0), 1 - T_s(E_0)], \) that is, \( u \equiv 1 \) in \([T_s(E_0), 1 - T_s(E_0)]. \)

(a)-(ii): According to part (ii) of Proposition 2.1, we have
\[ \inf_{E \in (0, E_0)} 2T_s(E) = 2T_s(0) \equiv 2C_{s,p} \lim_{a \to a^*} I(a) = 2C_{s,p} I_0 = \left( \frac{\lambda}{\lambda_0(1)} \right)^{1/2}, \]
where
\[ I_0 = \frac{1}{2} \left( \frac{p}{p-1} \right)^{1/2} \left( \frac{\lambda(p)}{\lambda(p)} \right)^{1/2}. \]
Here \( \lambda(p) = \frac{p}{p-1} (2 \int_0^1 (1 - e^t) \, dt)^p \) denotes the first eigenvalue of the \( p \)-Laplacian in \((0, 1)\) under homogeneous Dirichlet boundary conditions. Observe that there exists a \( \lambda_2, \) with \( 0 < \lambda < \lambda_0, \) such that for \( \lambda_2 < \lambda < \lambda_0 \) there is \( E^* \in (0, E_0) \) satisfying \( 2T_s(E^*) = 1, \) and furthermore that for \( 0 < \lambda \leq \lambda_2, \) we have \( 2T_s(E_0) \leq 1. \) Arguments similar to those of the proof of (a)-(i) show that for \( 0 < \lambda < \lambda_2, \) the solution \( u_E \) associated to \( T_s(E_0) \) has a flat core.

(b): By part (iii) of Proposition 2.1, there is an \( a^* \in (0, 1) \) such that
\[ \inf_{E \in (0, E_0)} 2T_s(E) = 2T_s(a^*) = 2C_{s,p} I(a^*) = \left( \frac{\lambda}{\lambda_0} \right)^{1/4}. \]

Since \( I \) is decreasing in \((0, a^*)\) and increasing in \((a^*, 1), \) there exists a \( \lambda_2 \) with \( 0 < \lambda_2 < \lambda_0, \) satisfying the following property: for \( \lambda_2 < \lambda < \lambda_0, \) there are \( E_1, E_2, \) with \( 0 < E_1 < a^* < E_2 < 1, \) satisfying \( 2T_s(E_1) = 2T_s(E_2) = 1. \) Now for \( 0 < \lambda \leq \lambda_2, \) there exists \( E_1, \) with \( 0 < E_1 < a^*, \) satisfying \( 2T_s(E_1) = 1. \) In this case, observe that we always have \( 2T_s(E_0) \leq 1. \) Therefore as in the preceding two cases, for \( 0 < \lambda < \lambda_2, \) the solution \( u_E \) associated to \( T_s(E_0) \) has a flat core.

As a consequence of the proof of Theorem 1.1, the flat core \( \mathcal{E}_s \) of the positive solution \( u_s, \) associated to \( T_s(E_0) \) spreads out toward all of \((0,1)\) as \( \lambda \to 0. \) \( \square \)

**Corollary 1.** Let \( u_s \) be the solution of \((P_s)\) associated to \( T_s(E_0) \) as in the proof of Theorem 1.1. Then \( m(\mathcal{E}_s(u_s)) \to 1, \) where \( m \) denotes the Lebesgue measure in \( \mathbb{R}. \)
3. Application to quasilinear elliptic equations with singular weights

As an application of our results, in this section we show the existence of a non-negative solution of the singular quasilinear elliptic problem

\[(P_\lambda) \begin{cases} \lambda \text{div} \left( |x|^{-ap} |\nabla u|^{p-2} \nabla u \right) + |x|^{-\left(q(a+1)p-c\right)} f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases} \]

where the nonlinearity \( f \) satisfies hypotheses (H1) through (H4), \( 1 < p < N, c > 0, -\infty < a < \min \left\{ \frac{n-p}{p}, \frac{n-p}{p} + \frac{c-1}{p} \right\} \), \( 0 \in \Omega, \Omega \subset \mathbb{R}^N \) is a bounded domain with \( \mathcal{C}^1 \)-boundary, and \( \lambda \) is a positive parameter.

Finally, a function \( v \) is said to be less than or equal to \( u \) on \( \partial \Omega \) if

\[ \frac{\lambda}{\mu} \int_{\Omega} |x|^{-ap} |\nabla u|^{p-2} \nabla u \nabla \phi \leq \int_{\Omega} |x|^{-\left(q(a+1)p-c\right)} f(u) \phi \]

for each \( \phi \in W^{1,p}_0(\Omega, |x|^{-ap}) \) with \( \phi \geq 0 \),

\[ u \leq 0 \quad \text{on } \partial \Omega. \]

(3.1) \( u(0) = 0 \) and center 0. In order to obtain a radial lower solution of the Problem (3.1)_\lambda, we will find a lower solution \( u(x) \) of the Problem (3.1)_\lambda in the “interval”

\[ [u, \bar{u}] = \{ u \in L^\infty(\Omega) : u(x) \leq u(x) \leq \bar{u}(x) \text{ a.e. on } \Omega \}. \]

In particular, every weak solution \( u \in [u, \bar{u}] \) of (3.1)_\lambda also satisfies \( u(x) \leq u(x) \leq u^*(x) \) for a.e. \( x \in \Omega \).

**Proof of Theorem 1.3.** The proof essentially consists of constructing suitable upper and lower solutions by the idea of García-Melián and Sabina de Lis [5] (see also [10]). Since \( \lambda = 1 \) is an upper solution of the Problem (3.1)_\lambda, it suffices to show the existence of a lower solution of the Problem (3.1)_\lambda satisfying \( 0 \leq u \leq 1 \) because we obtain the required solution by Theorem 3.1. For this, we take \( R > 0 \) such that \( B_R(0) \subset \Omega \), where \( B_R(0) \) is the ball with radius \( R \) and center 0. In order to obtain a radial lower solution of the Problem (3.1)_\lambda, we will find a lower solution \( u_0 \) of the Problem (3.1)_\lambda in \( [0, \bar{u}] \). In fact, it suffices to find a radially symmetrical lower solution, or in other words a solution \( v(\rho) = u_0(\rho \xi) \) satisfying the equation

\[ \xi = \frac{\rho^{1-\frac{1}{p}} - \rho^{1-\frac{p}{1-x}}}{1-\frac{1}{x}}, \]

where \( \rho = |x|, \xi = g(\rho) = \frac{\rho^{1-\frac{x}{p}} - \rho^{1-\frac{1}{p}}} {1-\frac{1}{p}} \),

Eq. (3.1) can be rewritten as

\[ \xi \left( \omega \right) \left( \frac{p-2}{p-2} \omega \right) \leq \xi + g^{-1}(\xi) p^{N-1-\left(q(a+1)p-c\right)} f(w) \geq 0 \quad \text{in } [0, \infty), \]

\[ w(0) = \lim_{x \to +\infty} w(x) = 0. \]

where \( w(0) = \lim_{x \to +\infty} w(x) = 0. \)
Now to find a function $w$ satisfying Eq. (3.2), we take any $b \in (0, +\infty)$ and consider the auxiliary boundary value problem

\[
\begin{aligned}
\dot{\lambda}\left(|\psi|^p - \psi\right) + g^{-1}(b)p^{(p-1)+c}f(\phi) &= 0 \quad \text{in } (0, b), \\
\phi(0) &= \phi(b) = 0.
\end{aligned}
\tag{3.3}
\]

A change of scale $\xi = b\eta$ yields

\[
\begin{aligned}
A \left(\psi_{\eta}^{p-2}\psi_{\eta}\right)_{\eta} + f(\psi) &= 0 \quad \text{in } (0, 1), \\
\psi(0) &= \psi(1) = 0,
\end{aligned}
\tag{3.4}
\]

where $\psi(\eta) = \phi(b\eta)$ and $A = \lambda b^p g^{-1}(b)p^{(1-x)-c}$.

Arguing as in the proof of Theorem 1.1 we study the problem

\[
(P)_{A,E} = \begin{cases} 
\left(\psi_{\eta}^{p-2}\psi_{\eta}\right)_{\eta} + f(\psi) = 0 & \text{in } (0, 1), \\
\psi(0) = \psi(1) = 0, \\
\end{cases}
\]

Using the time-mapping function

\[
T_+(A,E) = \int_0^{2\pi} \left( E^p - A^{-1} p^p F(t) \right)^{-1/p} dt,
\]

where $F(t) = \int_0^t f(s) ds$ and $z_{hi} = z_{hi}(A,E)$ is the first positive zero of

\[
H(A,E,t) = E^p - A^{-1} p^p F(t) \quad \text{where } p^* = p/(p-1),
\]

we obtain the existence of $\lambda_0 \in (0, +\infty)$ such that the Problem $(P)_{\lambda_0}$ always has a solution $0 \leq \psi \leq 1$ for $A \in (0, \lambda_0)$.

Setting $A^* = 2^{1/p} C(f,p)^{-1}$ where $C(f,p) = (p^*)^{1/p} \int_0^1 (F(1)-F(t))^{-1/p} dt$, in the case $A^* = \lambda_0$, we obtain $T_+(A,E_0) \leq 1/2$ for $A \leq A^*$. We thus find a positive solution $\psi$ of the Problem $(P)_{\lambda_0}$ in the interval $(0, T_+(A,E_0))$ so that $\psi(T_+(A,E_0)) = 1$ and

\[
\psi'(T_+(A,E_0)) = 0.
\]

Defining

\[
v(t) = \begin{cases} 
\psi(t) & \text{if } t \in (0, T_+(A,E_0)), \\
1 & \text{if } t \in [T_+(A,E_0), 1 - T_+(A,E_0)], \\
\psi(1-t) & \text{if } t \in [1 - T_+(A,E_0), 1],
\end{cases}
\]

we obtain a positive solution $v(t)$ of (3.4). Using $v$ and defining $C_{l,b} = bT_+(A,E_0) < b/2$, we see that the Problem (3.3) has a positive solution $\phi$ such that $\phi(t) = 1$ for $t \in [C_{l,b}, b - C_{l,b}]$ and that $0 < \phi(t) < 1$ elsewhere.

Let

\[
w(t) = \begin{cases} 
\phi(t) & \text{if } t \in [0, C_{l,b}], \\
1 & \text{if } t \in [C_{l,b}, T).
\end{cases}
\]

Then $w$ is a solution of Eq. (3.2). Indeed, since $g^{-1}$ is monotone decreasing, we have

\[
\dot{\lambda}\left(|w|^p - w\right) + g^{-1}(1/|w|^{p-1}+c)f(\phi) = g^{-1}(1/|w|^{p-1}+c) - g^{-1}(b)p^{(p-1)+c}
\]

in $[0, C_{l,b}]$ and the boundary conditions are satisfied trivially. Thus $n(\rho) = w(g(\rho))$ satisfies Eq. (3.1). Therefore, the function

\[
v(x) = \begin{cases} 
1 & \text{if } 0 \leq |x| \leq g^{-1}(C_{l,b}), \\
\phi(g(|x|)) & \text{if } g^{-1}(C_{l,b}) < |x| \leq R, \\
0 & \text{if } x \in \Omega \setminus B_R(0)
\end{cases}
\]

is a lower solution of $(P)_{A,E}$.

Finally when $A^* < \lambda_0$, for each $A^* < A < \lambda_0$ we can find a positive solution $\psi_A$ of the Problem $(P)_{A,E}$ in the interval $(0,1/2)$. Using a symmetrical argument, we extend this solution to the whole interval $(0,1)$. Observe that this solution satisfies $0 < \psi_A < 1$ in $(0,1)$. As in the preceding case, we may recover a lower solution of the Problem $(P)_{A}$ from the solution $\psi_A$. \qed

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